Quarter Squares Rule

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab.$$

Quartet

A SET of four, also called a TETRAD.

see also HEXAD, MONAD, QUINTET, TETRAD, TRIAD

Quartic Curve

A general plane quartic curve is a curve of the form

$$Ax^4 + By^4 + Cx^3y + Dx^2y^2 + Exy^3 + Fx^3 + Gy^3 + Hx^2y + Ixy^2 + Jx^2 + Ky^2 + Lxy + Mx + Ny + O = 0. \quad (1)$$

The incidence relations of the 28 bitangents of the general quartic curve can be put into a ONE-TO-ONE correspondence with the vertices of a particular POLYTOPE in 7-D space (Coxeter 1928, Du Val 1931). This fact is essentially similar to the discovery by Schoutte (1910) that the 27 SOLOMON'S SEAL LINES on a CUBIC SURFACE can be connected with a POLYTOPE in 6-D space (Du Val 1931). A similar but less complete relation exists between the tritangent planes of the canonical curve of genus 4 and an 8-D POLYTOPE (Du Val 1931).

The maximum number of DOUBLE POINTS for a nondegenerate quartic curve is three.

A quartic curve of the form

$$y^2 = (x-\alpha)(x-\xi)(x-\gamma)(x-\delta) \quad (2)$$

can be written

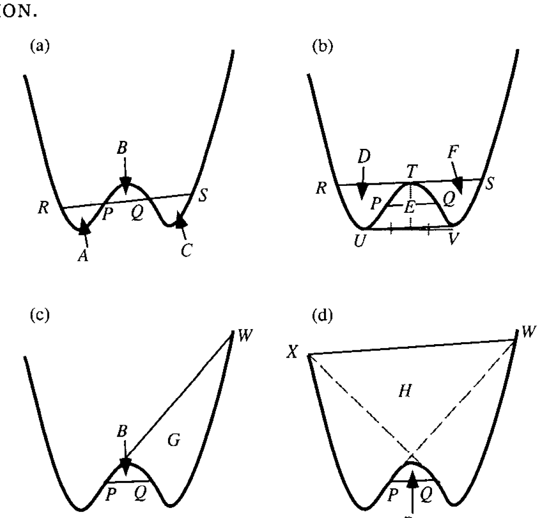
$$\left(\frac{y}{x-\alpha}\right)^2 = \left(1 - \frac{\beta-\alpha}{x-\alpha}\right)\left(1 - \frac{\gamma-\alpha}{x-\alpha}\right)\left(1 - \frac{\delta-\alpha}{x-\alpha}\right), \quad (3)$$

and so is CUBIC in the coordinates

$$X = \frac{1}{x-\alpha} \quad (4)$$

$$Y = \frac{y}{x-\alpha^2}. \quad (5)$$

This transformation is a BIRATIONAL TRANSFORMATION.



Let P and Q be the INFLECTION POINTS and R and S the intersections of the line PQ with the curve in Figure (a) above. Then

$$A = C \quad (6)$$

$$B = 2A. \quad (7)$$

In Figure (b), let UV be the double tangent, and T the point on the curve whose x coordinate is the average of the x coordinates of U and V. Then $UV||PQ||RS$ and

$$D = F \quad (8)$$

$$E = \sqrt{2}\,D. \quad (9)$$

In Figure (c), the tangent at P intersects the curve at W. Then

$$G = 8B. \quad (10)$$

Finally, in Figure (d), the intersections of the tangents at P and Q are W and X. Then

$$H = 27B \quad (11)$$

(Honsberger 1991).

see also CUBIC SURFACE, PEAR-SHAPED CURVE, SOLOMON'S SEAL LINES

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Quartic Equation

A general quartic equation (also called a BIQUADRATIC EQUATION) is a fourth-order POLYNOMIAL of the form

$$z^4 + a_3z^3 + a_2z^2 + a_1z + a_0 = 0. \quad (1)$$

The ROOTS of this equation satisfy NEWTON'S RELATIONS:

$$x_1 + x_2 + x_3 + x_4 = -a_3 \quad (2)$$

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = a_2 \quad (3)$$

$$x_1x_2x_3 + x_2x_3x_4 + x_1x_2x_4 + x_1x_3x_4 = -a_1 \quad (4)$$

$$x_1x_2x_3x_4 = a_0, \quad (5)$$

where the denominators on the right side are all $a_4 \equiv 1$.

Ferrari was the first to develop an algebraic technique for solving the general quartic. He applied his technique (which was stolen and published by Cardano) to the equation

$$x^4 + 6x^2 - 60x + 36 = 0 \tag{6}$$

(Smith 1994, p. 207).

The x^3 term can be eliminated from the general quartic (1) by making a substitution of the form

$$z \equiv x - \lambda, \tag{7}$$

so

$$\begin{aligned} x^4 + (a_3 - 4\lambda)x^3 + (a_2 - 3a_3\lambda + 6\lambda^2)x^2 \\ +(a_1 - 2a_2\lambda + 3a_3\lambda^2 - 4\lambda^3)x \\ +(a_0 - a_1\lambda + a_2\lambda^2 - a_3\lambda^3 + \lambda^4). \end{aligned} \tag{8}$$

Letting $\lambda = a_3/4$ so

$$z \equiv x - \tfrac{1}{4}\lambda \tag{9}$$

then gives

$$x^4 + px^2 + qx + r, \tag{10}$$

where

$$p \equiv a_2 - \tfrac{3}{8}a_3{}^2 \tag{11}$$

$$q \equiv a_1 - \tfrac{1}{2}a_2a_3 + \tfrac{1}{8}a_3{}^3 \tag{12}$$

$$r \equiv a_0 - \tfrac{1}{4}a_1a_3 + \tfrac{1}{16}a_2a_3{}^2 - \tfrac{3}{256}a_3{}^4. \tag{13}$$

Adding and subtracting $x^2u + u^2/4$ to (10) gives

$$x^4 + x^2u + \tfrac{1}{4}u^2 - x^2u - \tfrac{1}{4}u^2 + px^2 + qx + r = 0, \tag{14}$$

which can be rewritten

$$(x^2 - \tfrac{1}{2}u)^2 - [(u-p)x^2 - qx + (\tfrac{1}{4}u^2 - r)] = 0 \tag{15}$$

(Birkhoff and Mac Lane 1965). The first term is a perfect square P^2, and the second term is a perfect square Q^2 for those u such that

$$q^2 = 4(u-p)(\tfrac{1}{4}u^2 - r). \tag{16}$$

This is the resolvent CUBIC, and plugging a solution u_1 back in gives

$$P^2 - Q^2 = (P+Q)(P-Q), \tag{17}$$

so (15) becomes

$$(x^2 + \tfrac{1}{2}u_1 + Q)(x^2 + \tfrac{1}{2}u_1 - Q), \tag{18}$$

where

$$Q \equiv Ax - B \tag{19}$$

$$A \equiv \sqrt{u_1 - p} \tag{20}$$

$$B \equiv -\frac{q}{2A}. \tag{21}$$

Let y_1 be a REAL ROOT of the resolvent CUBIC EQUATION

$$y^3 - a_2y^2 + (a_1a_3 - 4a_0)y + (4a_2a_0 - a_1{}^2 - a_3{}^2a_0) = 0. \tag{22}$$

The four ROOTS are then given by the ROOTS of the equation

$$\begin{aligned} x^2 + \tfrac{1}{2}(a_3 \pm \sqrt{a_3{}^2 - 4a_2 + 4y_1}) \\ + \tfrac{1}{2}(y_1 \mp \sqrt{y_1{}^2 - 4a_0}\,) = 0, \end{aligned} \tag{23}$$

which are

$$z_1 = -\tfrac{1}{4}a_3 + \tfrac{1}{2}R + \tfrac{1}{2}D \tag{24}$$

$$z_2 = -\tfrac{1}{4}a_3 + \tfrac{1}{2}R - \tfrac{1}{2}D \tag{25}$$

$$z_3 = -\tfrac{1}{4}a_3 - \tfrac{1}{2}R + \tfrac{1}{2}E \tag{26}$$

$$z_4 = -\tfrac{1}{4}a_3 - \tfrac{1}{2}R - \tfrac{1}{2}E, \tag{27}$$

where

$$R \equiv \sqrt{\tfrac{1}{4}a_3{}^2 - a_2 + y_1} \tag{28}$$

$$D \equiv \begin{cases} \sqrt{\tfrac{3}{4}a_3{}^2 - R^2 - 2a_2 + \tfrac{1}{4}(4a_3a_2 - 8a_1 - a_3{}^3)R^{-1}} \\ R \neq 0 \\ \sqrt{\tfrac{3}{4}a_3{}^2 - 2a_2 + 2\sqrt{y_1{}^2 - 4a_0}} \\ R = 0 \end{cases} \tag{29}$$

$$E \equiv \begin{cases} \sqrt{\tfrac{3}{4}a_3{}^2 - R^2 - 2a_2 - \tfrac{1}{4}(4a_3a_2 - 8a_1 - a_3{}^3)R^{-1}} \\ R \neq 0 \\ \sqrt{\tfrac{3}{4}a_3{}^2 - 2a_2 - 2\sqrt{y_1{}^2 - 4a_0}} \\ R = 0. \end{cases} \tag{30}$$

Another approach to solving the quartic (10) defines

$$\alpha = (x_1 + x_2)(x_3 + x_4) = -(x_1 + x_2)^2 \tag{31}$$

$$\beta = (x_1 + x_3)(x_2 + x_4) = -(x_1 + x_3)^2 \tag{32}$$

$$\gamma = (x_1 + x_4)(x_2 + x_3) = -(x_2 + x_3)^2, \tag{33}$$

where use has been made of

$$x_1 + x_2 + x_3 + x_4 = 0 \tag{34}$$

(which follows since $a_3 = 0$), and

$$h(x) = (x-\alpha)(x-\beta)(x-\gamma) \tag{35}$$

$$= x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma. \tag{36}$$

Comparing with

$$\begin{aligned} P(x) &= x^3 + px^2 + qx + r & (37) \\ &= (x - x_1)(x - x_2)(x - x_3)(x - x_4) & (38) \\ &= x^3 + \left(\prod_{i \neq j}^{4} x_i x_j \right) x^2 \\ &\quad + (x_1 + x_2)(x_1 + x_3)(x_2 + x_3)x \\ &\quad - x_1 x_2 x_3 (x_1 + x_2 + x_3), & (39) \end{aligned}$$

gives

$$h(x) = x^3 - 2px^2 = (p^2 - r)z + q^2. \qquad (40)$$

Solving this CUBIC EQUATION gives α, β, and γ, which can then be solved for the roots of the quartic x_i (Faucette 1996).

see also CUBIC EQUATION, DISCRIMINANT (POLYNOMIAL), QUINTIC EQUATION

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Quartic Reciprocity Theorem

Gauss stated the case $n = 4$ using the GAUSSIAN INTEGERS.

see also RECIPROCITY THEOREM

References

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Quartic Surface

An ALGEBRAIC SURFACE of ORDER 4. Unlike CUBIC SURFACES, quartic surfaces have not been fully classified.

see also BOHEMIAN DOME, BURKHARDT QUARTIC, CASSINI SURFACE, CUSHION, CYCLIDE, DESMIC SURFACE, KUMMER SURFACE, MITER SURFACE, PIRIFORM, ROMAN SURFACE, SYMMETROID, TETRAHEDROID, TOOTH SURFACE

References

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Jessop, C. *Quartic Surfaces with Singular Points.* Cambridge, England: Cambridge University Press, 1916.

Quartile

One of the four divisions of observations which have been grouped into four equal-sized sets based on their RANK. The quartile including the top RANKED members is called the first quartile and denoted Q_1. The other quartiles are similarly denoted Q_2, Q_3, and Q_4. For N data points with N of the form $4n+5$ (for $n = 0$, 1, ...), the HINGES are identical to the first and third quartiles.

see also HINGE, INTERQUARTILE RANGE, QUARTILE DEVIATION, QUARTILE VARIATION COEFFICIENT

Quartile Deviation

$$\mathrm{QD} = \tfrac{1}{2}(Q_3 - Q_1),$$

where Q_1 and Q_2 are INTERQUARTILE RANGES.

see also QUARTILE VARIATION COEFFICIENT

Quartile Range

see INTERQUARTILE RANGE

Quartile Skewness Coefficient

see BOWLEY SKEWNESS

Quartile Variation Coefficient

$$V \equiv 100 \frac{Q_3 - Q_1}{Q_3 + Q_1},$$

where Q_1 and Q_2 are INTERQUARTILE RANGES.

Quasiamicable Pair

Let $\sigma(m)$ be the DIVISOR FUNCTION of m. Then two numbers m and n are a quasiamicable pair if

$$\sigma(m) = \sigma(n) = m + n + 1.$$

The first few are (48, 75), (140, 195), (1050, 1925), (1575, 1648), ... (Sloane's A005276). Quasiamicable numbers are sometimes called BETROTHED NUMBERS or REDUCED AMICABLE PAIRS.

see also AMICABLE PAIR

References

Beck, W. E. and Najar, R. M. "More Reduced Amicable Pairs." *Fib. Quart.* **15**, 331–332, 1977.

Guy, R. K. "Quasi-Amicable or Betrothed Numbers." §B5 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 59–60, 1994.
Hagis, P. and Lord, G. "Quasi-Amicable Numbers." *Math. Comput.* **31**, 608–611, 1977.
Sloane, N. J. A. Sequence A005276/M5291 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Quasiconformal Map

A generalized CONFORMAL MAP.

see also BELTRAMI DIFFERENTIAL EQUATION

References
Iyanaga, S. and Kawada, Y. (Eds.). "Quasiconformal Mappings." §347 in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, pp. 1086–1088, 1980.

Quasigroup

A GROUPOID S such that for all $a, b \in S$, there exist unique $x, y \in S$ such that

$$ax = b$$
$$ya = b.$$

No other restrictions are applied; thus a quasigroup need not have an IDENTITY ELEMENT, not be associative, etc. Quasigroups are precisely GROUPOIDS whose multiplication tables are LATIN SQUARES. A quasigroup can be empty.

see also BINARY OPERATOR, GROUPOID, LATIN SQUARE, LOOP (ALGEBRA), MONOID, SEMIGROUP

References
van Lint, J. H. and Wilson, R. M. *A Course in Combinatorics.* New York: Cambridge University Press, 1992.

Quasiperfect Number

A least ABUNDANT NUMBER, i.e., one such that

$$\sigma(n) = 2n + 1.$$

Quasiperfect numbers are therefore the sum of their nontrivial DIVISORS. No quasiperfect numbers are known, although if any exist, they must be greater than 10^{35} and have seven or more DIVISORS. Singh (1997) called quasiperfect numbers SLIGHTLY EXCESSIVE NUMBERS.

see also ABUNDANT NUMBER, ALMOST PERFECT NUMBER, PERFECT NUMBER

References
Guy, R. K. "Almost Perfect, Quasi-Perfect, Pseudoperfect, Harmonic, Weird, Multiperfect and Hyperperfect Numbers." §B2 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 45–53, 1994.
Singh, S. *Fermat's Enigma: The Epic Quest to Solve the World's Greatest Mathematical Problem.* New York: Walker, p. 13, 1997.

Quasiperiodic Function

see WEIERSTRAß SIGMA FUNCTION, WEIERSTRAß ZETA FUNCTION

Quasiperiodic Motion

The type of motion executed by a DYNAMICAL SYSTEM containing two incommensurate frequencies.

Quasirandom Sequence

A sequence of n-tuples that fills n-space more uniformly than uncorrelated random points. Such a sequence is extremely useful in computational problems where numbers are computed on a grid, but it is not known in advance how fine the grid must be to obtain accurate results. Using a quasirandom sequence allows stopping at any point where convergence is observed, whereas the usual approach of halving the interval between subsequent computations requires a huge number of computations between stopping points.

see also PSEUDORANDOM NUMBER, RANDOM NUMBER

References
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Quasi- (that is, Sub-) Random Sequences." §7.7 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 299–306, 1992.

Quasiregular Polyhedron

A quasiregular polyhedron is the solid region interior to two DUAL regular polyhedra with SCHLÄFLI SYMBOLS$\{p, q\}$ and $\{q, p\}$. Quasiregular polyhedra are denoted using a SCHLÄFLI SYMBOL of the form $\left\{ {p \atop q} \right\}$, with

$$\left\{ {p \atop q} \right\} = \left\{ {q \atop p} \right\}. \tag{1}$$

Quasiregular polyhedra have two kinds of regular faces with each entirely surrounded by faces of the other kind, equal sides, and equal dihedral angles. They must satisfy the Diophantine inequality

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1. \tag{2}$$

But $p, q \geq 3$, so r must be 2. This means that the possible quasiregular polyhedra have symbols $\left\{ {3 \atop 3} \right\}$, $\left\{ {3 \atop 4} \right\}$, and $\left\{ {3 \atop 5} \right\}$. Now

$$\left\{ {3 \atop 3} \right\} = \{3, 4\} \tag{3}$$

is the OCTAHEDRON, which is a regular PLATONIC SOLID and not considered quasiregular. This leaves only two convex quasiregular polyhedra: the CUBOCTAHEDRON $\left\{ {3 \atop 4} \right\}$ and the ICOSIDODECAHEDRON $\left\{ {3 \atop 5} \right\}$.

If nonconvex polyhedra are allowed, then additional quasiregular polyhedra are the GREAT DODECAHEDRON $\{5, \frac{5}{2}\}$ and the GREAT ICOSIDODECAHEDRON $\{3, \frac{5}{2}\}$ (Hart).

For faces to be equatorial $\{h\}$,

$$h = \sqrt{4N_1 + 1} - 1. \tag{4}$$

The EDGES of quasiregular polyhedra form a system of GREAT CIRCLES: the OCTAHEDRON forms three SQUARES, the CUBOCTAHEDRON four HEXAGONS, and the ICOSIDODECAHEDRON six DECAGONS. The VERTEX FIGURES of quasiregular polyhedra are RHOMBUSES (Hart). The EDGES are also all equivalent, a property shared only with the completely regular PLATONIC SOLIDS.

see also CUBOCTAHEDRON, GREAT DODECAHEDRON, GREAT ICOSIDODECAHEDRON, ICOSIDODECAHEDRON, PLATONIC SOLID

References

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Hart, G. W. "Quasi-Regular Polyhedra." http://www.li.net/~george/virtual-polyhedra/quasi-regular-info.html.

Quasirhombicosidodecahedron

see GREAT RHOMBICOSIDODECAHEDRON (UNIFORM)

Quasirhombicuboctahedron

see GREAT RHOMBICUBOCTAHEDRON (UNIFORM)

Quasisimple Group

A FINITE GROUP L is quasisimple if $L = [L, L]$ and $L/Z(L)$ is a SIMPLE GROUP.

see also COMPONENT, FINITE GROUP, SIMPLE GROUP

Quasithin Theorem

In the classical quasithin case of the QUASI-UNIPOTENT PROBLEM, if G does not have a "strongly embedded" SUBGROUP, then G is a GROUP of LIE-TYPE in characteristic 2 of Lie RANK 2 generated by a pair of parabolic SUBGROUPS P_1 and P_2, or G is one of a short list of exceptions.

see also LIE-TYPE GROUP, QUASI-UNIPOTENT PROBLEM

Quasitruncated Cuboctahedron

see GREAT TRUNCATED CUBOCTAHEDRON

Quasitruncated Dodecadocahedron

see TRUNCATED DODECADODECAHEDRON

Quasitruncated Dodecahedron

see TRUNCATED DODECAHEDRON

Quasitruncated Great Stellated Dodecahedron

see GREAT STELLATED TRUNCATED DODECAHEDRON

Quasitruncated Hexahedron

see STELLATED TRUNCATED HEXAHEDRON

Quasitruncated Small Stellated Dodecahedron

see SMALL STELLATED TRUNCATED DODECAHEDRON

Quasi-Unipotent Group

A GROUP G is quasi-unipotent if every element of G of order p is UNIPOTENT for all PRIMES p such that G has p-RANK ≥ 3.

Quasi-Unipotent Problem

see QUASITHIN THEOREM

Quaternary

The BASE 4 method of counting in which only the DIGITS 0, 1, 2, and 3 are used. These DIGITS have the following multiplication table.

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	10	12
3	0	3	12	21

see also BASE (NUMBER), BINARY, DECIMAL, HEXADECIMAL, OCTAL, TERNARY

References

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Weisstein, E. W. "Bases." http://www.astro.virginia.edu/~eww6n/math/notebooks/Bases.m.

Quaternary Tree

see QUADTREE

Quaternion

A member of a *noncommutative* DIVISION ALGEBRA first invented by William Rowan Hamilton. The quaternions are sometimes also known as HYPERCOMPLEX NUMBERS and denoted $\mathbb{H}$. While the quaternions are not commutative, they are associative.

The quaternions can be represented using complex 2×2 MATRICES

$$H = \begin{bmatrix} z & w \\ -w^* & z^* \end{bmatrix} = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}, \qquad (1)$$

where z and w are COMPLEX NUMBERS, a, b, c, and d are REAL, and z^* is the COMPLEX CONJUGATE of z. By analogy with the COMPLEX NUMBERS being representable as a sum of REAL and IMAGINARY PARTS, $a \cdot 1 + bi$, a quaternion can also be written as a linear combination

$$H = a\mathsf{U} + b\mathsf{I} + c\mathsf{J} + d\mathsf{K} \qquad (2)$$

of the four matrices

$$\mathsf{U} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3}$$

$$\mathsf{I} \equiv \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \tag{4}$$

$$\mathsf{J} \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{5}$$

$$\mathsf{K} \equiv \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \tag{6}$$

(Note that here, U is used to denote the IDENTITY MATRIX, not I.) The matrices are closely related to the PAULI SPIN MATRICES σ_x, σ_y, σ_z, combined with the IDENTITY MATRIX. From the above definitions, it follows that

$$\mathsf{I}^2 = -\mathsf{U} \tag{7}$$

$$\mathsf{J}^2 = -\mathsf{U} \tag{8}$$

$$\mathsf{K}^2 = -\mathsf{U}. \tag{9}$$

Therefore I, J, and K are three essentially different solutions of the matrix equation

$$\mathsf{X}^2 = -\mathsf{U}, \tag{10}$$

which could be considered the square roots of the negative identity matrix.

In $\mathbb{R}^4$, the basis of the quaternions can be given by

$$i \equiv \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{11}$$

$$j \equiv \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{12}$$

$$k \equiv \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{13}$$

$$1 \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{14}$$

The quaternions satisfy the following identities, sometimes known as HAMILTON'S RULES,

$$i^2 = j^2 = k^2 = -1 \tag{15}$$

$$ij = -ji = k \tag{16}$$

$$jk = -kj = i \tag{17}$$

$$ki = -ik = j. \tag{18}$$

They have the following multiplication table.

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

The quaternions ± 1, $\pm i$, $\pm j$, and $\pm k$ form a non-Abelian GROUP of order eight (with multiplication as the group operation) known as Q_8.

The quaternions can be written in the form

$$a = a_1 + a_2 i + a_3 j + a_4 k. \tag{19}$$

The conjugate quaternion is given by

$$a^* = a_1 - a_2 i - a_3 j - a_4 k. \tag{20}$$

The sum of two quaternions is then

$$a+b = (a_1+b_1)+(a_2+b_2)i+(a_3+b_3)j+(a_4+b_4)k, \tag{21}$$

and the product of two quaternions is

$$\begin{aligned} ab = (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4) \\ + (a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3)i \\ + (a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2)j \\ + (a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1)k, \end{aligned} \tag{22}$$

so the norm is

$$n(a) = \sqrt{aa^*} = \sqrt{a^*a} = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2 + a_4{}^2}. \tag{23}$$

In this notation, the quaternions are closely related to FOUR-VECTORS.

Quaternions can be interpreted as a SCALAR plus a VECTOR by writing

$$a = a_1 + a_2 i + a_3 j + a_4 k = (a_1, \mathbf{a}), \tag{24}$$

where $\mathbf{a} \equiv [a_2\, a_3\, a_4]$. In this notation, quaternion multiplication has the particularly simple form

$$\begin{aligned} q_1q_2 &= (s_1, \mathbf{v}_1) \cdot (s_2, \mathbf{v}_2) \\ &= (s_1s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, s_1\mathbf{v}_2 + s_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2). \end{aligned} \tag{25}$$

Division is uniquely defined (except by zero), so quaternions form a DIVISION ALGEBRA. The inverse of a quaternion is given by

$$a^{-1} = \frac{a^*}{aa^*}, \tag{26}$$

and the norm is multiplicative

$$n(ab) = n(a)n(b). \tag{27}$$

In fact, the product of two quaternion norms immediately gives the EULER FOUR-SQUARE IDENTITY.

A rotation about the UNIT VECTOR $\hat{\mathbf{n}}$ by an angle θ can be computing using the quaternion

$$q = (s, \mathbf{v}) = (\cos(\tfrac{1}{2}\theta), \hat{\mathbf{n}}\sin(\tfrac{1}{2}\theta)) \tag{28}$$

(Arvo 1994, Hearn and Baker 1996). The components of this quaternion are called EULER PARAMETERS. After rotation, a point $p = (0, \mathbf{p})$ is then given by

$$p' = qpq^{-1} = qpq^*, \tag{29}$$

since $n(q) = 1$. A concatenation of two rotations, first q_1 and then q_2, can be computed using the identity

$$q_2(q_1pq_1^*)q_2^* = (q_2q_1)p(q_1^*q_2^*) = (q_2q_1)p(q_2q_1)^* \tag{30}$$

(Goldstein 1980).

see also BIQUATERNION, CAYLEY-KLEIN PARAMETERS, COMPLEX NUMBER, DIVISION ALGEBRA, EULER PARAMETERS, FOUR-VECTOR, OCTONION

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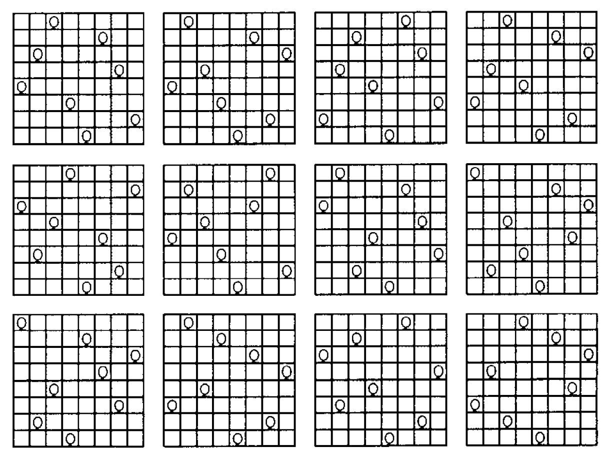
Quattuordecillion

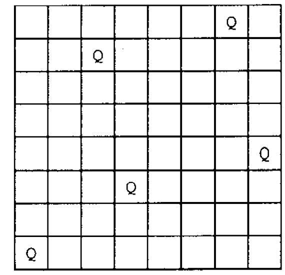
In the American system, 10^{45}.

see also LARGE NUMBER

Queens Problem



What is the maximum number of queens which can be placed on an $n \times n$ CHESSBOARD such that no two attack one another? The answer is n queens, which gives eight queens for the usual 8×8 board (Madachy 1979). The *number* of different ways the n queens can be placed on an $n \times n$ chessboard so that no two queens may attack each other for the first few n are 1, 0, 0, 2, 10, 4, 40, 92, ... (Sloane's A000170, Madachy 1979). The number of rotationally and reflectively distinct solutions are 1, 0, 0, 1, 2, 1, 6, 12, 46, 92, ... (Sloane's A002562; Dudeney 1970; p. 96). The 12 distinct solutions for $n = 8$ are illustrated above, and the remaining 80 are generated by ROTATION and REFLECTION (Madachy 1979).



The minimum number of queens needed to occupy or attack all squares of an 8×8 board is 5. Dudeney (1970, pp. 95–96) gave the following results for the number of distinct arrangements $N_p(k,n)$ of k queens attacking or occupying every square of an $n \times n$ board for which every queen is attacked ("protected") by at least one other.

k Queens	$n \times n$	$N_p(k,n)$
2	4	3
3	5	37
3	6	1
4	7	5

Dudeney (1970, pp. 95–96) also gave the following results for the number of distinct arrangements $N_u(k,n)$ of k queens attacking or occupying every square of an $n \times n$ board for which no two queens attack one another (they are "not protected").

k Queens	$n \times n$	$N_u(k,n)$
1	2	1
1	3	1
3	4	2
3	5	2
4	6	17
4	7	1
5	8	91

Vardi (1991) generalizes the problem from a square chessboard to one with the topology of the TORUS. The number of solutions for n queens with n ODD are 1, 0, 10, 28, 0, 88, ... (Sloane's A007705). Vardi (1991) also considers the toroidal "semiqueens" problem, in which a semiqueen can move like a rook or bishop, but only on POSITIVE broken diagonals. The number of solutions to this problem for n queens with n ODD are 1, 3, 15, 133, 2025, 37851, ... (Sloane's A006717), and 0 for EVEN n.

Chow and Velucchi give the solution to the question, "How many different arrangements of k queens are possible on an order n chessboard?" as 1/8th of the COEFFICIENT of $a^k b^{n^2-k}$ in the POLYNOMIAL

$$p(a,b,n) = \begin{cases} (a+b)^{n^2} + 2(a+b)^n (a^2+b^2)^{(n^2-n)/2} \\ \quad +3(a^2+b^2)^{n^2/2} + 2(a^4+b^4)^{n^2/4} & n \text{ even} \\ (a+b)^{n^2} + 2(a+b)(a^4+b^4)^{(n^2-1)/4} \\ \quad +(a+b)(a^2+b^2)^{(n^2-1)/2} \\ \quad +4(a+b)^n(a^2+b^2)^{(n^2-n)/2} & n \text{ odd.} \end{cases}$$

Velucchi also considers the nondominating queens problem, which consists of placing n queens on an order n chessboard to leave a maximum number $U(n)$ of unattacked vacant cells. The first few values are 0, 0, 0, 1, 3, 5, 7, 11, 18, 22, 30, 36, 47, 56, 72, 82, ... (Sloane's A001366). The results can be generalized to k queens on an $n \times n$ board.

see also BISHOPS PROBLEM, CHESS, KINGS PROBLEM, KNIGHTS PROBLEM, KNIGHT'S TOUR, ROOKS PROBLEM

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Queue

A queue is a special kind of LIST in which elements may only be removed from the bottom by a POP action or added to the top using a PUSH action. Examples of queues include people waiting in line, and submitted jobs waiting to be printed on a printer. The study of queues is called QUEUING THEORY.

see also LIST, QUEUING THEORY, STACK

Queuing Theory

The study of the waiting times, lengths, and other properties of QUEUES.

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Quicksort

The fastest known SORTING ALGORITHM (on average, and for a large number of elements), requiring $\mathcal{O}(n \lg n)$ steps. Quicksort is a recursive algorithm which first partitions an array $\{a_i\}_{i=1}^n$ according to several rules (Sedgewick 1978):

1. Some key ν is in its final position in the array (i.e., if it is the jth smallest, it is in position a_j).
2. All the elements to the left of a_j are less than or equal to a_j. The elements $a_1, a_2, \ldots, a_{j-1}$ are called the "left subfile."
3. All the elements to the right of a_j are greater than or equal to a_j. The elements $a_{j+1}, \ldots, a_n$ are called the "right subfile."

Quicksort was invented by Hoare (1961, 1962), has undergone extensive analysis and scrutiny (Sedgewick 1975, 1977, 1978), and is known to be about twice as fast as the next fastest SORTING algorithm. In the worst case, however, quicksort is a slow n^2 algorithm (and for quicksort, "worst case" corresponds to already sorted).

see also HEAPSORT, SORTING

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Quillen-Lichtenbaum Conjecture

A technical CONJECTURE which connects algebraic k-THEORY to Étale cohomology. The conjecture was made more precise by Dwyer and Friedlander (1982). Thomason (1985) established the first half of this conjecture, but the entire conjecture has not yet been established.

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Quincunx

The pattern ⁙ of dots on the "5" side of a 6-sided DIE. The word derives from the Latin words for both one and five.

see also DICE

References

Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 9 and 22, 1996.

Quindecillion

In the American system, 10^{48}.

see also LARGE NUMBER

Quintet

A SET of five.

see also HEXAD, MONAD, QUARTET, TETRAD, TRIAD

Quintic Equation

A general quintic cannot be solved algebraically in terms of finite additions, multiplications, and root extractions, as rigorously demonstrated by Abel and Galois.

Euler reduced the general quintic to

$$x^5 - 10qx^2 - p = 0. \tag{1}$$

A quintic also can be algebraically reduced to PRINCIPAL QUINTIC FORM

$$x^5 + a_2x^2 + a_1x + a_0 = 0. \tag{2}$$

By solving a quartic, a quintic can be algebraically reduced to the BRING QUINTIC FORM

$$x^5 - x - a = 0, \tag{3}$$

as was first done by Jerrard.

Consider the quintic

$$\prod_{j=0}^{4}[x - (\omega^j u_1 + \omega^{4j} u_2)] = 0, \tag{4}$$

where $\omega = e^{2\pi i/5}$ and u_1 and u_2 are COMPLEX NUMBERS. This is called DE MOIVRE'S QUINTIC. Generalize it to

$$\prod_{j=0}^{4}[x - (\omega^j u_1 + \omega^{2j} u_2 + \omega^{3j} u_3 + \omega^{4j} u_4)] = 0. \tag{5}$$

Expanding,

$$\begin{aligned}(\omega^j u_1 + \omega^{2j} u_2 + \omega^{3j} u_3 + \omega^{4j} u_4)^5 \\ -5U(\omega^j u_1 + \omega^{2j} u_2 + \omega^{3j} u_3 + \omega^{4j} u_4)^4 \\ -5V(\omega^j u_1 + \omega^{2j} u_2 + \omega^{3j} u_3 + \omega^{4j} u_4)^2 \\ +5W(\omega^j u_1 + \omega^{2j} u_2 + \omega^{3j} u_3 + \omega^{4j} u_4) \\ +[5(X-Y)-Z]=0, \quad (6)\end{aligned}$$

where

$$U = u_1 u_4 + u2u_3 \quad (7)$$
$$V = u_1 {u_2}^2 + u_2 {u_4}^2 + u_3 {u_1}^2 + u_4 {u_3}^2 \quad (8)$$
$$W = {u_1}^2 {u_4}^2 + {u_2}^2 {u_3}^2 - {u_1}^3 u_2 - {u_2}^3 u_4 - {u_3}^3 u_1 - {u_4}^3 u_3 - u_1 u_2 u_3 u_4 \quad (9)$$
$$X = {u_1}^3 u_3 u_4 + {u_2}^3 u_1 u_3 + {u_3}^3 u_2 u_4 + {u_4}^3 u_1 u_2 \quad (10)$$
$$Y = u_1 {u_3}^2 {u_4}^2 + u_2 {u_1}^2 {u_3}^2 + u_3 {u_2}^2 {u_4}^2 + u_4 {u_1}^2 {u_2}^2 \quad (11)$$
$$Z = {u_1}^5 + {u_2}^5 + {u_3}^5 + {u_4}^5. \quad (12)$$

The u_is satisfy

$$u_1 u_4 + u_2 u_3 = 0 \quad (13)$$
$$u_1 {u_2}^2 + u_2 {u_4}^2 + u_3 {u_1}^2 + u_4 {u_3}^2 = 0 \quad (14)$$
$${u_1}^2 {u_4}^2 + {u_2}^2 {u_3}^2 - {u_1}^3 u_2 - {u_2}^3 u_4 - {u_3}^3 u_1 - {u_4}^3 u_3 - u_1 u_2 u_3 u_4 = \tfrac{1}{5} a \quad (15)$$
$$\begin{aligned}5[({u_1}^3 u_3 u_4 + {u_2}^3 u_1 u_3 + {u_3}^3 u_3 u_4 + {u_4}^3 u_1 u_2) \\ - (u_1 {u_3}^2 {u_4}^2 + u_2 {u_1}^2 {u_3}^2 + u_3 {u_2}^2 {u_4}^2 + u_4 {u_1}^2 {u_2}^2)] \\ - ({u_1}^5 + {u_2}^5 + {u_3}^5 + {u_4}^5) = b. \quad (16)\end{aligned}$$

Spearman and Williams (1994) show that an irreducible quintic

$$x^5 + ax + b = 0 \quad (17)$$

with RATIONAL COEFFICIENTS is solvable by radicals IFF there exist rational numbers $\epsilon = \pm 1$, $c \geq 0$, and $e \neq 0$ such that

$$a = \frac{5e^4(3-4\epsilon c)}{c^2+1} \quad (18)$$
$$b = \frac{-4e^5(11\epsilon + 2c)}{c^2+1}. \quad (19)$$

The ROOTS are then

$$x_j = e(\omega^j u_1 + \omega^{2j} u_2 + \omega^{3j} u_3 + \omega^{4j} u_4), \quad (20)$$

where

$$u_1 = \left(\frac{{v_1}^2 v_3}{D^2}\right)^{1/5} \quad (21)$$
$$u_2 = \left(\frac{{v_3}^2 v_4}{D^2}\right)^{1/5} \quad (22)$$
$$u_3 = \left(\frac{{v_2}^2 v_1}{D^2}\right)^{1/5} \quad (23)$$
$$u_4 = \left(\frac{{v_4}^2 v_2}{D^2}\right)^{1/5} \quad (24)$$
$$v_1 = \sqrt{D} + \sqrt{D - \epsilon\sqrt{D}} \quad (25)$$
$$v_2 = -\sqrt{D} - \sqrt{D + \epsilon\sqrt{D}} \quad (26)$$
$$v_3 = -\sqrt{D} + \sqrt{D + \epsilon\sqrt{D}} \quad (27)$$
$$v_4 = \sqrt{D} - \sqrt{D - \epsilon\sqrt{D}} \quad (28)$$
$$D = c^2 + 1. \quad (29)$$

The general quintic can be solved in terms of THETA FUNCTIONS, as was first done by Hermite in 1858. Kronecker subsequently obtained the same solution more simply, and Brioshi also derived the equation. To do so, reduce the general quintic

$$a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0 \quad (30)$$

into BRING QUINTIC FORM

$$x^5 - x + \rho = 0. \quad (31)$$

Then define

$$k \equiv \tan\left[\tfrac{1}{4}\sin^{-1}\left(\frac{16}{25\sqrt{5}\,\rho^2}\right)\right] \quad (32)$$
$$s \equiv \begin{cases} -\operatorname{sgn}(\Im[\rho]) & \text{for } \Re[\rho] = 0 \\ \operatorname{sgn}(\Re[\rho]) & \text{for } \Re[\rho] \neq 0 \end{cases} \quad (33)$$
$$b = \frac{s(k^2)^{1/8}}{2 \cdot 5^{3/4}\sqrt{k(1-k^2)}} \quad (34)$$
$$q = q(k^2) = e^{i\pi K'(k^2)/K(k^2)}, \quad (35)$$

where k is the MODULUS, $m \equiv k^2$ is the PARAMETER, and q is the NOME. Solving

$$q(m) = e^{i\pi K'(m)/K(m)} \quad (36)$$

for m gives the inverse parameter

$$m(q) = \frac{{\vartheta_2}^4(q)}{{\vartheta_3}^4(q)}. \quad (37)$$

The ROOTS are then given by

$$\begin{aligned}x_1 &= (-1)^{3/4} b\{[m(e^{-2\pi i/5}q^{1/5})]^{1/8}\\ &\quad +i[m(e^{2\pi i/5}q^{1/5})]^{1/8}\}\\ &\quad \times\{[m(e^{-4\pi i/5}q^{1/5})]^{1/8} + [m(e^{4\pi i/5}q^{1/5})]^{1/8}\}\\ &\quad \times\{[m(q^{1/5})]^{1/8} + q^{5/8}(q^5)^{-1/8}[m(q^5)]^{1/8}\} \qquad (38)\end{aligned}$$

$$\begin{aligned}x_2 &= b\{-[m(q^{1/5})]^{1/8} + e^{3\pi i/4}[m(e^{2\pi i/5}q^{1/5})]^{1/8}\}\\ &\quad \times\{e^{-3\pi i/4}[m(e^{-2\pi i/5}q^{1/5})]^{1/8} + i[m(e^{4\pi i/5}q^{1/5})]^{1/8}\}\\ &\quad \times\{i[m(e^{-4\pi i/5}q^{1/5})]^{1/8} + q^{5/8}(q^5)^{-1/8}[m(q^5)]^{1/8}\} \qquad (39)\end{aligned}$$

$$\begin{aligned}x_3 &= b\{e^{-3\pi i/4}[m(e^{-2\pi i/5}q^{1/5})]^{1/8}\\ &\quad -i[m(e^{-4\pi i/5}q^{1/5})]^{1/8})\}\{-[m(q^{1/5})]^{1/8}\\ &\quad -i[m(e^{4\pi i/5}q^{1/5})]^{1/8}\}\\ &\quad \times\{e^{3\pi i/4}[m(e^{2\pi i/5}q^{1/5})]^{1/8} + q^{5/8}(q^5)^{-1/8}[m(q^5)]^{1/8}\} \qquad (40)\end{aligned}$$

$$\begin{aligned}x_4 &= b\{[m(q^{1/5})]^{1/8} - i[m(e^{-4\pi i/5}q^{1/5})]^{1/8})\}\\ &\quad \times\{-e^{3\pi i/4}[m(e^{2\pi i/5}q^{1/5})]^{1/8} - i[m(e^{4\pi i/5}q^{1/5})]^{1/8}\}\\ &\quad \times\{e^{-3\pi i/4}[m(e^{-2\pi i/5}q^{1/5})]^{1/8}\\ &\quad +q^{5/8}(q^5)^{-1/8}[m(q^5)]^{1/8}\} \qquad (41)\end{aligned}$$

$$\begin{aligned}x_5 &= b\{[m(q^{1/5})]^{1/8} - e^{-3\pi i/4}[m(e^{-2\pi i/5}q^{1/5})]^{1/8}\}\\ &\quad \times\{-e^{3\pi i/4}[m(e^{2\pi i/5}q^{1/5})]^{1/8} + i[m(e^{-4\pi i/5}q^{1/5})]^{1/8}\}\\ &\quad \times\{(-i[m(e^{4\pi i/5}q^{1/5})]^{1/8} + q^{5/8}(q^5)^{-1/8}[m(q^5)]^{1/8}\}. \qquad (42)\end{aligned}$$

Felix Klein used a TSCHIRNHAUSEN TRANSFORMATION to reduce the general quintic to the form

$$z^5 + 5az^2 + 5bz + c = 0. \qquad (43)$$

He then solved the related ICOSAHEDRAL EQUATION

$$\begin{aligned}I(z,1,Z) &= z^5(-1+11z^5+z^{10})^5\\ &\quad -[1+z^{30}-10005(z^{10}+z^{20})+522(-z^5+z^{25})]^2 Z = 0, \qquad (44)\end{aligned}$$

where Z is a function of radicals of a, b, and c. The solution of this equation can be given in terms of HYPERGEOMETRIC FUNCTIONS as

$$\frac{Z^{-1/60}{}_2F_1(-\frac{1}{60},\frac{29}{60},\frac{4}{5},1728Z)}{Z^{11/60}{}_2F_1(\frac{11}{60},\frac{41}{60},\frac{6}{5},1728Z)}. \qquad (45)$$

Another possible approach uses a series expansion, which gives one root (the first one in the list below) of

$$t^5 - t - \rho. \qquad (46)$$

All five roots can be derived using differential equations (Cockle 1860, Harley 1862). Let

$$F_1(\rho) = F_2(\rho) \qquad (47)$$

$$F_2(\rho) = {}_4F_3(\tfrac{1}{5},\tfrac{2}{5},\tfrac{3}{5},\tfrac{4}{5};\tfrac{1}{2},\tfrac{3}{4},\tfrac{5}{4};\tfrac{3125}{256}\rho^4) \qquad (48)$$

$$F_3(\rho) = {}_4F_3(\tfrac{9}{20},\tfrac{13}{20},\tfrac{17}{20},\tfrac{21}{20};\tfrac{3}{4},\tfrac{5}{4},\tfrac{3}{2};\tfrac{3125}{256}\rho^4) \qquad (49)$$

$$F_4(\rho) = {}_4F_3(\tfrac{7}{10},\tfrac{9}{10},\tfrac{11}{10},\tfrac{13}{10};\tfrac{5}{4},\tfrac{3}{2},\tfrac{7}{4};\tfrac{3125}{256}\rho^4), \qquad (50)$$

then the ROOTS are

$$t_1 = -\rho\,{}_4F_3(\tfrac{1}{5},\tfrac{2}{5},\tfrac{3}{5},\tfrac{4}{5};\tfrac{1}{2},\tfrac{3}{4},\tfrac{5}{4};\tfrac{3125}{256}\rho^4) \qquad (51)$$

$$t_2 = -F_1(\rho) + \tfrac{1}{4}\rho F_2(\rho) + \tfrac{5}{32}\rho^2 F_3(\rho) + \tfrac{5}{32}\rho^3 F_4(\rho) \qquad (52)$$

$$t_3 = -F_1(\rho) + \tfrac{1}{4}\rho F_2(\rho) - \tfrac{5}{32}\rho^2 F_3(\rho) + \tfrac{5}{32}\rho^3 F_4(\rho) \qquad (53)$$

$$t_4 = -iF_1(\rho) + \tfrac{1}{4}\rho F_2(\rho) - \tfrac{5}{32}i\rho^2 F_3(\rho) - \tfrac{5}{32}\rho^3 F_4(\rho) \qquad (54)$$

$$t_5 = iF_1(\rho) + \tfrac{1}{4}\rho F_2(\rho) + \tfrac{5}{32}i\rho^2 F_3(\rho) - \tfrac{5}{32}\rho^3 F_4(\rho). \qquad (55)$$

This technique gives closed form solutions in terms of HYPERGEOMETRIC FUNCTIONS in one variable for any POLYNOMIAL equation which can be written in the form

$$x^p + bx^q + c. \qquad (56)$$

Cadenhad, Young, and Runge showed in 1885 that all irreducible solvable quintics with COEFFICIENTS of x^4, x^3, and x^2 missing have the following form

$$x^5 + \frac{5\mu^4(4\nu+3)}{\nu^2+1}x + \frac{4\mu^5(2\nu+1)(4\nu+3)}{\nu^2+1} = 0, \qquad (57)$$

where μ and ν are RATIONAL.

see also BRING QUINTIC FORM, BRING-JERRARD QUINTIC FORM, CUBIC EQUATION, DE MOIVRE'S QUINTIC, PRINCIPAL QUINTIC FORM, QUADRATIC EQUATION, QUARTIC EQUATION, SEXTIC EQUATION

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Quintic Surface

A quintic surface is an ALGEBRAIC SURFACE of degree 5. Togliatti (1940, 1949) showed that quintic surfaces having 31 ORDINARY DOUBLE POINTS exist, although he did not explicitly derive equations for such surfaces. Beauville (1978) subsequently proved that 31 double points was the maximum possible, and quintic surfaces having 31 ORDINARY DOUBLE POINTS are therefore sometimes called TOGLIATTI SURFACES. van Straten (1993) subsequently constructed a 3-D family of solutions and in 1994, Barth derived the example known as the DERVISH.

see also ALGEBRAIC SURFACE, DERVISH, KISS SURFACE, ORDINARY DOUBLE POINT

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Quintillion

In the American system, 10^{18}.

see also LARGE NUMBER

Quintuple

A group of five elements, also called a QUINTUPLET or PENTAD.

see also MONAD, PAIR, PENTAD, QUADRUPLE, QUADRUPLET, QUINTUPLET, TETRAD, TRIAD, TRIPLET, TWINS

Quintuple Product Identity

A.k.a. the WATSON QUINTUPLE PRODUCT IDENTITY.

$$\prod_{n=1}^{\infty}(1-q^n)(1-zq^n)(1-z^{-1}q^{n-1})(1-z^2q^{2n-1})\times(1-z^{-2}q^{2n-1})=\sum_{m=-\infty}^{\infty}(z^{3m}-z^{-3m-1})q^{m(2m+1)/2}. \quad (1)$$

It can also be written

$$\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1}z)(1-q^{2n-1}z^{-1})\times(1-q^{4n-3}z^2)(1-q^{4n-4}z^{-2})=\sum_{n=-\infty}^{\infty}q^{3n^2-2n}[(z^{3n}+z^{-3n})-(z^{3n-2}+z^{-(3n-2)})] \quad (2)$$

or

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{(3k^2-k)/2}x^{3k}(1+zq^k)=\prod_{j=1}^{\infty}(1-q^j)(1+z^{-1}q^j)(1+zq^{j-1})\times(1+z^{-2}q^{2j-1})(1+z^2q^{2j-1}). \quad (3)$$

Using the NOTATION of the RAMANUJAN THETA FUNCTION (Berndt, p. 83),

$$f(B^3/q,q^5/B^3)-B^2f(q/B^3,B^3q^5)=f(-q^2)\frac{f(-B^2,-q^2/B^2)}{f(Bq,q/B)}. \quad (4)$$

see also JACOBI TRIPLE PRODUCT, RAMANUJAN THETA FUNCTIONS

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Quintuplet

A group of five elements, also called a QUINTUPLE or PENTAD.

see also MONAD, PAIR, PENTAD, QUADRUPLE, QUADRUPLET, QUINTUPLET, TETRAD, TRIAD, TRIPLET, TWINS

Quota Rule

A RECURRENCE RELATION between the function Q arising in QUOTA SYSTEMS,

$$Q(n,r) = Q(n-1,r-1) + Q(n-1,r).$$

References

Young, S. C.; Taylor, A. D.; and Zwicker, W. S. "Counting Quota Systems: A Combinatorial Question from Social Choice Theory." *Math. Mag.* **68**, 331–342, 1995.

Quota System

A generalization of simple majority voting in which a list of quotas $\{q_0, \ldots, q_n\}$ specifies, according to the number of votes, how many votes an alternative needs to win (Taylor 1995). The quota system declares a tie unless for some k, there are exactly k tie votes in the profile and one of the alternatives has at least q_k votes, in which case the alternative is the choice.

Let $Q(n)$ be the number of quota systems for n voters and $Q(n,r)$ the number of quota systems for which $q_0 = r+1$, so

$$Q(n) = \sum_{r=\lfloor n/2 \rfloor}^{n} Q(n,r) = \binom{n+1}{\lfloor \frac{n}{2} \rfloor + 1},$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION. This produces the sequence of CENTRAL BINOMIAL COEFFICIENTS 1, 2, 3, 6, 10, 20, 35, 70, 126, ... (Sloane's A001405). It may be defined recursively by $Q(0) = 1$ and

$$Q(n+1) = \begin{cases} 2Q(n) & \text{for } n \text{ even} \\ 2Q(n) - C_{(n+1)/2} & \text{for } n \text{ odd,} \end{cases}$$

where C_k is a CATALAN NUMBER (Young *et al.* 1995). The function $Q(n,r)$ satisfies

$$Q(n,r) = \binom{n+1}{r+1} - \binom{n+1}{r+2}$$

for $r > n/2 - 1$ (Young *et al.* 1995). $Q(n,r)$ satisfies the QUOTA RULE.

see also BINOMIAL COEFFICIENT, CENTRAL BINOMIAL COEFFICIENT

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Quotient

The ratio $q = r/s$ of two quantities r and s, where $s \neq 0$.

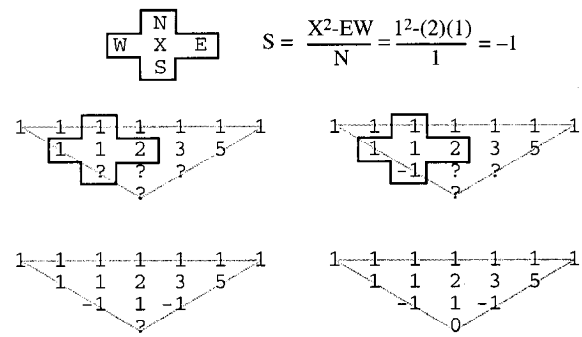
see also DIVISION, QUOTIENT GROUP, QUOTIENT RING, QUOTIENT SPACE

Quotient-Difference Algorithm

The ALGORITHM of constructing and interpreting a QUOTIENT-DIFFERENCE TABLE which allows interconversion of CONTINUED FRACTIONS, POWER SERIES, and RATIONAL FUNCTIONS approximations.

see also QUOTIENT-DIFFERENCE TABLE

Quotient-Difference Table



A quotient-difference table is a triangular ARRAY of numbers constructed by drawing a sequence of n numbers in a horizontal row and placing a 1 above each. An additional "1" is then placed at the beginning and end of the row of 1s, and the value of rows underneath the original row is then determined by looking at groups of adjacent numbers

$$\begin{array}{ccc} & N & \\ W & X & E \\ & S & \end{array}$$

and computing

$$S = \frac{X^2 - EW}{N}$$

for the elements falling within a triangle formed by the diagonals extended from the first and last "1," as illustrated above.

0s in quotient-difference tables form square "windows" which are bordered by GEOMETRIC PROGRESSIONS. Quotient-difference tables eventually yield a row of 0s IFF the starting sequence is defined by a linear RECURRENCE RELATION. For example, continuing the above example generated by the FIBONACCI NUMBERS

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 2 & 3 & 5 & \\ & & -1 & 1 & -1 & & \\ & & & 0 & & & \end{array}$$

$$\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 2 & 3 & 5 & 8 & \\ & & -1 & 1 & -1 & 1 & & \\ & & & 0 & 0 & & & \end{array}$$

$$\begin{array}{ccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 2 & 3 & 5 & 8 & 13 & \\ & & -1 & 1 & -1 & 1 & -1 & & \\ & & & 0 & 0 & 0 & & & \\ & & & & 0 & & & & \end{array}$$

$$\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & \\
 & & -1 & 1 & -1 & 1 & -1 & 1 & & \\
 & & & 0 & 0 & 0 & 0 & & & \\
 & & & & 0 & 0 & & & &
\end{array}$$

and it can be seen that a row of 0s emerges (and furthermore that an attempt to extend the table will result in division by zero). This verifies that the FIBONACCI NUMBERS satisfy a linear recurrence, which is in fact given by the well-known formula

$$F_n = F_{n-1} + F_{n-2}.$$

However, construction of a quotient-difference table for the CATALAN NUMBERS, MOTZKIN NUMBERS, etc., does not lead to a row of zeros, suggesting that these numbers cannot be generated using a linear recurrence.

see also DIFFERENCE TABLE, FINITE DIFFERENCE

References

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Quotient Group

The quotient group of G with respect to a SUBGROUP H is denoted G/H and is read "G modulo H." The slash NOTATION conflicts with that for a FIELD EXTENSION, but the meaning can be determined based on context.

see also ABHYANKAR'S CONJECTURE, FIELD EXTENSION, OUTER AUTOMORPHISM GROUP, SUBGROUP

Quotient Ring

The quotient ring of R with respect to a RING modulo some INTEGER n is denoted R/nR and is read "the ring R modulo n." If n is a PRIME p, then $\mathbb{Z}/p\mathbb{Z}$ is the FINITE FIELD $\mathbb{F}_p$. For COMPOSITE

$$n = \prod_{i=1}^{k} p_i$$

with distinct p_i, $\mathbb{Z}/p\mathbb{Z}$ is ISOMORPHIC to the DIRECT SUM

$$\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_{p_1} \otimes \mathbb{F}_{p_2} \otimes \cdots \otimes \mathbb{F}_{p_k}.$$

see also FINITE FIELD, RING

Quotient Rule

The DERIVATIVE rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

see also CHAIN RULE, DERIVATIVE, POWER RULE, PRODUCT RULE

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Quotient Space

The quotient space $X/\sim$ of a TOPOLOGICAL SPACE X and an EQUIVALENCE RELATION $\sim$ on X is the set of EQUIVALENCE CLASSES of points in X (under the EQUIVALENCE RELATION $\sim$) together with the topology given by a SUBSET U of $X/\sim$. U of $X/\sim$ is open IFF $\cup_{a\in U} a$ is open in X.

This can be stated in terms of MAPS as follows: if $q : X \to X/\sim$ denotes the MAP that sends each point to its EQUIVALENCE CLASS in $X/\sim$, the topology on $X/\sim$ can be specified by prescribing that a subset of $X/\sim$ is open IFF q^{-1}[the set] is open.

In general, quotient spaces are not well behaved, and little is known about them. However, it is known that any compact metrizable space is a quotient of the CANTOR SET, any compact connected n-dimensional MANIFOLD for $n > 0$ is a quotient of any other, and a function out of a quotient space $f : X/\sim \to Y$ is continuous IFF the function $f \circ q : X \to Y$ is continuous.

Let $\mathbb{D}^n$ be the closed n-D DISK and $\mathbb{S}^{n-1}$ its boundary, the $(n-1)$-D sphere. Then $\mathbb{D}^n/\mathbb{S}^{n-1}$ (which is homeomorphic to $\mathbb{S}^n$), provides an example of a quotient space. Here, $\mathbb{D}^n/\mathbb{S}^{n-1}$ is interpreted as the space obtained when the boundary of the n-DISK is collapsed to a point, and is formally the "quotient space by the equivalence relation generated by the relations that all points in $\mathbb{S}^{n-1}$ are equivalent."

see also EQUIVALENCE RELATION, TOPOLOGICAL SPACE

References

Munkres, J. R. *Topology: A First Course.* Englewood Cliffs, NJ: Prentice-Hall, 1975.

R

$\mathbb{R}$

The FIELD of REAL NUMBERS.

see also $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{I}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}^-$, $\mathbb{R}^+$, $\mathbb{Z}$

$\mathbb{R}^-$

The REAL NEGATIVE numbers.

see also $\mathbb{R}$, $\mathbb{R}^+$

$\mathbb{R}^+$

The REAL POSITIVE numbers.

see also $\mathbb{R}$, $\mathbb{R}^-$

$r_k(n)$

The number of representations of n by k squares is denoted $r_k(n)$. The *Mathematica*® (Wolfram Research, Champaign, IL) function `NumberTheory`NumberTheoryFunctions`SumOfSquaresR[k,n]` gives $r_k(n)$.

$r_2(n)$ is often simply written $r(n)$. Jacobi solved the problem for $k = 2$, 4, 6, and 8. The first cases $k = 2$, 4, and 6 were found by equating COEFFICIENTS of the THETA FUNCTION $\vartheta_3(x)$, ${\vartheta_3}^2(x)$, and ${\vartheta_3}^4(x)$. The solutions for $k = 10$ and 12 were found by Liouville and Eisenstein, and Glaisher (1907) gives a table of $r_k(n)$ for $k = 2s = 18$. $r_3(n)$ was found as a finite sum involving quadratic reciprocity symbols by Dirichlet. $r_5(n)$ and $r_7(n)$ were found by Eisenstein, Smith, and Minkowski.

$r(n) = r_2(n)$ is 0 whenever n has a PRIME divisor of the form $4k+3$ to an ODD POWER; it doubles upon reaching a new PRIME of the form $4k+1$. It is given explicitly by

$$r(n) = 4 \sum_{d=1,3,\ldots|n} (-1)^{(d-1)/2} = 4[d_1(n) - d_3(n)], \quad (1)$$

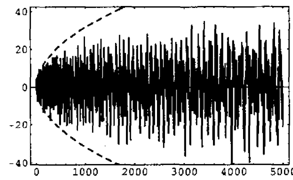
where $d_k(n)$ is the number of DIVISORS of n of the form $4m + k$. The first few values are 4, 4, 0, 4, 8, 0, 0, 4, 4, 8, 0, 0, 8, 0, 0, 4, 8, 4, 0, 8, 0, 0, 0, 0, 12, 8, 0, 0, ... (Sloane's A004018). The first few values of the summatory function

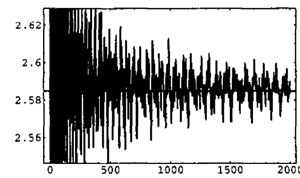
$$R(n) \equiv \sum_{k=1}^{n} r(n), \quad (2)$$

where are 0, 4, 8, 8, 12, 20, 20, 20, 24, 28, 36, ... (Sloane's A014198). Shanks (1993) defines instead $R'(n) = 1 + R(n)$, with $R'(0) = 1$. A LAMBERT SERIES for $r(n)$ is

$$\sum_{n=1}^{\infty} \frac{4(-1)^{n+1}x^n}{1-x^n} = \sum_{n=1}^{\infty} r(n)x^n \quad (3)$$

(Hardy and Wright 1979).





Asymptotic results include

$$\sum_{k=1}^{n} r_2(k) = \pi n + \mathcal{O}(\sqrt{n}) \quad (4)$$

$$\sum_{k=1}^{n} \frac{r_2(k)}{k} = K + \pi \ln n + \mathcal{O}(n^{-1/2}), \quad (5)$$

where K is a constant known as the SIERPIŃSKI CONSTANT. The left plot above

$$\left[\sum_{k=1}^{n} r_2(k)\right] - \pi n, \quad (6)$$

with $\pm\sqrt{n}$ illustrated by the dashed curve, and the right plot shows

$$\left[\sum_{k=1}^{n} \frac{r_2(k)}{k}\right] - \pi \ln n, \quad (7)$$

with the value of K indicated as the solid horizontal line.

The number of solutions of

$$x^2 + y^2 + z^2 = n \quad (8)$$

for a given n without restriction on the signs or relative sizes of x, y, and z is given by $r_3(n)$. If $n > 4$ is SQUAREFREE, then Gauss proved that

$$r_3(n) = \begin{cases} 24h(-n) & \text{for } n \equiv 3 \pmod 8 \\ 12h(-4n) & \text{for } n \equiv 1, 2, 5, 6 \pmod 8 \\ 0 & \text{for } n \equiv 7 \pmod 8 \end{cases} \quad (9)$$

(Arno 1992), where $h(x)$ is the CLASS NUMBER of x.

Additional higher-order identities are given by

$$r_4(n) = 8 \sum_{d|n} d = 8\sigma(n) \quad (10)$$

$$= 24 \sum_{d=1,3,\ldots|n} d = 24\sigma_0(n) \quad (11)$$

$$r_{10}(n) = \tfrac{4}{5}[E_4(n) + 16E_4'(n) + 8\chi_4(n)] \quad (12)$$

$$r_{24}(n) = \rho_{24}(n) + \tfrac{128}{691}[(-1)^{n-1}259\tau(n) - 512\tau(\tfrac{1}{2}n)], \quad (13)$$

where

$$E_4(n) = \sum_{d=1,3,\ldots|n} (-1)^{(d-1)/2} d^4 \quad (14)$$

$$E_4'(n) = \sum_{d'=1,3,\ldots|n} (-1)^{(d'-1)/2} d^4 \quad (15)$$

$$\chi_4(n) = \tfrac{1}{4} \sum_{a^2+b^2=n} (a+bi)^4, \quad (16)$$

$d' \equiv n/d$, $d_k(n)$ is the number of divisors of n of the form $4m+k$, $\rho_{24}(n)$ is a SINGULAR SERIES, $\sigma(n)$ is the DIVISOR FUNCTION, $\sigma_0(n)$ is the DIVISOR FUNCTION of order 0 (i.e., the number of DIVISORS), and τ is the TAU FUNCTION.

Similar expressions exist for larger EVEN k, but they quickly become extremely complicated and can be written simply only in terms of expansions of modular functions.

see also CLASS NUMBER, LANDAU-RAMANUJAN CONSTANT, PRIME FACTORS, SIERPIŃSKI CONSTANT, TAU FUNCTION

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R-Estimate

A ROBUST ESTIMATION based on RANK tests. Examples include the statistic of the KOLMOGOROV-SMIRNOV TEST, SPEARMAN RANK CORRELATION, and WILCOXON SIGNED RANK TEST.

see also L-ESTIMATE, M-ESTIMATE, ROBUST ESTIMATION

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Robust Estimation." §15.7 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 694–700, 1992.

Raabe's Test

Given a SERIES of POSITIVE terms u_i and a SEQUENCE of POSITIVE constants $\{a_i\}$, use KUMMER'S TEST

$$\rho' \equiv \lim_{n\to\infty}\left(a_n\frac{u_n}{u_{n+1}} - a_{n+1}\right).$$

with $a_n = n$, giving

$$\begin{aligned}\rho' &\equiv \lim_{n\to\infty}\left(n\frac{u_n}{u_{n+1}} - (n+1)\right)\\ &= \lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right].\end{aligned}$$

Defining

$$\rho \equiv \rho' + 1 = \lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}} - 1\right)\right],$$

then gives Raabe's test:

1. If $\rho > 1$, the SERIES CONVERGES.
2. If $\rho < 1$, the SERIES DIVERGES.
3. If $\rho = 1$, the SERIES may CONVERGE or DIVERGE.

see also CONVERGENT SERIES, CONVERGENCE TESTS, DIVERGENT SERIES, KUMMER'S TEST

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Bromwich, T. J. I'a and MacRobert, T. M. *An Introduction to the Theory of Infinite Series, 3rd ed.* New York: Chelsea, p. 39, 1991.

Rabbit Constant

The limiting RABBIT SEQUENCE written as a BINARY FRACTION $0.1011010110110\ldots_2$ (Sloane's A005614), where b_2 denotes a BINARY number (a number in base-2). The DECIMAL value is

$$R = 0.709803442861291314\ldots$$

(Sloane's A014565).

Amazingly, the rabbit constant is also given by the CONTINUED FRACTION $[0, 2^{F_0}, 2^{F_1}, 2^{F_2}, 2^{F_3}, \ldots]$, where F_n are FIBONACCI NUMBERS with F_0 taken as 0 (Gardner 1989, Schroeder 1991). Another amazing connection was discovered by S. Plouffe. Define the BEATTY SEQUENCE $\{a_i\}$ by

$$a_i \equiv \lfloor i\phi \rfloor,$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION and ϕ is the GOLDEN RATIO. The first few terms are 1, 3, 4, 6, 8, 9, 11, ... (Sloane's A000201). Then

$$R = \sum_{i=1}^{\infty} 2^{-a_i}.$$

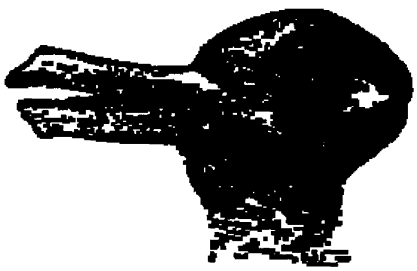
see also RABBIT SEQUENCE, THUE CONSTANT, THUE-MORSE CONSTANT

References

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/cntfrc/cntfrc.html`.

Gardner, M. *Penrose Tiles and Trapdoor Ciphers... and the Return of Dr. Matrix, reissue ed.* New York: W. H. Freeman, pp. 21–22, 1989.
Plouffe, S. "The Rabbit Constant to 330 Digits." http://lacim.uqam.ca/piDATA/rabbit.txt.
Schroeder, M. *Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise.* New York: W. H. Freeman, p. 55, 1991.
Sloane, N. J. A. Sequences A005614, A014565, and A000201/M2322 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Rabbit-Duck Illusion



A perception ILLUSION in which the brain switches between seeing a rabbit and a duck.

see also YOUNG GIRL-OLD WOMAN ILLUSION

Rabbit Sequence

A SEQUENCE which arises in the hypothetical reproduction of a population of rabbits. Let the SUBSTITUTION MAP $0 \to 1$ correspond to young rabbits growing old, and $1 \to 10$ correspond to old rabbits producing young rabbits. Starting with 0 and iterating using STRING REWRITING gives the terms 1, 10, 101, 10110, 10110101, 1011010110110, ... The limiting sequence written as a BINARY FRACTION $0.1011010110110\ldots_2$ (Sloane's A005614), where b_2 denotes a BINARY number (a number in base-2) is called the RABBIT CONSTANT.

see also RABBIT CONSTANT, THUE-MORSE SEQUENCE

References

Schroeder, M. *Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise.* New York: W. H. Freeman, p. 55, 1991.
Sloane, N. J. A. Sequence A005614 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Rabdology

see NAPIER'S BONES

Rabin-Miller Strong Pseudoprime Test

A PRIMALITY TEST which provides an efficient probabilistic ALGORITHM for determining if a given number is PRIME. It is based on the properties of STRONG PSEUDOPRIMES. Given an ODD INTEGER n, let $n = 2^r s + 1$ with s ODD. Then choose a random integer a with $1 \leq a \leq n-1$. If $a^s \equiv 1 \pmod{n}$ or $a^{2^j s} \equiv -1 \pmod{n}$ for some $0 \leq j \leq r-1$, then n passes the test. A PRIME will pass the test for all a.

The test is very fast and requires no more than $(1 + o(1)) \lg n$ multiplications (mod n), where LG is the LOGARITHM base 2. Unfortunately, a number which passes the test is not necessarily PRIME. Monier (1980) and Rabin (1980) have shown that a COMPOSITE NUMBER passes the test for at most 1/4 of the possible bases a.

The Rabin-Miller test (combined with a LUCAS PSEUDOPRIME test) is the PRIMALITY TEST used by *Mathematica*® versions 2.2 and later (Wolfram Research, Champaign, IL). As of 1991, the combined test had been proven correct for all $n < 2.5 \times 10^{10}$, but not beyond. The test potentially could therefore incorrectly identify a large COMPOSITE NUMBER as PRIME (but not vice versa). STRONG PSEUDOPRIME tests have been subsequently proved valid for every number up to 3.4×10^{14}.

see also LUCAS-LEHMER TEST, MILLER'S PRIMALITY TEST, PSEUDOPRIME, STRONG PSEUDOPRIME

References

Arnault, F. "Rabin-Miller Primality Test: Composite Numbers Which Pass It." *Math. Comput.* **64**, 355–361, 1995.
Miller, G. "Riemann's Hypothesis and Tests for Primality." *J. Comp. Syst. Sci.* **13**, 300–317, 1976.
Monier, L. "Evaluation and Comparison of Two Efficient Probabilistic Primality Testing Algorithms." *Theor. Comput. Sci.* **12**, 97–108, 1980.
Rabin, M. O. "Probabilistic Algorithm for Testing Primality." *J. Number Th.* **12**, 128–138, 1980.
Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, pp. 15–17, 1991.

Rabinovich-Fabrikant Equation

The 3-D MAP

$$\begin{aligned}\dot{x} &= y(z - 1 + x^2) + \gamma x\\ \dot{y} &= x(3z + 1 - x^2) + \gamma y\\ \dot{z} &= -2z(\alpha + xy)\end{aligned}$$

(Rabinovich and Fabrikant 1979). The parameters are most commonly taken as $\gamma = 0.87$ and $\alpha = 1.1$. It has a CORRELATION EXPONENT of 2.19 ± 0.01.

References

Grassberger, P. and Procaccia, I. "Measuring the Strangeness of Strange Attractors." *Physica D* **9**, 189–208, 1983.
Rabinovich, M. I. and Fabrikant, A. L. *Sov. Phys. JETP* **50**, 311–317, 1979.

Racah V-Coefficient

The Racah V-COEFFICIENTS are written

$$V(j_1 j_2 j; m_1 m_2 m) \tag{1}$$

and are sometimes expressed using the related CLEBSCH-GORDON COEFFICIENTS

$$C^j_{m_1 m_2} = (j_1 j_2 m_1 m_2 | j_1 j_2 j m), \tag{2}$$

or WIGNER $3j$-SYMBOLS. Connections among the three are

$$(j_1j_2m_1m_2|j_1j_2m) = (-1)^{-j_1+j_2-m}\sqrt{2j+1}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} \quad (3)$$

$$(j_1j_2m_1m_2|j_1j_2jm) = (-1)^{j+m}\sqrt{2j+1}\,V(j_1j_2j;m_1m_2\ -m) \quad (4)$$

$$V(j_1j_2j;m_1m_2m) = (-1)^{-j_1+j_2+j}\begin{pmatrix} j_1 & j_2 & j_1 \\ m_2 & m_1 & m_2 \end{pmatrix}. \quad (5)$$

see also CLEBSCH-GORDON COEFFICIENT, RACAH W-COEFFICIENT, WIGNER $3j$-SYMBOL, WIGNER $6j$-SYMBOL, WIGNER $9j$-SYMBOL

References

Sobel'man, I. I. "Angular Momenta." Ch. 4 in *Atomic Spectra and Radiative Transitions, 2nd ed.* Berlin: Springer-Verlag, 1992.

Racah W-Coefficient

Related to the CLEBSCH-GORDON COEFFICIENTS by

$$(J_1J_2[J']J_3|J_1,J_2J_3[J'']) = \sqrt{(2J'+1)(2J''+1)}\,W(J_1J_2JJ_3;J'J'')$$

and

$$(J_1J_2[J']J_3|J_1J_3[J'']J_2) = \sqrt{(2J'+1)(2J''+1)}\,W(J_1'J_3J_2J'';JJ_1).$$

see also CLEBSCH-GORDON COEFFICIENT, RACAH V-COEFFICIENT, WIGNER $3j$-SYMBOL, WIGNER $6j$-SYMBOL, WIGNER $9j$-SYMBOL

References

Messiah, A. "Racah Coefficients and '6j' Symbols." Appendix C.II in *Quantum Mechanics, Vol. 2.* Amsterdam, Netherlands: North-Holland, pp. 1061–1066, 1962.

Sobel'man, I. I. "Angular Momenta." Ch. 4 in *Atomic Spectra and Radiative Transitions, 2nd ed.* Berlin: Springer-Verlag, 1992.

Radau Quadrature

A GAUSSIAN QUADRATURE-like formula for numerical estimation of integrals. It requires $m+1$ points and fits all POLYNOMIALS to degree $2m$, so it effectively fits exactly all POLYNOMIALS of degree $2m-1$. It uses a WEIGHTING FUNCTION $W(x)=1$ in which the endpoint -1 in the interval $[-1,1]$ is included in a total of n ABSCISSAS, giving $r=n-1$ free abscissas. The general formula is

$$\int_{-1}^1 f(x)\,dx = w_1f(-1)+\sum_{i=2}^n w_if(x_i). \quad (1)$$

The free abscissas x_i for $i=2, \ldots, n$ are the roots of the POLYNOMIAL

$$\frac{P_{n-1}(x)+P_n(x)}{1+x}, \quad (2)$$

where $P(x)$ is a LEGENDRE POLYNOMIAL. The weights of the free abscissas are

$$w_i = \frac{1-x_i}{n^2[P_{n-1}(x_i)]^2} = \frac{1}{(1-x_i)[P'_{n-1}(x_i)]^2}, \quad (3)$$

and of the endpoint

$$w_1 = \frac{2}{n^2}. \quad (4)$$

The error term is given by

$$E = \frac{2^{2n-1}n[(n-1)!]^4}{[(2n-1)!]^3}f^{(2n-1)}(\xi), \quad (5)$$

for $\xi \in (-1,1)$.

n	x_i	w_i
2	-1	0.5
	0.333333	1.5
3	-1	0.222222
	-0.289898	1.02497
	0.689898	0.752806
4	-1	0.125
	-0.575319	0.657689
	0.181066	0.776387
	0.822824	0.440924
5	-1	0.08
	-0.72048	0.446208
	-0.167181	0.623653
	0.446314	0.562712
	0.885792	0.287427

The ABSCISSAS and weights can be computed analytically for small n.

n	x_i	w_i
2	-1	$\frac{1}{2}$
	$\frac{1}{3}$	$\frac{3}{2}$
3	-1	$\frac{2}{9}$
	$\frac{1}{5}(1-\sqrt{6})$	$\frac{1}{18}(16+\sqrt{6})$
	$\frac{1}{5}(1+\sqrt{6})$	$\frac{1}{18}(16-\sqrt{6})$

see also CHEBYSHEV QUADRATURE, LOBATTO QUADRATURE

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 888, 1972.

Chandrasekhar, S. *Radiative Transfer.* New York: Dover, p. 61, 1960.

Hildebrand, F. B. *Introduction to Numerical Analysis.* New York: McGraw-Hill, pp. 338–343, 1956.

Rademacher Function

see SQUARE WAVE

Radial Curve

Let C be a curve and let O be a fixed point. Let P be on C and let Q be the CURVATURE CENTER at P. Let P_1 be the point with P_1O a line segment PARALLEL and of equal length to PQ. Then the curve traced by P_1 is the radial curve of C. It was studied by Robert Tucker in 1864. The parametric equations of a curve (f, g) with RADIAL POINT (x_0, y_0) are

$$x = x_0 - \frac{g'(f'^2 + g'^2)}{f'g'' - f''g'}$$
$$y = y_0 + \frac{f'(f'^2 + g'^2)}{f'g'' - f''g'}.$$

Curve	Radial Curve
astroid	quadrifolium
catenary	kampyle of Eudoxus
cycloid	circle
deltoid	trifolium
logarithmic spiral	logarithmic spiral
tractrix	kappa curve

References

Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 40 and 202, 1972.

Yates, R. C. "Radial Curves." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 172–174, 1952.

Radial Point

The point with respect to which a RADIAL CURVE is computed.

see also RADIANT POINT

Radian

A unit of angular measure in which the ANGLE of an entire CIRCLE is 2π radians. There are therefore 360° per 2π radians, equal to $180/\pi$ or $57.29577951°$/radian. A RIGHT ANGLE is $\pi/2$ radians.

see also ANGLE, ARC MINUTE, ARC SECOND, DEGREE, GRADIAN, STERADIAN

Radiant Point

The point of illumination for a CAUSTIC.

see also CAUSTIC, RADIAL POINT

Radical

The symbol $\sqrt[n]{x}$ used to indicate a root is called a radical. The expression $\sqrt[n]{x}$ is therefore read "x radical n," or "the nth ROOT of x." $n = 2$ is written $\sqrt{x}$ and is called the SQUARE ROOT of x. $n = 3$ corresponds to the CUBE ROOT. The quantity under the root is called the RADICAND.

Some interesting radical identities are due to Ramanujan, and include the equivalent forms

$$(2^{1/3} + 1)(2^{1/3} - 1)^{1/3} = 3^{1/3}$$

and

$$(2^{1/3} - 1)^{1/3} = (\tfrac{1}{9})^{1/3} - (\tfrac{2}{9})^{1/3} + (\tfrac{4}{9})^{1/3}.$$

Another such identity is

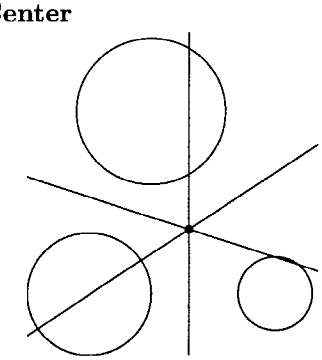
$$(5^{1/3} - 4^{1/3})^{1/2} = \tfrac{1}{3}(2^{1/3} + 20^{1/3} - 25^{1/3}).$$

see also CUBE ROOT, NESTED RADICAL, POWER, RADICAL INTEGER, RADICAND, ROOT (RADICAL), SQUARE ROOT, VINCULUM

Radical Axis

see RADICAL LINE

Radical Center



The RADICAL LINES of three CIRCLES are CONCURRENT in a point known as the radical center (also called the POWER CENTER). This theorem was originally demonstrated by Monge (Dörrie 1965, p. 153).

see also APOLLONIUS' PROBLEM, CONCURRENT, MONGE'S PROBLEM, RADICAL LINE

References

Dörrie, H. *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, 1965.

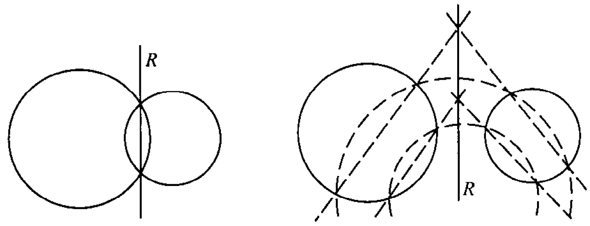
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, p. 32, 1929.

Radical Integer

A radical integer is a number obtained by closing the INTEGERS under ADDITION, DIVISION, MULTIPLICATION, SUBTRACTION, and ROOT extraction. An example of such a number is $\sqrt[3]{7} + \sqrt{-2} - \sqrt{3 + \sqrt[4]{1} + \sqrt{2}}$. The radical integers are a subring of the ALGEBRAIC INTEGERS. If there are ALGEBRAIC INTEGERS which are not radical integers, they must at least be cubic.

see also ALGEBRAIC INTEGER, ALGEBRAIC NUMBER, EUCLIDEAN NUMBER

Radical Line



The LOCUS of points of equal POWER with respect to two nonconcentric CIRCLES which is PERPENDICULAR to the line of centers (the CHORDAL THEOREM; Dörrie 1965). Let the circles have RADII r_1 and r_2 and their centers be separated by a distance d. If the CIRCLES intersect in two points, then the radical line is the line passing through the points of intersection. If not, then draw any two CIRCLES which cut each original CIRCLE twice. Draw lines through each pair of points of intersection of each CIRCLE. The line connecting their two points of intersection is then the radical line.

The radical line is located at distances

$$d_1 = \frac{d^2 + {r_1}^2 - {r_2}^2}{2d} \tag{1}$$

$$d_2 = -\frac{d^2 + {r_2}^2 - {r_1}^2}{2d} \tag{2}$$

along the line of centers from C_1 and C_2, respectively, where

$$d \equiv d_1 - d_2. \tag{3}$$

The radical line of any two POLAR CIRCLES is the ALTITUDE from the third vertex.

see also CHORDAL THEOREM, COAXAL CIRCLES, INVERSE POINTS, INVERSION, POWER (CIRCLE), RADICAL CENTER

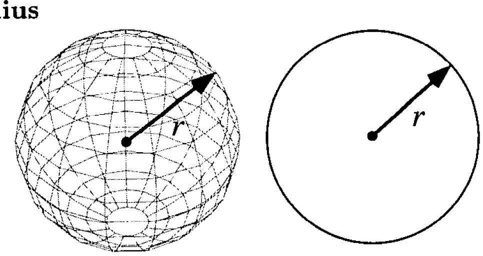
References
Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 31–34, 1967.
Dixon, R. *Mathographics.* New York: Dover, p. 68, 1991.
Dörrie, H. *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, p. 153, 1965.
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 28–34 and 176–177, 1929.

Radicand

The quantity under a RADICAL sign.

see also RADICAL, VINCULUM

Radius



The distance from the center of a CIRCLE to its PERIMETER, or from the center of a SPHERE to its surface. The radius is equal to half the DIAMETER.

see also BERTRAND'S PROBLEM, CIRCLE, CIRCUMFERENCE, DIAMETER, EXTENT, INVERSION RADIUS, KINNEY'S SET, PI, RADIUS OF CONVERGENCE, RADIUS OF CURVATURE, RADIUS (GRAPH), RADIUS OF GYRATION, RADIUS OF TORSION, RADIUS VECTOR, SPHERE

Radius of Convergence

The RADIUS (or 1-D distance in the 1-D case) over which series expansion CONVERGES.

Radius of Curvature

The radius of curvature is given by

$$R \equiv \frac{1}{\kappa}, \tag{1}$$

where κ is the CURVATURE. At a given point on a curve, R is the radius of the OSCULATING CIRCLE. The symbol ρ is sometimes used instead of R to denote the radius of curvature.

Let x and y be given parametrically by

$$x = x(t) \tag{2}$$

$$y = y(t), \tag{3}$$

then

$$R = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}, \tag{4}$$

where $x' = dx/dt$ and $y' = dy/dt$. Similarly, if the curve is written in the form $y = f(x)$, then the radius of curvature is given by

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}. \tag{5}$$

see also BEND (CURVATURE), CURVATURE, OSCULATING CIRCLE, TORSION (DIFFERENTIAL GEOMETRY)

References
Kreyszig, E. *Differential Geometry.* New York: Dover, p. 34, 1991.

Radius (Graph)

The minimum ECCENTRICITY of any VERTEX of a GRAPH.

Radius of Gyration

A function quantifying the spatial extent of the structure of a curve. It is defined by

$$R_g = \frac{\sqrt{\int_0^\infty r^2 p(r)\, dr}}{2\int_0^\infty p(r)\, dr},$$

where $p(r)$ is the LENGTH DISTRIBUTION FUNCTION. Small compact patterns have small R_g.

References
Pickover, C. A. *Keys to Infinity.* New York: W. H. Freeman, pp. 204–206, 1995.

Radius of Torsion

$$\sigma \equiv \frac{1}{\tau},$$

where τ is the TORSION. The symbol ϕ is also sometimes used instead of σ.

see also TORSION (DIFFERENTIAL GEOMETRY)

References

Kreyszig, E. *Differential Geometry.* New York: Dover, p. 39, 1991.

Radius Vector

The VECTOR $\mathbf{r}$ from the ORIGIN to the current position. It is also called the POSITION VECTOR. The derivative of $\mathbf{r}$ satisfies

$$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = \frac{1}{3}\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = \frac{1}{2}\frac{d}{dt}(r^2) = r\frac{dr}{dt} = rv,$$

where v is the magnitude of the VELOCITY (i.e., the SPEED).

Radix

The BASE of a number system, i.e., 2 for BINARY, 8 for OCTAL, 10 for DECIMAL, and 16 for HEXADECIMAL. The radix is sometimes called the BASE or SCALE.

see also BASE (NUMBER)

Rado's Sigma Function

see BUSY BEAVER

Radon-Nikodym Theorem

A THEOREM which gives NECESSARY and SUFFICIENT conditions for a countably additive function of sets can be expressed as an integral over the set.

References

Doob, J. L. "The Development of Rigor in Mathematical Probability (1900–1950)." *Amer. Math. Monthly* **103**, 586–595, 1996.

Radon Transform

An INTEGRAL TRANSFORM whose inverse is used to reconstruct images from medical CT scans. A technique for using Radon transforms to reconstruct a map of a planet's polar regions using a spacecraft in a polar orbit has also been devised (Roulston and Muhleman 1997).

The Radon transform can be defined by

$$R(p,\tau)[f(x,y)] = \int_{-\infty}^{\infty} f(x,\tau+px)\,dx$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta[y-(\tau+px)]\,dy\,dx \equiv U(p,\tau), \quad (1)$$

where p is the SLOPE of a line and τ is its intercept. The inverse Radon transform is

$$f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d}{dy} H[U(p,y-px)]\,dp, \quad (2)$$

where H is a HILBERT TRANSFORM. The transform can also be defined by

$$R'(r,\alpha)[f(x,y)]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(r - x\cos\alpha - y\sin\alpha)\,dx\,dy, \quad (3)$$

where r is the PERPENDICULAR distance from a line to the origin and α is the ANGLE formed by the distance VECTOR.

Using the identity

$$\mathcal{F}[R[f(\omega,\alpha)]] = \mathcal{F}^2[f(u,v)], \quad (4)$$

where $\mathcal{F}$ is the FOURIER TRANSFORM, gives the inversion formula

$$f(x,y) =$$
$$c\int_0^{\pi}\int_{-\infty}^{\infty} \mathcal{F}[R[f(\omega,\alpha)]]|\omega|e^{i\omega(x\cos\alpha+y\sin\alpha)}\,d\omega\,d\alpha. \quad (5)$$

The FOURIER TRANSFORM can be eliminated by writing

$$f(x,y) = \int_0^{\pi}\int_{-\infty}^{\infty} R[f(r,\alpha)]W(r,\alpha,x,y)\,dr\,d\alpha, \quad (6)$$

where W is a WEIGHTING FUNCTION such as

$$W(r,\alpha,x,y) = h(x\cos\alpha + y\sin\alpha - r) = \mathcal{F}^{-1}[|\omega|]. \quad (7)$$

Nievergelt (1986) uses the inverse formula

$$f(x,y) = \frac{1}{\pi}\lim_{c\to 0}$$
$$\int_0^{\pi}\int_{-\infty}^{\infty} R[f(r + x\cos\alpha + y\sin\alpha,\alpha)]G_c(r)\,dr\,d\alpha, \quad (8)$$

where

$$G_c(r) = \begin{cases} \frac{1}{\pi c^2} & \text{for } |r| \leq c \\ \frac{1}{\pi c^2}\left(1 - \frac{1}{\sqrt{1-c^2/r^2}}\right) & \text{for } |r| > c. \end{cases} \quad (9)$$

LUDWIG'S INVERSION FORMULA expresses a function in terms of its Radon transform. $R'(r,\alpha)$ and $R(p,\tau)$ are related by

$$p = \cot\alpha \qquad \tau = r\csc\alpha \quad (10)$$
$$r = \frac{\tau}{1+p^2} \qquad \alpha = \cot^{-1} p. \quad (11)$$

The Radon transform satisfies superposition

$$R(p,\tau)[f_1(x,y) + f_2(x,y)] = U_1(p,\tau) + U_2(p,\tau), \quad (12)$$

linearity

$$R(p,\tau)[af(x,y)] = aU(p,\tau), \tag{13}$$

scaling

$$R(p,\tau)\left[f\left(\frac{x}{a},\frac{y}{b}\right)\right] = |a|U\left(p\frac{a}{b},\frac{\tau}{b}\right), \tag{14}$$

ROTATION, with R_ϕ ROTATION by ANGLE ϕ

$$R(p,\tau)[R_\phi f(x,y)] = \frac{1}{|\cos\phi + p\sin\phi|} U\left(\frac{p-\tan\phi}{1+p\tan\phi}, \frac{\tau}{\cos\phi + p\sin\phi}\right), \tag{15}$$

and skewing

$$R(p,\tau)[f(ax+by,cx+dy)] = \frac{1}{|a+bp|}U\left[\frac{c+dp}{a+bp},\tau\frac{d-b(c+bd)}{a+bp}\right] \tag{16}$$

(Durrani and Bisset 1984).

The line integral along p,τ is

$$I = \sqrt{1+p^2}\,U(p,\tau). \tag{17}$$

The analog of the 1-D CONVOLUTION THEOREM is

$$R(p,\tau)[f(x,y)*g(y)] = U(p,\tau)*g(\tau), \tag{18}$$

the analog of PLANCHEREL'S THEOREM is

$$\int_{-\infty}^{\infty} U(p,\tau)\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy, \tag{19}$$

and the analog of PARSEVAL'S THEOREM is

$$\int_{-\infty}^{\infty} R(p,\tau)[f(x,y)]^2\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^2(x,y)\,dx\,dy. \tag{20}$$

If f is a continuous function on $\mathbb{C}$, integrable with respect to a plane LEBESGUE MEASURE, and

$$\int_l f\,ds = 0 \tag{21}$$

for every (doubly) infinite line l where s is the length measure, then f must be identically zero. However, if the global integrability condition is removed, this result fails (Zalcman 1982, Goldstein 1993).

see also TOMOGRAPHY

References

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Armitage, D. H. and Goldstein, M. "Nonuniqueness for the Radon Transform." *Proc. Amer. Math. Soc.* **117**, 175–178, 1993.

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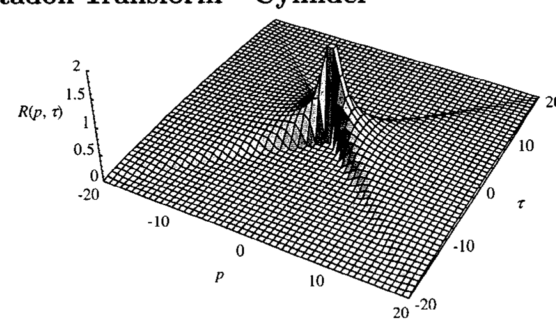
Radon Transform—Cylinder



Let the 2-D cylinder function be defined by

$$f(x,y) \equiv \begin{cases} 1 & \text{for } r < R \\ 0 & \text{for } r > R. \end{cases} \tag{1}$$

Then the Radon transform is given by

$$R(p,\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta[y-(\tau+px)]\,dy\,dx, \tag{2}$$

where

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} \tag{3}$$

is the DELTA FUNCTION.

$$\begin{aligned} R(p,\tau) &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^R\int_{-\infty}^{\infty} e^{-ik(r\sin\theta - pr\cos\theta)}r\,dr\,d\theta\,dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\tau}\int_0^{2\pi}\int_0^R e^{-ikr(\sin\theta - p\cos\theta)}r\,dr\,d\theta\,dk. \end{aligned} \tag{4}$$

Now write

$$\sin\theta - p\cos\theta = \sqrt{1+p^2}\,\cos(\theta+\phi) \equiv \sqrt{1+p^2}\,\cos\theta', \tag{5}$$

with ϕ a phase shift. Then

$$R(p,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\tau}\int_0^R \left(\int_0^{2\pi} e^{-ik\sqrt{1+p^2}\,r\cos\theta'}\,d\theta'\right) r\,dr\,dk$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\tau}\int_0^R 2\pi J_0(k\sqrt{1+p^2}\,r)r\,dr\,dk$$
$$= \int_{-\infty}^{\infty} e^{ik\tau}\int_0^R J_0(k\sqrt{1+p^2}\,r)r\,dr\,dk. \tag{6}$$

Then use

$$\int_0^z t^{n+1}J_n(t)\,dt = z^{n+1}J_{n+1}(z), \tag{7}$$

which, with $n=0$, becomes

$$\int_0^z tJ_0(t)\,dt = zJ_1(z). \tag{8}$$

Define

$$t \equiv k\sqrt{1+p^2}\,r \tag{9}$$
$$dt = k\sqrt{1+p^2}\,dr \tag{10}$$
$$r\,dr = \frac{t\,dt}{k^2(1+p^2)}, \tag{11}$$

so the inner integral is

$$\int_0^{R\sqrt{1+p^2}} J_0(t)\frac{t\,dt}{k^2(1+p^2)}$$
$$= \frac{1}{k^2(1+p^2)}kR\sqrt{1+p^2}J_1(kR\sqrt{1+p^2}\,)$$
$$= \frac{J_1(kR\sqrt{1+p^2}\,)}{k\sqrt{1+p^2}}R, \tag{12}$$

and the Radon transform becomes

$$R(p,\tau) = \frac{R}{\sqrt{1+p^2}}\int_{-\infty}^{\infty}\frac{e^{ik\tau}J_1(kR\sqrt{1+p^2}\,)}{k}\,dk$$
$$= \frac{2R}{\sqrt{1+p^2}}\int_0^{\infty}\frac{\cos(k\tau)J_1(kR\sqrt{1+p^2}\,)}{k}\,dk$$
$$= \begin{cases} \frac{2}{1+p^2}\sqrt{R^2(1+p^2)-\tau^2} & \text{for } \tau^2 < R^2(1+p^2) \\ 0 & \text{for } \tau^2 \geq R^2(1+p^2). \end{cases} \tag{13}$$

Converting to R' using $p=\cot\alpha$,

$$R'(r,\alpha) = \frac{2}{\sqrt{1+\cot^2\alpha}}\sqrt{(1+\cot^2\alpha)R^2 - r^2\csc^2\alpha}$$
$$= \frac{2}{\csc\alpha}\sqrt{\csc^2\alpha R^2 - r^2\csc^2\alpha}$$
$$= 2\sqrt{R^2-r^2}, \tag{14}$$

which could have been derived more simply by

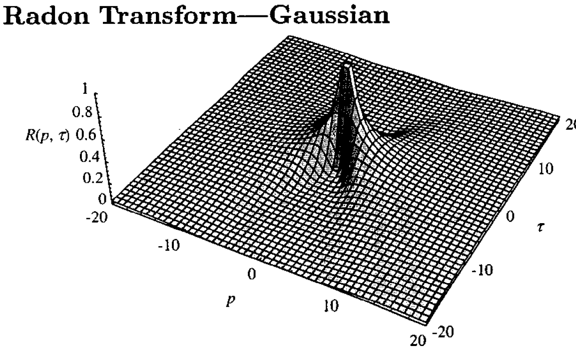
$$R'(r,\alpha) = \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} dy. \tag{15}$$

Radon Transform—Delta Function

For a DELTA FUNCTION at (x_0, y_0),

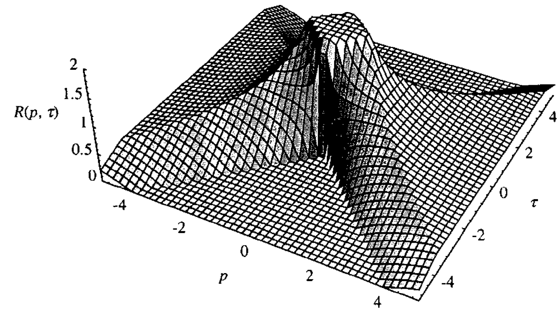
$$R(p,\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-x_0)\delta(y-y_0)\delta[y-(\tau+px)]\,dy\,dx$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ik[y-(\tau+px)]}\delta(x-x_0)\delta(y-y_0) \times dk\,dy\,dx$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\tau}\left[\int_{-\infty}^{\infty} e^{-iky}\delta(y-y_0)\,dy \times \int_{-\infty}^{\infty} e^{ikpx}\delta(x-x_0)\,dx\right] dk$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\tau}e^{-iky_0}e^{ikpx_0}\,dk$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(\tau+px_0-y_0)}\,dk = \delta(\tau+px_0-y_0).$$

Radon Transform—Gaussian



$$R(p,\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{1}{\sigma\sqrt{2\pi}}e^{-(x^2+y^2)/2\sigma^2}\right] \times\delta[y-(\tau+px)]\,dy\,dx$$
$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-[x^2+(\tau+px)^2]/2\sigma^2}\,dx$$
$$= \frac{1}{\sqrt{1+p^2}}e^{-t^2/[2(1+p^2)\sigma^2]}.$$

Radon Transform—Square



$$R(p,\tau)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta[y-(\tau+px)]\,dy\,dx, \quad (1)$$

where

$$f(x,y)\equiv\begin{cases}1 & \text{for } x,y\in[-a,a]\\ 0 & \text{otherwise}\end{cases} \quad (2)$$

and

$$\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} \quad (3)$$

is the DELTA FUNCTION.

$$\begin{aligned}
R(p,\tau) &= \frac{1}{2\pi}\int_{-a}^{a}\int_{-a}^{a}\int_{-\infty}^{\infty} e^{-ik[y-(\tau+px)]}\,dk\,dy\,dx\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\tau}\left[\int_{-a}^{a} e^{-ky}\,dy\int_{-a}^{a} e^{ikpx}\,dx\right]dk\\
&= \frac{1}{2\pi}e^{ik\tau}\frac{1}{-ik}[e^{-iky}]_{-a}^{a}\frac{1}{ikp}[e^{ikpx}]_{-a}^{a}\,dk\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\tau}\frac{1}{k^2p}[-2i\sin(ka)][2i\sin(kpa)]\,dk\\
&= \frac{2}{\pi p}\int_{-\infty}^{\infty}\frac{\sin(ka)\sin(kpa)e^{ik\tau}}{k^2}\,dk\\
&= \frac{4}{\pi p}\int_{0}^{\infty}\frac{\sin(ka)\sin(kpa)\cos(k\tau)}{k^2}\,dk\\
&= \frac{2}{\pi p}\int_{0}^{\infty}\frac{\sin[k(\tau+a)]-\sin[k(\tau-a)]}{k^2}\sin(kpa)\,dk\\
&= \frac{2}{\pi p}\left\{\int_{0}^{\infty}\frac{\sin[k(\tau+a)]\sin(kpa)}{k^2}\,dk\right.\\
&\quad\left.-\int_{0}^{\infty}\frac{\sin[k(\tau-a)]\sin(kpa)}{k^2}\,dk\right\}. \quad (4)
\end{aligned}$$

From Gradshteyn and Ryzhik (1979, equation 3.741.3),

$$\int_{0}^{\infty}\frac{\sin(ax)\sin(bx)}{x^2}\,dx=\tfrac{1}{2}\pi\,\mathrm{sgn}(ab)\min(|a|,|b|), \quad (5)$$

so

$$\begin{aligned}
R(p,\tau) = \frac{1}{p}\{&\mathrm{sgn}[(\tau+a)pa]\min(|\tau+a|,|pa|)\\
&-\mathrm{sgn}[(\tau-a)pa]\min(|\tau-a|,|pa|)\}. \quad (6)
\end{aligned}$$

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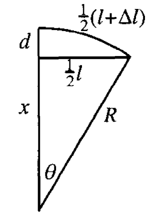
Railroad Track Problem



Given a straight segment of track of length l, add a small segment Δl so that the track bows into a circular ARC. Find the maximum displacement d of the bowed track. The PYTHAGOREAN THEOREM gives

$$R^2=x^2+(\tfrac{1}{2}l)^2, \quad (1)$$

But R is simply $x+d$, so

$$R^2=(x+d)^2=x^2+2xd+d^2. \quad (2)$$

Solving (1) and (2) for x gives

$$x=\frac{\frac{1}{4}l^2-d^2}{2d}. \quad (3)$$

Expressing the length of the ARC in terms of the central angle,

$$\begin{aligned}
\tfrac{1}{2}(l+\Delta l) &= \theta(d+x)=\theta\left(d+\frac{\frac{1}{4}l^2-d^2}{2d}\right)\\
&= \theta\left(\frac{2d^2+\frac{1}{4}l^2-d^2}{2d}\right)=\theta\left(\frac{d^2+\frac{1}{4}l^2}{2d}\right). \quad (4)
\end{aligned}$$

But θ is given by

$$\tan\theta=\frac{\frac{1}{2}l}{x}=\frac{\frac{1}{2}l(2d)}{\frac{1}{4}l^2-d^2}=\frac{dl}{\frac{1}{4}l^2-d^2}, \quad (5)$$

so plugging θ in gives

$$\tfrac{1}{2}(l+\Delta l)=\left(\frac{d^2+\frac{1}{4}l^2}{2d}\right)\tan^{-1}\left(\frac{dl}{\frac{1}{4}l^2-d^2}\right) \quad (6)$$

$$d(l+\Delta l)=(d^2+\tfrac{1}{4}l^2)\tan^{-1}\left(\frac{dl}{\frac{1}{4}l^2-d^2}\right). \quad (7)$$

For $l\gg d$,

$$\frac{dl}{\frac{1}{4}l^2\left(1-\frac{d^2}{4l^2}\right)}=\frac{4d}{l}\left(1-\frac{4d^2}{l^2}\right)^{-1}\approx\frac{4d}{l}\left(1+\frac{4d}{l^2}\right). \quad (8)$$

Therefore,

$$\begin{aligned}
&d(l+\Delta l)\\
&\approx (d^2+\tfrac{1}{4}l^2)\left\{\frac{4d}{l}\left(1+\frac{4d^2}{l^2}\right)-\frac{1}{3}\left[\frac{4d}{l}\left(1+\frac{4d^2}{l^2}\right)\right]^3\right\}\\
&\approx (d^2+\tfrac{1}{4}l^2)\left[\frac{4d}{l}+\frac{16d^3}{l^3}-\frac{1}{3}\left(\frac{4d}{l}\right)^3\left(1+3\frac{4d^2}{l^2}\right)\right]. \quad (9)
\end{aligned}$$

Keeping only terms to order $(d/l)^3$,

$$dl + \Delta l \approx \frac{4d^3}{l} + dl + \frac{4d^3}{l} - \frac{16}{3}\frac{d^3}{l} \qquad (10)$$

$$\Delta l \approx \left(8 - \tfrac{16}{3}\right)\frac{d^3}{l} = \frac{24-16}{3}\frac{d^3}{l} = \frac{8}{3}\frac{d^3}{l}, \qquad (11)$$

so

$$d^2 = \tfrac{3}{8} l \Delta l \qquad (12)$$

and

$$d \approx \tfrac{1}{2}\sqrt{\tfrac{3}{2} l \Delta l} = \tfrac{1}{4}\sqrt{6 l \Delta l}. \qquad (13)$$

If we take $l = 1$ mile $= 5280$ feet and $\Delta l = 1$ foot, then $d \approx 44.450$ feet.

Ramanujan 6-10-8 Identity

Let $ad = bc$, then

$$\begin{aligned}64[(a+b+c)^6 + (b+c+d)^6 - (c+d+a)^6 \\ -(d+a+b)^6 + (a-d)^6 - (b-c)^6] \\ \times[(a+b+c)^{10} + (b+c+d)^{10} - (c+d+a)^{10} \\ -(d+a+b)^{10} + (a-d)^{10} - (b-c)^{10}] \\ = 45[(a+b+c)^8 + (b+c+d)^8 - (c+d+a)^8 \\ -(d+a+b)^8 + (a-d)^8 - (b-c)^8]^2. \quad (1)\end{aligned}$$

This can also be expressed by defining

$$\begin{aligned}F_{2m}(a,b,c,d) = (a+b+c)^{2m} + (b+c+d)^{2m} \\ -(c+d+a)^{2m} - (d+a+b)^{2m} + (a-d)^{2m} - (b-c)^{2m} \quad (2) \\ f_{2m}(x,y) = (1+x+y)^{2m} + (x+y+xy)^{2m} - (y+xy+1)^{2m} \\ -(xy+1+x)^{2m} + (1-xy)^{2m} - (x-y)^{2m}. \quad (3)\end{aligned}$$

Then

$$F_{2m}(a,b,c,d) = a^{2m} f_{2m}(x,y), \qquad (4)$$

and identity (1) can then be written

$$64 f_6(x,y) f_{10}(x,y) = 45 {f_8}^2(x,y). \qquad (5)$$

Incidentally,

$$f_2(x,y) = 0 \qquad (6)$$

$$f_4(x,y) = 0. \qquad (7)$$

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Ramanujan Constant

The IRRATIONAL constant

$$R \equiv e^{\pi\sqrt{163}} = 262537412640768743.99999999999925\ldots$$

which is very close to an INTEGER. Numbers such as the Ramanujan constant can be found using the theory of MODULAR FUNCTIONS. A few rather spectacular examples are given by Ramanujan (1913–14), including the one above, and can be generated using some amazing properties of the j-FUNCTION.

M. Gardner (Apr. 1975) played an April Fool's joke on the readers of *Scientific American* by claiming that this number was exactly an INTEGER. He admitted the hoax a few months later (Gardner, July 1975).

see also ALMOST INTEGER, CLASS NUMBER, j-FUNCTION

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Ramanujan Continued Fraction

Let $f(a,b)$ be a RAMANUJAN THETA FUNCTION. Then

$$\frac{f(-q,-q^4)}{f(-q^2,-q^3)} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+\ldots},$$

where the quantity on the right is a CONTINUED FRACTION.

see also RAMANUJAN THETA FUNCTIONS

Ramanujan Cos/Cosh Identity

$$\left[1 + 2\sum_{n=1}^{\infty}\frac{\cos(n\theta)}{\cosh(n\pi)}\right]^{-2} + \left[1 + 2\sum_{n=1}^{\infty}\frac{\cosh(n\theta)}{\cosh(n\pi)}\right]^{-2} = \frac{2\Gamma^4(\frac{3}{4})}{\pi},$$

where $\Gamma(z)$ is the GAMMA FUNCTION.

Ramanujan-Eisenstein Series

Let t be a discriminant,

$$q \equiv -e^{-\pi\sqrt{t}}, \tag{1}$$

then

$$\begin{aligned} E_2(q) \equiv L(q) &\equiv 1 - 24\sum_{k=1}^{\infty} \frac{(2k+1)q^{2k+1}}{1-q^{2k+1}} \\ &= \left(\frac{2K}{\pi}\right)^2 (1-2k^2) \end{aligned} \tag{2}$$

$$\begin{aligned} E_4(q) \equiv M(q) &\equiv 1 + 240\sum_{k=1}^{\infty} \frac{k^3 q^{2k}}{1-q^{2k}} \\ &= \left(\frac{2K}{\pi}\right)^4 (1-k^2k'^2) \end{aligned} \tag{3}$$

$$\begin{aligned} E_6(q) \equiv N(q) &\equiv 1 - 504\sum_{k=1}^{\infty} \frac{k^5 q^{2k}}{1-q^{2k}} \\ &= \left(\frac{2K}{\pi}\right)^6 (1-2k^2)(1+\tfrac{1}{2}k^2k'^2). \end{aligned} \tag{4}$$

see also KLEIN'S ABSOLUTE INVARIANT, PI

References

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Ramanujan Function

$$\phi(a,n) \equiv 1 + 2\sum_{k=1}^{n} \frac{1}{(ak)^3 - ak}$$

$$\phi(a) \equiv \lim_{n\to\infty} \phi(a,n) = 1 + 2\sum_{k=1}^{\infty} \frac{1}{(ak)^3 - ak}.$$

The values of $\phi(n)$ for $n = 2, 3, \ldots$ are

$$\begin{aligned} \phi(2) &= 2\ln 2 \\ \phi(3) &= \ln 3 \\ \phi(4) &= \tfrac{3}{2}\ln 2 \\ \phi(6) &= \tfrac{1}{2}\ln 3 + \tfrac{1}{3}\ln 4. \end{aligned}$$

Ramanujan g- and G-Functions

Following Ramanujan (1913–14), write

$$\prod_{k=1,3,5,\ldots}^{\infty} (1 + e^{-k\pi\sqrt{n}}) = 2^{1/4} e^{-\pi\sqrt{n}/24} G_n \tag{1}$$

$$\prod_{k=1,3,5,\ldots}^{\infty} (1 - e^{-k\pi\sqrt{n}}) = 2^{1/4} e^{-\pi\sqrt{n}/24} g_n. \tag{2}$$

These satisfy the equalities

$$g_{4n} = 2^{1/4} g_n G_n \tag{3}$$

$$G_n = G_{1/n} \tag{4}$$

$$g_n{}^{-1} = g_{4/n} \tag{5}$$

$$\tfrac{1}{4} = (g_n G_n)^8 (G_n{}^8 - g_n{}^8). \tag{6}$$

G_n and g_n can be derived using the theory of MODULAR FUNCTIONS and can always be expressed as roots of algebraic equations when n is RATIONAL. For simplicity, Ramanujan tabulated g_n for n EVEN and G_n for n ODD. However, (6) allows G_n and g_n to be solved for in terms of g_n and G_n, giving

$$g_n = \tfrac{1}{2}\left(G_n{}^8 + \sqrt{G_n{}^{16} - G_n{}^{-8}}\right)^{1/8} \tag{7}$$

$$G_n = \tfrac{1}{2}\left(g_n{}^8 + \sqrt{g_n{}^{16} + Gg_n{}^{-8}}\right)^{1/8}. \tag{8}$$

Using (3) and the above two equations allows g_{4n} to be computed in terms of g_n or G_n

$$g_{4n} = \begin{cases} 2^{1/8} g_n \left(g_n{}^8 + \sqrt{g_n{}^{16} + g_n{}^{-8}}\right)^{1/8} & \text{for } n \text{ even} \\ 2^{1/8} G_n \left(G_n{}^8 + \sqrt{G_n{}^{16} - G_n{}^{-8}}\right)^{1/8} & \text{for } n \text{ odd.} \end{cases} \tag{9}$$

In terms of the PARAMETER k and complementary PARAMETER k',

$$G_n = (2k_n k'_n)^{-1/12} \tag{10}$$

$$g_n = \left(\frac{k'^2_n}{2k}\right)^{1/12}. \tag{11}$$

Here,

$$k_n = \lambda^*(n) \tag{12}$$

is the ELLIPTIC LAMBDA FUNCTION, which gives the value of k for which

$$\frac{K'(k)}{K(k)} = \sqrt{n}. \tag{13}$$

Solving for $\lambda^*(n)$ gives

$$\lambda^*(n) = \tfrac{1}{2}[\sqrt{1 + G_n{}^{-12}} - \sqrt{1 - G_n{}^{-12}}] \tag{14}$$

$$\lambda^*(n) = g_n{}^6[\sqrt{g_n{}^{12} + g_n{}^{-12}} - g_n{}^6]. \tag{15}$$

Analytic values for small values of n can be found in Ramanujan (1913–1914) and Borwein and Borwein (1987), and have been compiled in Weisstein (1996). Ramanujan (1913–1914) contains a typographical error labeling G_{465} as G_{265}.

see also G-FUNCTION

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Ramanujan's Hypergeometric Identity

$$1-\left(\frac{1}{2}\right)^3+\left(\frac{1\cdot 3}{2\cdot 4}\right)^3+\ldots = {}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{matrix};-1\right)$$
$$=\left[{}_2F_1\left(\begin{matrix}\frac{1}{4},\frac{1}{4}\\ 1\end{matrix};-1\right)\right]^2=\frac{\Gamma^2(\frac{9}{8})}{\Gamma^2(\frac{5}{4})\Gamma^2(\frac{7}{8})},$$

where ${}_2F_1(a,b;c;x)$ is a HYPERGEOMETRIC FUNCTION, ${}_3F_2(a,b,c;d;e;x)$ is a GENERALIZED HYPERGEOMETRIC FUNCTION, and $\Gamma(z)$ is a GAMMA FUNCTION.

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Ramanujan's Hypothesis

see TAU CONJECTURE

Ramanujan's Identity

$$5\frac{\phi^5(x^5)}{\phi^6(x)}=\sum_{m=0}^{\infty}P(5m+4)x^m,$$

where

$$\phi(x)=\prod_{m=1}^{\infty}(1-x^m)$$

and $P(n)$ is the PARTITION FUNCTION P.

see also RAMANUJAN'S SUM IDENTITY

Ramanujan's Integral

$$\int_{-\infty}^{\infty}\frac{J_{\mu+\xi}(x)}{x^{\mu+\xi}}\frac{J_{\nu-\xi}(y)}{y^{\nu-\xi}}e^{it\xi}\,d\xi$$
$$=\left[\frac{2\cos\left(\frac{1}{2}t\right)}{x^2e^{-it/2}+y^2e^{it/2}}\right]^{(\mu+\nu)/2}$$
$$\times J_{\mu+\nu}\left[\sqrt{2\cos\left(\tfrac{1}{2}t\right)(x^2e^{-it/2}+y^2e^{it/2})}\right]e^{it(\nu-\mu)/2},$$

where $J_n(z)$ is a BESSEL FUNCTION OF THE FIRST KIND.

References
Watson, G. N. *A Treatise on the Theory of Bessel Functions, 2nd ed.* Cambridge, England: Cambridge University Press, 1966.

Ramanujan's Interpolation Formula

$$\int_0^\infty x^{s-1}\sum_{k=0}^{\infty}(-1)^k x^k\phi(k)\,dx=\frac{\pi\phi(-s)}{\sin(s\pi)} \tag{1}$$
$$\int_0^\infty x^{s-1}\sum_{k=0}^{\infty}(-1)^k\frac{x^k}{k!}\lambda(k)\,dx=\Gamma(s)\lambda(-s), \tag{2}$$

where $\lambda(z)$ is the DIRICHLET LAMBDA FUNCTION and $\Gamma(z)$ is the GAMMA FUNCTION. Equation (2) is obtained from (1) by defining

$$\phi(u)=\frac{\lambda(u)}{\Gamma(1+u)}. \tag{3}$$

These formulas give valid results only for certain classes of functions.

References
Hardy, G. H. *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work, 3rd ed.* New York: Chelsea, pp. 15 and 186–195, 1959.

Ramanujan's Master Theorem

Suppose that in some NEIGHBORHOOD of $x=0$,

$$F(x)=\sum_{k=0}^{\infty}\frac{\phi(k)(-x)^k}{k!}.$$

Then

$$\int_0^\infty x^{n-1}F(x)\,dx=\Gamma(n)\phi(-n).$$

References
Berndt, B. C. *Ramanujan's Notebooks: Part I.* New York: Springer-Verlag, p. 298, 1985.

Ramanujan-Petersson Conjecture

A CONJECTURE for the EIGENVALUES of modular forms under HECKE OPERATORS.

Ramanujan Psi Sum

A sum which includes both the JACOBI TRIPLE PRODUCT and the q-BINOMIAL THEOREM as special cases. Ramanujan's sum is

$$\sum_{n=-\infty}^{\infty}\frac{(a)_n}{(b)_n}x^n=\frac{(ax)_\infty(q/ax)_\infty(q)_\infty(b/a)_\infty}{(x)_\infty(b/ax)_\infty(b)_\infty(q/a)_\infty},$$

where the NOTATION $(q)_k$ denotes q-SERIES. For $b=q$, this becomes the q-BINOMIAL THEOREM.

see also JACOBI TRIPLE PRODUCT, q-BINOMIAL THEOREM, q-SERIES

Ramanujan's Square Equation

It has been proved that the only solutions to the DIOPHANTINE EQUATION

$$2^n - 7 = x^2$$

are $n = 3$, 4, 5, 7, and 15 (Beeler *et al.* 1972, Item 31).

References

Beeler, M.; Gosper, R. W.; and Schroeppel, R. *HAKMEM*. Cambridge, MA: MIT Artificial Intelligence Laboratory, Memo AIM-239, Feb. 1972.

Ramanujan's Sum

The sum

$$c_q(m) = \sum_{h^*(q)} e^{2\pi i h m/q}, \tag{1}$$

where h runs through the residues RELATIVELY PRIME to q, which is important in the representation of numbers by the sums of squares. If $(q, q') = 1$ (i.e., q and q' are RELATIVELY PRIME), then

$$c_{qq'}(m) = c_q(m) c_{q'}(m). \tag{2}$$

For argument 1,

$$c_b(1) = \mu(b), \tag{3}$$

where μ is the MÖBIUS FUNCTION, and for general m,

$$c_b(m) = \mu\left(\frac{b}{(b,m)}\right) \frac{\phi(b)}{\phi\left(\frac{b}{(b,m)}\right)}. \tag{4}$$

see also MÖBIUS FUNCTION, WEYL'S CRITERION

References

Vardi, I. *Computational Recreations in Mathematica*. Redwood City, CA: Addison-Wesley, p. 254, 1991.

Ramanujan's Sum Identity

If

$$\frac{1 + 53x + 9x^2}{1 - 82x - 82x^2 + x^3} = \sum_{n=1}^{\infty} a_n x^n \tag{1}$$

$$\frac{2 - 26x - 12x^2}{1 - 82x - 82x^2 + x^3} = \sum_{n=0}^{\infty} b_n x^n \tag{2}$$

$$\frac{2 + 8x - 10x^2}{1 - 82x - 82x^2 + x^3} = \sum_{n=0}^{\infty} c_n x^n, \tag{3}$$

then

$$a_n{}^3 + b_n{}^3 = c_n{}^3 + (-1)^n. \tag{4}$$

Hirschhorn (1995) showed that

$$a_n = \tfrac{1}{85}[(64 + 8\sqrt{85})\alpha^n + (64 - 8\sqrt{85})\beta^n - 43(-1)^n] \tag{5}$$

$$b_n = \tfrac{1}{85}[(77 + 7\sqrt{85})\alpha^n + (77 - 7\sqrt{85})\beta^n + 16(-1)^n] \tag{6}$$

$$c_n = \tfrac{1}{85}[(93 + 9\sqrt{85})\alpha^n + (93 - 9\sqrt{85})\beta^n - 16(-1)^n], \tag{7}$$

where

$$\alpha = \tfrac{1}{2}(83 + 9\sqrt{85}) \tag{8}$$

$$\beta = \tfrac{1}{2}(83 - 9\sqrt{85}). \tag{9}$$

Hirschhorn (1996) showed that checking the first seven cases $n = 0$ to 6 is sufficient to prove the result.

References

Hirschhorn, M. D. "An Amazing Identity of Ramanujan." *Math. Mag.* **68**, 199–201, 1995.
Hirschhorn, M. D. "A Proof in the Spirit of Zeilberger of an Amazing Identity of Ramanujan." *Math. Mag.* **69**, 267–269, 1996.

Ramanujan's Tau-Dirichlet Series

see TAU-DIRICHLET SERIES

Ramanujan's Tau Function

see TAU FUNCTION

Ramanujan Theta Functions

Ramanujan's one-variable theta function is defined by

$$\varphi(x) \equiv \sum_{m=-\infty}^{\infty} x^{m^2}. \tag{1}$$

It is equal to the function in the JACOBI TRIPLE PRODUCT with $z = 1$,

$$\varphi(x) = G(1) = \prod_{n=1}^{\infty} (1 + x^{2n-1})^2 (1 - x^{2n})$$
$$= \sum_{m=-\infty}^{\infty} x^{m^2} = 1 + 2\sum_{m=0}^{\infty} x^{m^2}. \tag{2}$$

Special values include

$$\varphi(e^{-\pi\sqrt{2}}) = \frac{\Gamma(\frac{9}{8})}{\Gamma(\frac{5}{4})} \sqrt{\frac{\Gamma(\frac{1}{4})}{2^{1/4}\pi}} \tag{3}$$

$$\varphi(e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma(\frac{3}{4})} \tag{4}$$

$$\varphi(e^{-2\pi}) = \frac{\sqrt{2+\sqrt{2}}}{2} \frac{\pi}{\Gamma(\frac{3}{4})}. \tag{5}$$

Ramanujan's two-variable theta function is defined by

$$f(a, b) \equiv \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \tag{6}$$

for $|ab| < 1$. It satisfies

$$f(-1, a) = 0 \tag{7}$$

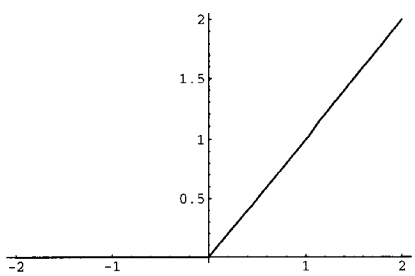
$$f(a, b) = f(b, a) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty \tag{8}$$

$$f(-q) \equiv f(-q, -q^2)$$
$$= \sum_{k=0}^{\infty}(-1)^k q^{k(2k-1)/2} + \sum_{k=1}^{\infty}(-1)^k q^{k(2k+1)/2}$$
$$= (q;q)_\infty, \quad (9)$$

where $(q)_\infty$ are q-SERIES.

see also JACOBI TRIPLE PRODUCT, SCHRÖTER'S FORMULA, q-SERIES

Ramp Function



$$R(x) \equiv xH(x) \quad (1)$$
$$= \int_{-\infty}^{x} H(x')\,dx' \quad (2)$$
$$= \int_{-\infty}^{\infty} H(x')H(x-x')\,dx' \quad (3)$$
$$= H(x) * H(x), \quad (4)$$

where $H(x)$ is the HEAVISIDE STEP FUNCTION and $*$ is the CONVOLUTION. The DERIVATIVE is

$$R'(x) = -H(x). \quad (5)$$

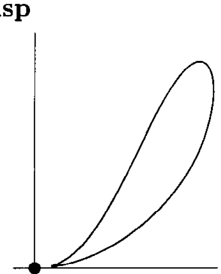
The FOURIER TRANSFORM of the ramp function is given by

$$\mathcal{F}[R(x)] = \int_{-\infty}^{\infty} e^{-2\pi ikx} R(x)\,dx = \pi i\delta'(2\pi k) - \frac{1}{4\pi^2k^2}, \quad (6)$$

where $\delta(x)$ is the DELTA FUNCTION and $\delta'(x)$ its DERIVATIVE.

see also FOURIER TRANSFORM—RAMP FUNCTION, HEAVISIDE STEP FUNCTION, RECTANGLE FUNCTION, SGN, SQUARE WAVE

Ramphoid Cusp



A type of CUSP as illustrated above for the curve $x^4 + x^2y^2 - 2x^2y - xy^2 + y^2 = 0$.

see also CUSP

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Walker, R. J. *Algebraic Curves.* New York: Springer-Verlag, pp. 57–58, 1978.

Ramsey Number

The Ramsey number $R(m,n)$ gives the solution to the PARTY PROBLEM, which asks the minimum number of guests $R(m,n)$ that must be invited so that at least m will know each other (i.e., there exists a CLIQUE of order m) or at least n will not know each other (i.e., there exists an independent set of order n). By symmetry, it is true that

$$R(m,n) = R(n,m). \quad (1)$$

It also must be true that

$$R(2,m) = m. \quad (2)$$

A generalized Ramsey number is written

$$R(m_1, \ldots, m_k; n) \quad (3)$$

and is the smallest INTEGER R such that, no matter how each n-element SUBSET of an r-element SET are colored with k colors, there exists an i such that there is a SUBSET of size m_i, all of whose n-element SUBSETS are color i. The usual Ramsey numbers are then equivalent to $R(m,n) = R(m,n;2)$.

Bounds are given by

$$R(k,l) \leq \begin{cases} R(k-1,l) + R(k,l-1) - 1 & \text{for } R(k-1,l) \text{ and } R(k,l-1) \text{ even} \\ R(k-1,l) + R(k,l-1) & \text{otherwise} \end{cases} \quad (4)$$

and

$$R(k,k) \leq 4R(k-2,k) + 2 \quad (5)$$

(Chung and Grinstead 1983). Erdős proved that for diagonal Ramsey numbers $R(k,k)$,

$$\frac{k2^{k/2}}{e\sqrt{2}} < R(k,k). \quad (6)$$

This result was subsequently improved by a factor of 2 by Spencer (1975). $R(3,k)$ was known since 1980 to be bounded from above by $c_2k^2/\ln k$, and Griggs (1983) showed that $c_2 = 5/12$ was an acceptable limit. J.-H. Kim (Cipra 1995) subsequently bounded $R(3,k)$ by a similar expression from below, so

$$c_1\frac{k^2}{\ln k} \leq R(3,k) \leq c_2\frac{k^2}{\ln k}. \quad (7)$$

Burr (1983) gives Ramsey numbers for all 113 graphs with no more than 6 EDGES and no isolated points.

A summary of known results up to 1983 for $R(m,n)$ is given in Chung and Grinstead (1983). Radziszowski maintains an up-to-date list of the best current bounds, reproduced in part in the following table for $R(m,n;2)$.

m	n	$R(m,n)$
3	3	6
3	4	9
3	5	14
3	6	18
3	7	23
3	8	28
3	9	36
3	10	[40, 43]
3	11	[46, 51]
3	12	[52, 60]
3	13	[60, 69]
3	14	[66, 78]
3	15	[73, 89]
3	16	[79, ∞]
3	17	[92, ∞]
3	18	[98, ∞]
3	19	[106, ∞]
3	20	[109, ∞]
3	21	[122, ∞]
3	22	[125, ∞]
3	23	[136, ∞]

m	n	$R(m,n)$
4	4	18
4	5	25
4	6	[35, 41]
4	7	[49, 62]
4	8	[55, 85]
4	9	[69, 116]
4	10	[80, 151]
4	11	[93, 191]
4	12	[98, 238]
4	13	[112, 291]
4	14	[119, 349]
4	15	[128, 417]

m	n	$R(m,n)$
5	5	[43, 49]
5	6	[58, 87]
5	7	[80, 143]
5	8	[95, 216]
5	9	[116, 371]
5	10	[1, 445]

m	n	$R(m,n)$
6	6	[102, 165]
6	7	[109, 300]
6	8	[122, 497]
6	9	[153, 784]
6	10	[167, 1180]

m	n	$R(m,n)$
7	7	[205, 545]
7	8	[1, 1035]
7	9	[1, 1724]
7	10	[1, 2842]

m	n	$R(m,n)$
8	8	[282, 1874]
8	9	[1, 3597]
8	10	[1, 6116]

m	n	$R(m,n)$
9	9	[565, 6680]
9	10	[1, 12795]

m	n	$R(m,n)$
10	10	[798, 23981]

m	n	$R(m,n)$
11	11	[522, ∞]

Known values for generalized Ramsey numbers are given in the following table.

$R(\ldots;2)$	Bounds
$R(3,3,3;2)$	17
$R(3,3,4;2)$	[30, 32]
$R(3,3,5;2)$	[45, 59]
$R(3,4,4;2)$	[55, 81]
$R(3,4,5;2)$	≥ 80
$R(4,4,4;2)$	[128, 242]
$R(3,3,3,3;2)$	[51, 64]
$R(3,3,3,4;2)$	[87, 159]
$R(3,3,3,3,3;2)$	[162, 317]
$R(3,3,3,3,3,3;2)$	[1, 500]

$R(\ldots;3)$	Bounds
$R(4,4;3)$	[14, 15]

see also CLIQUE, COMPLETE GRAPH, EXTREMAL GRAPH, IRREDUNDANT RAMSEY NUMBER, SCHUR NUMBER

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Ramsey's Theorem

A generalization of DILWORTH'S LEMMA. For each $m, n \in \mathbb{N}$ with $m, n \geq 2$, there exists a least INTEGER $R(m,n)$ (the RAMSEY NUMBER) such that no matter how the COMPLETE GRAPH $K_{R(m,n)}$ is two-colored, it will contain a green SUBGRAPH K_m or a red subgroup K_n. Furthermore,

$$R(m,n) \leq R(m-1,n) + R(m,n-1)$$

if $m, n \geq 3$. The theorem can be equivalently stated that, for all $\in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that any complete DIGRAPH on n VERTICES contains a complete transitive SUBGRAPH of m VERTICES. Ramsey's theorem is a generalization of the PIGEONHOLE PRINCIPLE since

$$R(\underbrace{2,2,\ldots,2}_{t}) = t+1.$$

see also DILWORTH'S LEMMA, NATURAL INDEPENDENCE PHENOMENON, PIGEONHOLE PRINCIPLE, RAMSEY NUMBER

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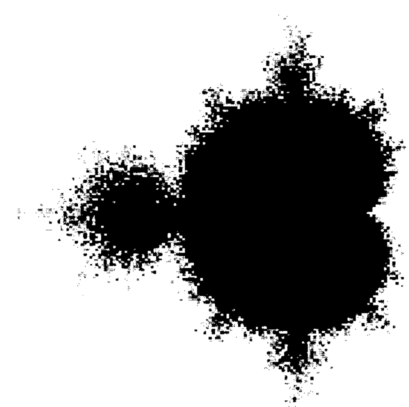
Randelbrot Set



The FRACTAL-like figure obtained by performing the same iteration as for the MANDELBROT SET, but adding a random component R,

$$z_{n+1} = {z_n}^2 + c + R.$$

In the above plot, $R \equiv R_x + iR_y$, where $R_x, R_y \in [-0.05, 0.05]$.

see also MANDELBROT SET

References

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Random Distribution

A DISTRIBUTION in which the variates occur with PROBABILITIES asymptotically matching their "true" underlying DISTRIBUTION is said to be random.

see also DISTRIBUTION, RANDOM NUMBER

Random Dot Stereogram

see STEREOGRAM

Random Graph

A random graph is a GRAPH in which properties such as the number of NODES, EDGES, and connections between them are determined in some random way. Erdős and Rényi showed that for many monotone-increasing properties of random graphs, graphs of a size slightly less than a certain threshold are very unlikely to have the property, whereas graphs with a few more EDGES are almost certain to have it. This is known as a PHASE TRANSITION.

see also GRAPH (GRAPH THEORY), GRAPH THEORY

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Random Matrix

A random matrix is a MATRIX of given type and size whose entries consist of random numbers from some specified distribution.

see also MATRIX

Random Number

Computer-generated random numbers are sometimes called PSEUDORANDOM NUMBERS, while the term "random" is reserved for the output of unpredictable physical processes. It is impossible to produce an arbitrarily long string of random digits and prove it is random. Strangely, it is very difficult for humans to produce a string of random digits, and computer programs can be written which, on average, actually predict some of the digits humans will write down based on previous ones.

The LINEAR CONGRUENCE METHOD is one algorithm for generating PSEUDORANDOM NUMBERS. The initial number used as the starting point in a random number generating algorithm is known as the SEED. The goodness of random numbers generated by a given ALGORITHM can be analyzed by examining its NOISE SPHERE.

see also BAYS' SHUFFLE, CLIFF RANDOM NUMBER GENERATOR, QUASIRANDOM SEQUENCE, SCHRAGE'S ALGORITHM, STOCHASTIC

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Random Percolation

see PERCOLATION THEORY

Random Polynomial

A POLYNOMIAL having random COEFFICIENTS.

see also KAC FORMULA

Random Variable

A random variable is a measurable function from a PROBABILITY SPACE $(S, \mathbb{S}, P)$ into a MEASURABLE SPACE $(S', \mathbb{S}')$ known as the STATE SPACE.

see also PROBABILITY SPACE, RANDOM DISTRIBUTION, RANDOM NUMBER, STATE SPACE, VARIATE

References

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Random Walk

A random process consisting of a sequence of discrete steps of fixed length. The random thermal perturbations in a liquid are responsible for a random walk phenomenon known as Brownian motion, and the collisions of molecules in a gas are a random walk responsible for diffusion. Random walks have interesting mathematical properties that vary greatly depending on the dimension in which the walk occurs and whether it is confined to a lattice.

see also RANDOM WALK—1-D, RANDOM WALK—2-D, RANDOM WALK—3-D, SELF-AVOIDING WALK

References

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Random Walk—1-D

Let N steps of equal length be taken along a LINE. Let p be the probability of taking a step to the right, q the probability of taking a step to the left, n_1 the number of steps taken to the right, and n_2 the number of steps taken to the left. The quantities p, q, n_1, n_2, and N are related by

$$p+q=1 \tag{1}$$

and

$$n_1+n_2=N. \tag{2}$$

Now examine the probability of taking exactly n_1 steps out of N to the right. There are $\binom{N}{n_1}=\binom{n_1+n_2}{n_1}$ ways of taking n_1 steps to the right and n_2 to the left, where $\binom{n}{m}$ is a BINOMIAL COEFFICIENT. The probability of taking a particular ordered sequence of n_1 and n_2 steps is $p^{n_1}q^{n_2}$. Therefore,

$$P(n_1)=\frac{(n_1+n_2)!}{n_1!n_2!}p^{n_1}q^{n_2}=\frac{N!}{n_1!(N-n_1)!}p^{n_1}q^{N-n_1}, \tag{3}$$

where $n!$ is a FACTORIAL. This is a BINOMIAL DISTRIBUTION and satisfies

$$\sum_{n_1=0}^{N}P(n_1)=(p+q)^N=1^N=1. \tag{4}$$

The MEAN number of steps n_1 to the right is then

$$\langle n_1\rangle\equiv\sum_{n_1=0}^{N}n_1P(n_1)=\sum_{n_1=0}^{N}\frac{N!}{n_1!(N-n_1)!}p^{n_1}q^{N-n_1}n_1, \tag{5}$$

but

$$n_1p^{n_1}=p\frac{\partial}{\partial p}p^{n_1}, \tag{6}$$

so

$$\begin{aligned}\langle n_1\rangle&=\sum_{n_1=0}^{N}\frac{N!}{n_1!(N-n_1)!}\left(p\frac{\partial}{\partial p}p^{n_1}\right)q^{N-n_1}\\&=p\frac{\partial}{\partial p}\sum_{n_1=0}^{N}\frac{N!}{n_1!(N-n_1)!}p^{n_1}q^{N-n_1}\\&=p\frac{\partial}{\partial p}(p+q)^N=pN(p+q)^{N-1}=pN.\end{aligned} \tag{7}$$

From the BINOMIAL THEOREM,

$$\langle n_2\rangle=N-\langle n_1\rangle=N(1-p)=qN. \tag{8}$$

The VARIANCE is given by

$${\sigma_{n_1}}^2=\langle {n_1}^2\rangle-\langle n_1\rangle^2. \tag{9}$$

But

$$\langle {n_1}^2\rangle=\sum_{n_1=0}^{N}\frac{N!}{n_1!(N-n_1)!}p^{n_1}q^{N-n_1}{n_1}^2, \tag{10}$$

so

$$\begin{aligned}{n_1}^2p^{n_1}&=n_1\left(p\frac{\partial}{\partial p}\right)p^{n_1}=\left(p\frac{\partial}{\partial p}\right)^2p^{n_1}\\&=\sum_{n_1=0}^{N}\frac{N!}{n_1!(N-n_1)!}\left(p\frac{\partial}{\partial p}\right)^2p^{n_1}q^{N-n_1}\\&=\left(p\frac{\partial}{\partial p}\right)^2\sum_{n_1=0}^{N}\frac{N!}{n_1!(N-n_1)!}p^{n_1}q^{N-n_1}\\&=\left(p\frac{\partial}{\partial p}\right)^2(p+q)^N=\frac{\partial}{\partial p}[pN(p+q)N-1]\\&=p[N(p+q)^{N-1}+pN(N-1)(p+q)^{N-2}]\\&=p[N+pN(N-1)]\\&=pN[1+pN-p]=(Np)^2+Npq\\&=\langle n_1\rangle^2+Npq.\end{aligned} \tag{11}$$

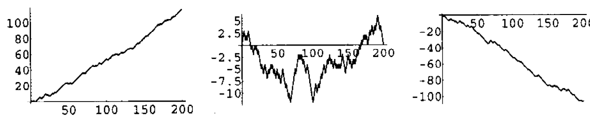
Therefore,

$${\sigma_{n_1}}^2=\langle {n_1}^2\rangle-\langle n_1\rangle^2=Npq, \tag{12}$$

and the ROOT-MEAN-SQUARE deviation is

$$\sigma_{n_1}=\sqrt{Npq}. \tag{13}$$

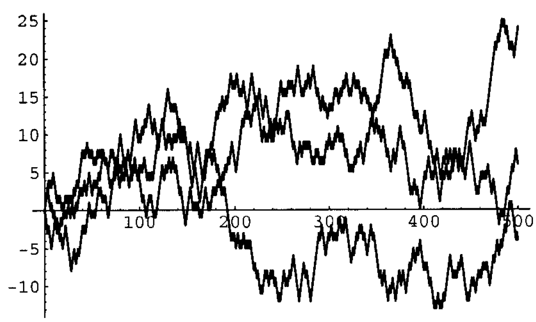
For a large number of total steps N, the BINOMIAL DISTRIBUTION characterizing the distribution approaches a GAUSSIAN DISTRIBUTION.



Consider now the distribution of the distances d_N traveled after a given number of steps,

$$d_N \equiv n_1 - n_2 = 2n_1 - N, \tag{14}$$

as opposed to the *number* of steps in a given direction. The above plots show $d_N(p)$ for $N = 200$ and three values $p = 0.1$, $p = 0.5$, and $p = 0.9$, respectively. Clearly, weighting the steps toward one direction or the other influences the overall trend, but there is still a great deal of random scatter, as emphasized by the plot below, which shows three random walks all with $p = 0.5$.



Surprisingly, the most probable number of sign changes in a walk is 0, followed by 1, then 2, etc.

For a random walk with $p = 1/2$, the probability $P_N(d)$ of traveling a given distance d after N steps is given in the following table.

steps	-5	-4	-3	-2	-1	0	1	2	3	4	5
0						1					
1					$\frac{1}{2}$	0	$\frac{1}{2}$				
2				$\frac{1}{4}$	0	$\frac{2}{4}$	0	$\frac{1}{4}$			
3			$\frac{1}{8}$	0	$\frac{3}{8}$	0	$\frac{3}{8}$	0	$\frac{1}{8}$		
4		$\frac{1}{16}$	0	$\frac{4}{16}$	0	$\frac{6}{16}$	0	$\frac{4}{16}$	0	$\frac{1}{16}$	
5	$\frac{1}{32}$	0	$\frac{5}{32}$	0	$\frac{10}{32}$	0	$\frac{10}{32}$	0	$\frac{5}{32}$	0	$\frac{1}{32}$

In this table, subsequent rows are found by adding HALF of each cell in a given row to each of the two cells diagonally below it. In fact, it is simply PASCAL'S TRIANGLE padded with intervening zeros and with each row multiplied by an additional factor of $1/2$. The COEFFICIENTS in this triangle are given by

$$P_N(d) = \frac{1}{2^N}\binom{N}{\frac{d+N}{2}}. \tag{15}$$

The expectation value of the distance after N steps is therefore

$$\begin{aligned}\langle d_N\rangle &= \sum_{d=-N,-(N-2),\ldots}^{N} |d|\,P_N(d)\\ &= \frac{1}{2^N}\sum_{d=-N,-(N-2),\ldots}^{N} \frac{|d|\,N!}{\left(\frac{N+d}{2}\right)!\left(\frac{N-d}{2}\right)!}.\end{aligned} \tag{16}$$

This sum can be done symbolically by separately considering the cases N EVEN and N ODD. First, consider EVEN N so that $N \equiv 2J$. Then

$$\begin{aligned}\langle d_{2J}\rangle &= \frac{N!}{2^N}\left[\sum_{\substack{d=-2J,\\-2(J-1),\ldots}}^{-2} \frac{|d|}{\left(\frac{2J+d}{2}\right)!\left(\frac{2J-d}{2}\right)!}\right.\\ &\quad\left.+\sum_{d=0} \frac{|d|}{\left(\frac{2J+d}{2}\right)!\left(\frac{2J-d}{2}\right)!} + \sum_{d=2,4,\ldots}^{2J} \frac{|d|}{\left(\frac{2J+d}{2}\right)!\left(\frac{2J-d}{2}\right)!}\right]\\ &= \frac{N!}{2^N}\left[\sum_{d=-J,-(J-1),\ldots}^{-1} \frac{|2d|}{\left(\frac{2J+2d}{2}\right)!\left(\frac{2J-2d}{2}\right)!}\right.\\ &\quad\left.+\sum_{d=1,2,\ldots}^{J} \frac{|2d|}{\left(\frac{2J+2d}{2}\right)!\left(\frac{2J-2d}{2}\right)!}\right]\\ &= \frac{N!}{2^N}\left[2\sum_{d=1}^{J}\frac{2d}{(J+d)!(J-d)!}\right]\\ &= \frac{N!}{2^{N-2}}\sum_{d=1}^{J}\frac{d}{(J+d)!(J-d)!}.\end{aligned} \tag{17}$$

But this sum can be evaluated analytically as

$$\sum_{d=1}^{J}\frac{d}{(J+d)!(J-d)!} = \frac{J}{2\Gamma^2(1+J)}, \tag{18}$$

which, when combined with $N = 2J$ and plugged back in, gives

$$\langle d_{2J}\rangle = \frac{\Gamma(2J+1)J}{2^{2J-1}\Gamma^2(1+J)} = \frac{\Gamma(2J)}{2^{2J-2}\Gamma^2(J)}. \tag{19}$$

But the LEGENDRE DUPLICATION FORMULA gives

$$\Gamma(2J) = \frac{2^{2J-1/2}\Gamma(J)\Gamma(J+\frac{1}{2})}{\sqrt{2\pi}}, \tag{20}$$

so

$$\langle d_{2J}\rangle = \frac{\frac{1}{\sqrt{2\pi}}2^{2J-1/2}\Gamma(J)\Gamma(J+\frac{1}{2})}{2^{2J-2}\Gamma^2(J)} = \frac{2}{\sqrt{\pi}}\frac{\Gamma(J+\frac{1}{2})}{\Gamma(J)}. \tag{21}$$

Now consider N ODD, so $N \equiv 2J - 1$. Then

$$\begin{aligned}
\langle d_{2J-1}\rangle &= \frac{N!}{2^N}\left[\sum_{\substack{d=-(2J-1),\\ -(2J+1),\ldots}}^{-1} \frac{|d|}{\left(\frac{2J-1+d}{2}\right)!\left(\frac{2J-1-d}{2}\right)!}\right.\\
&\quad\left.+\sum_{d=1,3,\ldots}^{2J-1}\frac{|d|}{\left(\frac{2J-1+d}{2}\right)!\left(\frac{2J-1-d}{2}\right)!}\right]\\
&= \frac{N!}{2^{N-1}}\left[\sum_{d=1,3,\ldots}^{2J-1}\frac{d}{\left(\frac{2J-1+d}{2}\right)!\left(\frac{2J-1-d}{2}\right)!}\right]\\
&= \frac{N!}{2^{N-1}}\left[\sum_{d=2,4,\ldots}^{2J}\frac{d-1}{\left(\frac{2J-2+d}{2}\right)!\left(\frac{2J-d}{2}\right)!}\right]\\
&= \frac{\Gamma(2J)}{2^{2J-2}}\left[\sum_{d=1}^{J}\frac{2d-1}{(J+d-1)!(J-d)!}\right]\\
&= \Gamma(2J)\left[\frac{1+J-{}_2F_1(1,-J;J;1)}{2^{2J-2}\Gamma(J)\Gamma(1+J)}+\frac{1}{\Gamma(2J)}\right]\\
&= \frac{\frac{2^{2J-1/2}}{\sqrt{2\pi}}\Gamma(J)\Gamma(J+1/2)}{2^{2J-2}\Gamma^2(J)J}[1+J-{}_2F_1(1,-J;J;-1)]+1\\
&= \frac{2}{\sqrt{\pi}}\frac{\Gamma(J+\frac{1}{2})}{J\Gamma(J)}[1+J-{}_2F_1(1,-J;J;-1)]+1. \qquad (22)
\end{aligned}$$

But the HYPERGEOMETRIC FUNCTION ${}_2F_1$ has the special value

$${}_2F_1(1,-J;J;-1) = \frac{\sqrt{\pi}}{2}\frac{J\Gamma(J)}{\Gamma(J+\frac{1}{2})}+1, \qquad (23)$$

so

$$\langle d_{2J-1}\rangle = \frac{2}{\sqrt{\pi}}\frac{\Gamma(J+\frac{1}{2})}{\Gamma(J)}. \qquad (24)$$

Summarizing the EVEN and ODD solutions,

$$\langle d_N\rangle = \frac{2}{\sqrt{\pi}}\frac{\Gamma(J+\frac{1}{2})}{\Gamma(J)}, \qquad (25)$$

where

$$\begin{cases} J=\frac{1}{2}N & \text{for } N \text{ even}\\ J=\frac{1}{2}(N+1) & \text{for } N \text{ odd.}\end{cases} \qquad (26)$$

Written explicitly in terms of N,

$$\langle d_N\rangle = \begin{cases} \frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}N+\frac{1}{2})}{\Gamma(\frac{1}{2}N)} & \text{for } N \text{ even}\\ \frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}N+1)}{\Gamma(\frac{1}{2}N+\frac{1}{2})} & \text{for } N \text{ odd.}\end{cases} \qquad (27)$$

The first few values of $\langle d_N\rangle$ are then

$$\begin{aligned}
\langle d_0\rangle &= 0\\
\langle d_1\rangle = \langle d_2\rangle &= 1\\
\langle d_3\rangle = \langle d_4\rangle &= \tfrac{3}{2}\\
\langle d_5\rangle = \langle d_6\rangle &= \tfrac{15}{8}\\
\langle d_7\rangle = \langle d_8\rangle &= \tfrac{35}{16}\\
\langle d_9\rangle = \langle d_{10}\rangle &= \tfrac{315}{128}\\
\langle d_{11}\rangle = \langle d_{12}\rangle &= \tfrac{693}{256}\\
\langle d_{13}\rangle = \langle d_{14}\rangle &= \tfrac{3003}{1024}.
\end{aligned}$$

Now, examine the asymptotic behavior of $\langle d_N\rangle$. The asymptotic expansion of the GAMMA FUNCTION ratio is

$$\frac{\Gamma(J+\frac{1}{2})}{\Gamma(J)} = \sqrt{J}\left(1-\frac{1}{8J}+\frac{1}{128J^2}+\ldots\right) \qquad (28)$$

(Graham *et al.* 1994), so plugging in the expression for $\langle d_N\rangle$ gives the asymptotic series

$$\langle d_N\rangle = \sqrt{\frac{2N}{\pi}}\left(1\mp\frac{1}{4N}+\frac{1}{32N^2}\pm\frac{5}{128N^3}-\frac{21}{2048N^4}\mp\ldots\right), \qquad (29)$$

where the top signs are taken for N EVEN and the bottom signs for N ODD. Therefore, for large N,

$$\langle d_N\rangle \sim \sqrt{\frac{2N}{\pi}}, \qquad (30)$$

which is also shown in Mosteller *et al.* (1961, p. 14).

see also BINOMIAL DISTRIBUTION, CATALAN NUMBER, p-GOOD PATH, PÓLYA'S RANDOM WALK CONSTANTS, RANDOM WALK—2-D, RANDOM WALK—3-D, SELF-AVOIDING WALK

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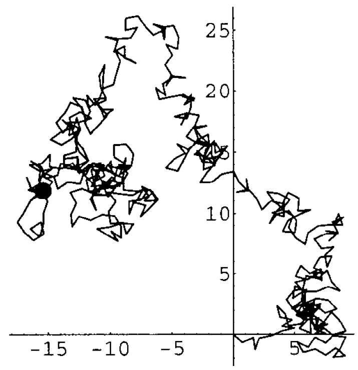
Random Walk—2-D



In a PLANE, consider a sum of N 2-D VECTORS with random orientations. Use PHASOR notation, and let the phase of each VECTOR be RANDOM. Assume N unit steps are taken in an arbitrary direction (i.e., with the angle θ uniformly distributed in $[0, 2\pi)$ and *not* on a LATTICE), as illustrated above. The position z in the COMPLEX PLANE after N steps is then given by

$$z = \sum_{j=1}^{N} e^{i\theta_j}, \tag{1}$$

which has ABSOLUTE SQUARE

$$\begin{aligned} |z|^2 &= \sum_{j=1}^{N} e^{i\theta_j} \sum_{k=1}^{N} e^{-i\theta_k} = \sum_{j=1}^{N} \sum_{k=1}^{N} e^{i(\theta_j - \theta_k)} \\ &= N + \sum_{\substack{j,k=1 \\ k \neq j}}^{N} e^{i(\theta_j - \theta_k)}. \end{aligned} \tag{2}$$

Therefore,

$$\left\langle |z|^2 \right\rangle = N + \left\langle \sum_{\substack{j,k=1 \\ k \neq j}}^{N} e^{i(\theta_j - \theta_k)} \right\rangle. \tag{3}$$

Each step is likely to be in any direction, so both θ_j and θ_k are RANDOM VARIABLES with identical MEANS of zero, and their difference is also a random variable. Averaging over this distribution, which has equally likely POSITIVE and NEGATIVE values yields an expectation value of 0, so

$$\left\langle |z|^2 \right\rangle = N. \tag{4}$$

The root-mean-square distance after N unit steps is therefore

$$|z|_{\text{rms}} = \sqrt{N}, \tag{5}$$

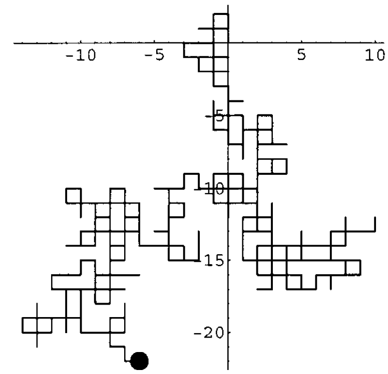
so with a step size of l, this becomes

$$d_{\text{rms}} = l\sqrt{N}. \tag{6}$$

In order to travel a distance d

$$N \approx \left(\frac{d}{l}\right)^2 \tag{7}$$

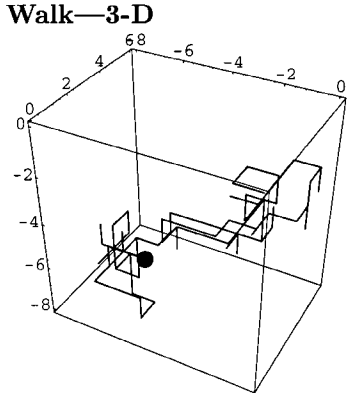
steps are therefore required.



Amazingly, it has been proven that on a 2-D LATTICE, a random walk has unity probability of reaching any point (including the starting point) as the number of steps approaches INFINITY.

see also PÓLYA'S RANDOM WALK CONSTANTS, RANDOM WALK—1-D, RANDOM WALK—3-D

Random Walk—3-D



On a 3-D LATTICE, a random walk has *less than* unity probability of reaching any point (including the starting point) as the number of steps approaches infinity. The probability of reaching the starting point again is 0.3405373296.... This is one of PÓLYA'S RANDOM WALK CONSTANTS.

see also PÓLYA'S RANDOM WALK CONSTANTS, RANDOM WALK—1-D, RANDOM WALK—2-D

Range (Image)

If T is MAP over a DOMAIN D, then the range of T is defined as

$$\text{Range}(T) = T(D) = \{T(\mathbf{X}) : \mathbf{X} \in D\}.$$

The range $T(D)$ is also called the IMAGE of D under T.

see also DOMAIN, MAP

Range (Line Segment)

The set of all points on a LINE SEGMENT, also called a PENCIL.

see also PERSPECTIVITY, SECTION (PENCIL)

References

Woods, F. S. *Higher Geometry: An Introduction to Advanced Methods in Analytic Geometry.* New York: Dover, p. 8, 1961.

Range (Statistics)

$$R \equiv \max(x_i) - \min(x_i). \tag{1}$$

For small samples, the range is a good estimator of the population STANDARD DEVIATION (Kenney and Keeping 1962, pp. 213–214). For a continuous UNIFORM DISTRIBUTION

$$P(x) = \begin{cases} \frac{1}{C} & \text{for } 0 < x < C \\ 0 & \text{for } |x| < C, \end{cases} \tag{2}$$

the distribution of the range is given by

$$D(R) = N\left(\frac{R}{C}\right)^{N-1} - (N-1)\left(\frac{R}{C}\right)^{N}. \tag{3}$$

Given two samples with sizes m and n and ranges R_1 and R_2, let $u \equiv R_1/R_2$. Then

$$D(u) = \begin{cases} \frac{m(m-1)n(n-1)}{(m+n)(m+n-1)(m+n-2)} \\ \times[(m+n)u^{m-2} - (m+n-2)u^{m-1}] \\ \qquad \text{for } 0 \leq u \leq 1 \\ \frac{m(m-1)n(n-1)}{(m+n)(m+n-1)(m+n-2)} \\ \times[(m+n)u^{-n} - (m+n-2)u^{-n-1}] \\ \qquad \text{for } 1 \leq u < \infty. \end{cases} \tag{4}$$

The MEAN is

$$\mu_u = \frac{(m-1)n}{(m+1)(n-2)}, \tag{5}$$

and the MODE is

$$\hat{u} = \begin{cases} \frac{(m-2)(m+n)}{(m-1)(m+n-2)} & \text{for } m-n \leq 2 \\ \frac{(n+1)(m+n-2)}{n(m+n)} & \text{for } m-n \geq 2. \end{cases} \tag{6}$$

References

Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics, Pt. 1, 3rd ed.* Princeton, NJ: Van Nostrand, pp. 213–214, 1962.

Rank

In a total generality, the "rank" of a mathematical object is defined whenever that object is FREE. In general, the rank of a FREE object is the CARDINALITY of the FREE generating SUBSET G. The word "rank" also refers to several unrelated concepts in mathematics involving groups, quadratic forms, sequences, statistics, and tensors.

see also RANK (GROUP), RANK (QUADRATIC FORM), RANK (SEQUENCE), RANK (STATISTICS), RANK (TENSOR)

Rank (Group)

For an arbitrary finitely generated ABELIAN GROUP G, the rank of G is defined to be the rank of the FREE generating SUBSET G modulo its TORSION SUBGROUP. For a finitely generated GROUP, the rank is defined to be the rank of its "Abelianization."

see also ABELIAN GROUP, BETTI NUMBER, BURNSIDE PROBLEM, QUASITHIN THEOREM, QUASI-UNIPOTENT GROUP, TORSION (GROUP THEORY)

Rank (Quadratic Form)

For a QUADRATIC FORM Q in the canonical form

$$Q = {y_1}^2 + {y_2}^2 + \ldots + {y_p}^2 - {y_{p+1}}^2 - {y_{p+2}}^2 - \ldots - {y_r}^2,$$

the rank is the total number r of square terms (both POSITIVE and NEGATIVE).

see also SIGNATURE (QUADRATIC FORM)

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1105, 1979.

Rank (Sequence)

The position of a RATIONAL NUMBER in the SEQUENCE 1, $\frac{1}{2}$, 2, $\frac{1}{3}$, 3, $\frac{1}{4}$, $\frac{2}{3}$, $\frac{3}{2}$, 4, $\frac{1}{5}$, ..., ordered in terms of increasing NUMERATOR+DENOMINATOR.

see also ENCODING, FAREY SERIES

Rank (Statistics)

The ORDINAL NUMBER of a value in a list arranged in a specified order (usually decreasing).

see also SPEARMAN RANK CORRELATION, WILCOXON RANK SUM TEST, WILCOXON SIGNED RANK TEST, ZIPF'S LAW

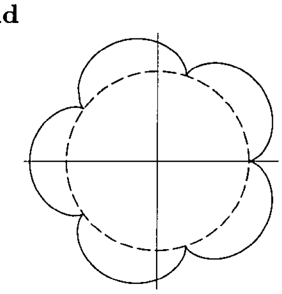
Rank (Tensor)

The total number of CONTRAVARIANT and COVARIANT indices of a TENSOR. The rank of a TENSOR is independent of the number of DIMENSIONS of the SPACE.

Rank	Object
0	scalar
1	vector
≥ 2	tensor

see also CONTRAVARIANT TENSOR, COVARIANT TENSOR, SCALAR, TENSOR, VECTOR

Ranunculoid



An EPICYCLOID with $n = 5$ cusps, named after the buttercup genus *Ranunculus* (Madachy 1979).

see also EPICYCLOID.

References

Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, p. 223, 1979.

Pickover, C. A. *Keys to Infinity.* New York: Wiley, pp. 79–80, 1995.

RAT-Free Set

A RAT-free set is a set of points, no three of which determine a RIGHT TRIANGLE. Let $f(n)$ be the smallest RAT-free subset guaranteed to be contained in a planar set of n points, then the function $f(n)$ is bounded by

$$\sqrt{n} \leq f(n) \leq 2\sqrt{n}.$$

References

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Ratio

The ratio of two numbers r and s is written r/s, where r is the NUMERATOR and s is the DENOMINATOR. The ratio of r to s is equivalent to the QUOTIENT r/s. Betting ODDS written as $r : s$ correspond to $s/(r+s)$. A number which can be expressed as a ratio of INTEGERS is called a RATIONAL NUMBER.

see also DENOMINATOR, FRACTION, NUMERATOR, ODDS, QUOTIENT, RATIONAL NUMBER

Ratio Distribution

Given two distributions Y and X with joint probability density function $f(x,y)$, let $U = Y/X$ be the ratio distribution. Then the distribution function of u is

$$\begin{aligned} D(u) &= P(U \leq u) \\ &= P(Y \leq uX \mid X > 0) + P(Y \geq uX \mid X < 0) \\ &= \int_0^\infty \int_0^{ux} f(x,y)\,dy\,dx + \int_{-\infty}^0 \int_{ux}^0 f(x,y)\,dy\,dx. \end{aligned} \tag{1}$$

The probability function is then

$$\begin{aligned} P(u) = D'(u) &= \int_0^\infty x f(x,ux)\,dx - \int_{-\infty}^0 x f(x,ux)\,dx \\ &= \int_{-\infty}^\infty |x| f(x,ux)\,dx. \end{aligned} \tag{2}$$

For variates with a standard NORMAL DISTRIBUTION, the ratio distribution is a CAUCHY DISTRIBUTION. For a UNIFORM DISTRIBUTION

$$f(x,y) = \begin{cases} 1 & \text{for } x, y \in [0,1] \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

$$P(u) = \begin{cases} 0 & u < 0 \\ \int_0^1 x\,dx = [\tfrac{1}{2}x^2] = \tfrac{1}{2} & \text{for } 0 \leq u \leq 1 \\ \int_0^{1/u} x\,dx = [\tfrac{1}{2}x^2]_0^{1/u} = \tfrac{1}{2u^2} & \text{for } u > 1. \end{cases} \tag{4}$$

see also CAUCHY DISTRIBUTION

Ratio Test

Let u_k be a SERIES with POSITIVE terms and suppose

$$\rho \equiv \lim_{k \to \infty} \frac{u_{k+1}}{u_k}.$$

Then

1. If $\rho < 1$, the SERIES CONVERGES.
2. If $\rho > 1$ or $\rho = \infty$, the SERIES DIVERGES.
3. If $\rho = 1$, the SERIES may CONVERGE or DIVERGE.

The test is also called the CAUCHY RATIO TEST or D'ALEMBERT RATIO TEST.

see also CONVERGENCE TESTS

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Rational Approximation

If r is any number and n is any INTEGER, then there is a RATIONAL NUMBER m/n for which

$$0 \leq r - \frac{m}{n} < \frac{1}{n}. \tag{1}$$

If r is IRRATIONAL and k is any WHOLE NUMBER, there is a FRACTION m/n with $n \leq k$ and for which

$$0 \leq r - \frac{m}{n} < \frac{1}{nk}. \tag{2}$$

Furthermore, there are an infinite number of FRACTIONS m/n for which

$$0 \leq r - \frac{m}{n} < \frac{1}{n^2}. \tag{3}$$

Hurwitz has shown that for an IRRATIONAL NUMBER ζ

$$\left|\zeta - \frac{h}{k}\right| < \frac{1}{ck^2}, \tag{4}$$

there are infinitely RATIONAL NUMBERS h/k if $0 < c \leq \sqrt{5}$, but if $c > \sqrt{5}$, there are some ζ for which this approximation holds for only finitely many h/k.

Rational Canonical Form

There is an invertible matrix Q such that

$$\mathsf{Q}^{-1}\mathsf{T}\mathsf{Q} = \operatorname{diag}[L(\psi_1), L(\psi_2), \ldots, L(\psi_s)],$$

where $L(f)$ is the companion MATRIX for any MONIC POLYNOMIAL

$$f(\lambda) = f_0 + f_1\lambda + \ldots + f_n\lambda^n$$

with $f_n = 1$. The POLYNOMIALS ψ_i are called the "invariant factors" of T, and satisfy $\psi_{i+1}|\psi_i$ for $i = s-1$, ..., 1 (Hartwig 1996).

References
Gantmacher, F. R. *The Theory of Matrices, Vol. 1.* New York: Chelsea, 1960.
Hartwig, R. E. "Roth's Removal Rule and the Rational Canonical Form." *Amer. Math. Monthly* **103**, 332–335, 1996.
Herstein, I. N. *Topics in Algebra, 2nd ed.* New York: Springer-Verlag, p. 162, 1975.
Hoffman, K. and Kunze, K. *Linear Algebra, 3rd ed.* Englewood Cliffs, NJ: Prentice-Hall, 1996.
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Turnbull, H. W. and Aitken, A. C. *An Introduction to the Theory of Canonical Matrices, 2nd impression.* New York: Blackie and Sons, 1945.

Rational Cuboid

see EULER BRICK

Rational Distances

It is possible to find six points in the PLANE, no three on a LINE and no four on a CIRCLE (i.e., none of which are COLLINEAR or CONCYCLIC), such that all the mutual distances are RATIONAL. An example is illustrated by Guy (1994, p. 185).

It is not known if a TRIANGLE with INTEGER sides, MEDIANS, and AREA exists (although there are incorrect PROOFS of the impossibility in the literature). However, R. L. Rathbun, A. Kemnitz, and R. H. Buchholz have showed that there are infinitely many triangles with RATIONAL sides (HERONIAN TRIANGLES) with *two* RATIONAL MEDIANS (Guy 1994, p. 188).

see also COLLINEAR, CONCYCLIC, CYCLIC QUADRILATERAL, EQUILATERAL TRIANGLE, EULER BRICK, HERONIAN TRIANGLE, RATIONAL QUADRILATERAL, RATIONAL TRIANGLE, SQUARE, TRIANGLE

References
Guy, R. K. "Six General Points at Rational Distances" and "Triangles with Integer Sides, Medians, and Area." §D20 and D21 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 185–190 and 188–190, 1994.

Rational Domain

see FIELD

Rational Double Point

There are nine possible types of ISOLATED SINGULARITIES on a CUBIC SURFACE, eight of them rational double points. Each type of ISOLATED SINGULARITY has an associated normal form and COXETER-DYNKIN DIAGRAM (A_1, A_2, A_3, A_4, A_5, D_4, D_5, E_6 and $\tilde{E}_6$).

The eight types of rational double points (the $\tilde{E}_6$ type being the one excluded) can occur in only 20 combinations on a CUBIC SURFACE (of which Fischer 1986 gives 19): A_1, $2A_1$, $3A_1$, $4A_1$, A_2, (A_2, A_1), $2A_2$, $(2A_2, A_1)$, $3A_2$, A_3, (A_3, A_1), $(A_3, 2A_1)$, A_4, (A_4, A_1), A_5, (A_5, A_1), D_4, D_5, and E_6 (Looijenga 1978, Bruce and Wall 1979, Fischer 1986).

In particular, on a CUBIC SURFACE, precisely those configurations of rational double points occur for which the disjoint union of the COXETER-DYNKIN DIAGRAM is a SUBGRAPH of the COXETER-DYNKIN DIAGRAM $\tilde{E}_6$. Also, a surface specializes to a more complicated one precisely when its graph is contained in the graph of the other one (Fischer 1986).

see also COXETER-DYNKIN DIAGRAM, CUBIC SURFACE, ISOLATED SINGULARITY

References
Bruce, J. and Wall, C. T. C. "On the Classification of Cubic Surfaces." *J. London Math. Soc.* **19**, 245–256, 1979.
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Rational Function

A quotient of two polynomials $P(z)$ and $Q(z)$,

$$R(z) \equiv \frac{P(z)}{Q(z)},$$

is called a rational function. More generally, if P and Q are POLYNOMIALS in multiple variables, their quotient is a rational function.

see also ABEL'S CURVE THEOREM, CLOSED FORM, FUNDAMENTAL THEOREM OF SYMMETRIC FUNCTIONS, QUOTIENT-DIFFERENCE ALGORITHM, RATIONAL INTEGER, RATIONAL NUMBER, RIEMANN CURVE THEOREM

Rational Integer

A synonym for INTEGER. The word "rational" is sometimes used for emphasis to distinguish it from other types of "integers" such as CYCLOTOMIC INTEGERS, EISENSTEIN INTEGERS, and GAUSSIAN INTEGERS.

see also CYCLOTOMIC INTEGER, EISENSTEIN INTEGER, GAUSSIAN INTEGER, INTEGER, RATIONAL NUMBER

References
Hardy, G. H. and Wright, E. M. *An Introduction to the Theory of Numbers, 5th ed.* Oxford, England: Clarendon Press, p. 1, 1979.

Rational Number

A number that can be expressed as a FRACTION p/q where p and q are INTEGERS, is called a rational number with NUMERATOR p and DENOMINATOR q. Numbers which are not rational are called IRRATIONAL NUMBERS. Any rational number is trivially also an ALGEBRAIC NUMBER.

For a, b, and c any different rational numbers, then

$$\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}$$

is the SQUARE of a rational number (Honsberger 1991). The probability that a random rational number has an EVEN DENOMINATOR is 1/3 (Beeler *et al.* 1972, Item 54).

see also ALGEBRAIC INTEGER, ALGEBRAIC NUMBER, ANOMALOUS CANCELLATION, DENOMINATOR, DIRICHLET FUNCTION, FRACTION, INTEGER, IRRATIONAL NUMBER, NUMERATOR, QUOTIENT, TRANSCENDENTAL NUMBER

References
Beeler, M.; Gosper, R. W.; and Schroeppel, R. *HAKMEM.* Cambridge, MA: MIT Artificial Intelligence Laboratory, Memo AIM-239, Feb. 1972.
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Rational Point

A K-rational point is a point (X, Y) on an ALGEBRAIC CURVE, where X and Y are in a FIELD K.

The rational point may also be a POINT AT INFINITY. For example, take the ELLIPTIC CURVE

$$Y^2 = X^3 + X + 42$$

and homogenize it by introducing a third variable Z so that each term has degree 3 as follows:

$$ZY^2 = X^3 + XZ^2 + 42Z^3.$$

Now, find the points at infinity by setting $Z = 0$, obtaining

$$0 = X^3.$$

Solving gives $X = 0$, Y equal to any value, and (by definition) $Z = 0$. Despite freedom in the choice of Y, there is only a single POINT AT INFINITY because the two triples (X_1, Y_1, Z_1), (X_2, Y_2, Z_2) are considered to be equivalent (or identified) only if one is a scalar multiple of the other. Here, (0, 0, 0) is not considered to be a valid point. The triples (a, b, 1) correspond to the ordinary points (a, b), and the triples (a, b, 0) correspond to the POINTS AT INFINITY, usually called the LINE AT INFINITY.

The rational points on ELLIPTIC CURVES over the GALOIS FIELD GF(q) are 5, 7, 9, 10, 13, 14, 16, ... (Sloane's A005523).

see also ELLIPTIC CURVE, LINE AT INFINITY, POINT AT INFINITY

References
Sloane, N. J. A. Sequence A005523/M3757 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Rational Quadrilateral

A rational quadrilateral is a QUADRILATERAL for which the sides, DIAGONALS, and AREA are RATIONAL. The simplest case has sides $a = 52$, $b = 25$, $c = 39$, and $d = 60$ and DIAGONALS of length $p = 63$ and $q = 56$.

see also AREA, DIAGONAL (POLYGON), RATIONAL QUADRILATERAL

Rational Triangle

A rational triangle is a TRIANGLE all of whose sides are RATIONAL NUMBERS and all of whose ANGLES are RATIONAL numbers of DEGREES. The only such triangle is the EQUILATERAL TRIANGLE (Conway and Guy 1996).

see also EQUILATERAL TRIANGLE, FERMAT'S RIGHT TRIANGLE THEOREM, RIGHT TRIANGLE

References
Conway, J. H. and Guy, R. K. "The Only Rational Triangle." In *The Book of Numbers.* New York: Springer-Verlag, pp. 201 and 228–239, 1996.

RATS Sequence

A sequence produced by the instructions "reverse, add, then sort the digits," where zeros are suppressed. For example, after 668 we get

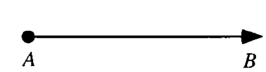
$$668 + 866 = 1534,$$

so the next term is 1345. Applied to 1, the sequence gives 1, 2, 4, 8, 16, 77, 145, 668, 1345, 6677, 13444, 55778, ... (Sloane's A004000)

see also 196-ALGORITHM, KAPREKAR ROUTINE, REVERSAL, SORT-THEN-ADD SEQUENCE

References
Sloane, N. J. A. Sequence A004000/M1137 in "An On-Line Version of the Encyclopedia of Integer Sequences."

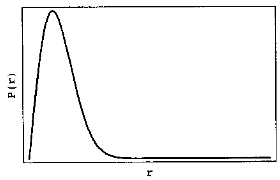
Ray

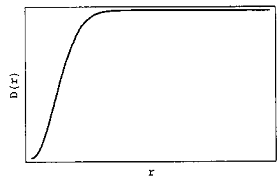


A VECTOR $\overrightarrow{AB}$ from a point A to a point B. In GEOMETRY, a ray is usually taken as a half-infinite LINE with one of the two points A and B taken to be at INFINITY.

see also LINE, VECTOR

Rayleigh Distribution





The distribution with PROBABILITY FUNCTION

$$P(r) = \frac{re^{-r^2/2s^2}}{s^2} \tag{1}$$

for $r \in [0, \infty)$. The MOMENTS about 0 are given by

$$\begin{aligned} \mu'_m &\equiv \int_0^\infty r^m P(r)\,dr = s^{-2} \int_0^\infty r^{m+1} e^{-r^2/2s^2}\,dr \\ &= s^{-2} I_{m+1}\left(\frac{1}{2s^2}\right), \end{aligned} \tag{2}$$

where $I(x)$ is a GAUSSIAN INTEGRAL. The first few of these are

$$I_1(a^{-1}) = \tfrac{1}{2}a \tag{3}$$
$$I_2(a^{-1}) = \tfrac{1}{4}a\sqrt{a\pi} \tag{4}$$
$$I_3(a^{-1}) = \tfrac{1}{2}a^2 \tag{5}$$
$$I_4(a^{-1}) = \tfrac{3}{8}a^2\sqrt{a\pi} \tag{6}$$
$$I_5(a^{-1}) = a^3, \tag{7}$$

so

$$\mu'_0 = s^{-2}\tfrac{1}{2}(2s^2) = 1 \tag{8}$$
$$\mu'_1 = s^{-2}\tfrac{1}{4}(2s^2)\sqrt{2s^2\pi} = \tfrac{1}{2}s\sqrt{2\pi} = s\sqrt{\frac{\pi}{2}} \tag{9}$$
$$\mu'_2 = s^{-2}\tfrac{1}{2}(2s^2)^2 = 2s^2 \tag{10}$$
$$\mu'_3 = s^{-2}\tfrac{3}{8}(2s^2)^2\sqrt{2s^2\pi} = \tfrac{3}{2}s^3\sqrt{2\pi} = 3s^3\sqrt{\frac{\pi}{2}} \tag{11}$$
$$\mu'_4 = s^{-2}(2s^2)^3 = 8s^4. \tag{12}$$

The MOMENTS about the MEAN are

$$\mu_2 = \mu'_2 - (\mu'_1)^2 = \frac{4-\pi}{2}s^2 \tag{13}$$
$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3 = \sqrt{\frac{\pi}{2}}(\pi-3)s^3 \tag{14}$$
$$\begin{aligned} \mu_4 &= \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu-1')^4 \\ &= \frac{32-3\pi^2}{4}s^4, \end{aligned} \tag{15}$$

so the MEAN, VARIANCE, SKEWNESS, and KURTOSIS are

$$\mu = \mu'_1 = s\sqrt{\frac{\pi}{2}} \tag{16}$$
$$\sigma^2 = \mu_2 = \frac{4-\pi}{2}s^2 \tag{17}$$
$$\gamma_1 = \frac{\mu_3}{\sigma^3} = \frac{2(\pi-3)\sqrt{\pi}}{(4-\pi)^{3/2}} \tag{18}$$
$$\gamma_2 = \frac{\mu_4}{\sigma^4} - 3 = \frac{2(-3\pi^2+12\pi-8)}{(\pi-4)^2}. \tag{19}$$

Rayleigh Differential Equation

$$y'' - \mu(1 - \tfrac{1}{3}y'^2)y' + y = 0,$$

where $\mu > 0$. Differentiating and setting $y = y'$ gives the VAN DER POL EQUATION.

see also VAN DER POL EQUATION

Rayleigh's Formulas

The formulas

$$j_n(z) = z^n \left(-\frac{1}{z}\frac{d}{dz}\right)^n \frac{\sin z}{z}$$
$$y_n(z) = -z^n \left(-\frac{1}{z}\frac{d}{dz}\right)^n \frac{\cos z}{z}$$

for $n = 0, 1, 2, \ldots$, where $j_n(z)$ is a SPHERICAL BESSEL FUNCTION OF THE FIRST KIND and $y_n(z)$ is a SPHERICAL BESSEL FUNCTION OF THE SECOND KIND.

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 439, 1972.

Rayleigh-Ritz Variational Technique

A technique for computing EIGENFUNCTIONS and EIGENVALUES. It proceeds by requiring

$$J = \int_a^b [p(x){y_x}^2 - q(x)y^2]\,dx \tag{1}$$

to have a STATIONARY VALUE subject to the normalization condition

$$\int_a^b y^2 w(x)\,dx = 1 \tag{2}$$

and the boundary conditions

$$py_x y|_a^b = 0. \tag{3}$$

This leads to the STURM-LIOUVILLE EQUATION

$$\frac{d}{dx}\left(p\frac{dy}{dx}\right) + qy + \lambda wy = 0, \tag{4}$$

which gives the stationary values of

$$F[y(x)] = \frac{\int_a^b (p{y_x}^2 - qy^2)\,dx}{\int_a^b y^2 w\,dx} \tag{5}$$

as

$$F[y_n(x)] = \lambda_n, \tag{6}$$

where λ_n are the EIGENVALUES corresponding to the EIGENFUNCTION y_n.

References

Arfken, G. "Rayleigh-Ritz Variational Technique." §17.8 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 957–961, 1985.

Rayleigh's Theorem

see PARSEVAL'S THEOREM

Re-Entrant Circuit

A CYCLE in a GRAPH which terminates at the starting point.

see also CYCLE (GRAPH), EULERIAN CIRCUIT, HAMILTONIAN CYCLE

Real Analysis

That portion of mathematics dealing with functions of real variables. While this includes some portions of TOPOLOGY, it is most commonly used to distinguish that portion of CALCULUS dealing with real as opposed to COMPLEX NUMBERS.

Real Axis

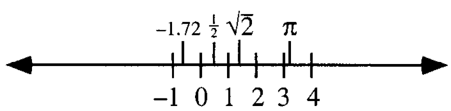
see REAL LINE

Real Function

A FUNCTION whose RANGE is in the REAL NUMBERS is said to be a real function.

see also COMPLEX FUNCTION, SCALAR FUNCTION, VECTOR FUNCTION

Real Line



A LINE with a fixed scale so that every REAL NUMBER corresponds to a unique POINT on the LINE. The generalization of the real line to 2-D is called the COMPLEX PLANE.

see also ABSCISSA, COMPLEX PLANE

References

Courant, R. and Robbins, H. *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, p. 57, 1996.

Real Matrix

A MATRIX whose elements consist entirely of REAL NUMBERS.

Real Number

The set of all RATIONAL and IRRATIONAL numbers is called the real numbers, or simply the "reals," and denoted $\mathbb{R}$. The set of real numbers is also called the CONTINUUM, denoted C.

The real numbers can be extended with the addition of the IMAGINARY NUMBER i, equal to $\sqrt{-1}$. Numbers of the form $x+iy$, where x and y are both real, are then called COMPLEX NUMBERS. Another extension which includes both the real numbers and the infinite ORDINAL NUMBERS of Georg Cantor is the SURREAL NUMBERS.

Pick two real numbers x and y at random in $(0,1)$ with a UNIFORM DISTRIBUTION. What is the PROBABILITY P_{even} that $[x/y]$, where $[r]$ denotes NINT, the nearest INTEGER to r, is EVEN? The answer may be found as follows (Putnam Exam).

$$P\left(a<\frac{x}{y}<b\right)=\begin{cases}P(ay<x<by)\\ P\left(\frac{x}{b}<y<\frac{x}{a}\right)\end{cases}$$
$$=\begin{cases}\int_0^1\int_{ay}^{by}dx\,dy=\frac{1}{2}(b-a) & \text{for } 0\le a<b<1\\ \int_0^1\int_{x/b}^{x/a}dy\,dx=\frac{1}{2a}-\frac{1}{2b} & \text{for } 1<a<b\end{cases}\tag{1}$$

$$\begin{aligned}P_{\text{even}}&=P\left(0<\frac{x}{y}<\tfrac{1}{2}\right)+\sum_{n=1}^{\infty}P\left(2n-\tfrac{1}{2}<\frac{x}{y}<2n+\tfrac{1}{2}\right)\\&=\tfrac{1}{2}(\tfrac{1}{2}-0)+\sum_{n=1}^{\infty}\left[\frac{1}{2(2n-\frac{1}{2})}-\frac{1}{2(2n+\frac{1}{2})}\right]\\&=\frac{1}{4}+\sum_{n=1}^{\infty}\left(\frac{1}{4n-1}+\frac{1}{4n-1}\right)\\&=\tfrac{1}{4}+(\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{9}+\ldots)=\tfrac{1}{4}+(1-\tan^{-1}1)\\&=\frac{5}{4}-\frac{\pi}{4}=\tfrac{1}{4}(5-\pi)\approx 46.460\%.\end{aligned}\tag{2}$$

Plouffe's "Inverse Symbolic Calculator" includes a huge database of 54 million real numbers which are algebraically related to fundamental mathematical constants and functions.

see also COMPLEX NUMBER, CONTINUUM, i, IMAGINARY NUMBER, INTEGER RELATION, RATIONAL NUMBER, REAL PART, SURREAL NUMBER

References

Plouffe, S. "Inverse Symbolic Calculator." `http://www.cecm.sfu.ca/projects/ISC/`.
Plouffe, S. "Plouffe's Inverter." `http://www.lacim.uqam.ca/pi/`.
Putnam Exam. Problem B-3 in the 54th Putnam Exam.

Real Part

The real part $\Re$ of a COMPLEX NUMBER $z=x+iy$ is the REAL NUMBER *not* multiplying i, so $\Re[x+iy]=x$. In terms of z itself,

$$\Re[z]=\tfrac{1}{2}(z+z^*),$$

where z^* is the COMPLEX CONJUGATE of z.

see also ABSOLUTE SQUARE, COMPLEX CONJUGATE, IMAGINARY PART

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 16, 1972.

Real Polynomial

A POLYNOMIAL having only REAL NUMBERS as COEFFICIENTS.

see also POLYNOMIAL

Real Projective Plane

The closed topological MANIFOLD, denoted $\mathbb{R}P^2$, which is obtained by projecting the points of a plane E from a fixed point P (not on the plane), with the addition of the LINE AT INFINITY, is called the real projective plane. There is then a one-to-one correspondence between points in E and lines through P. Since each line through P intersects the sphere $\mathbb{S}^2$ centered at P and tangent to E in two ANTIPODAL POINTS, $\mathbb{R}P^2$ can be described as a QUOTIENT SPACE of $\mathbb{S}^2$ by identifying any two such points. The real projective plane is a NONORIENTABLE SURFACE.

The BOY SURFACE, CROSS-CAP, and ROMAN SURFACE are all homeomorphic to the real projective plane and, because $\mathbb{R}P^2$ is nonorientable, these surfaces contain self-intersections (Kuiper 1961, Pinkall 1986).

see also BOY SURFACE, CROSS-CAP, NONORIENTABLE SURFACE, PROJECTIVE PLANE, ROMAN SURFACE

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Geometry Center. "The Projective Plane." http://www.geom.umn.edu/zoo/toptype/pplane/.

Gray, A. "Realizations of the Real Projective Plane." §12.5 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 241–245, 1993.

Klein, F. §1.2 in *Vorlesungen über nicht-euklidische Geometrie.* Berlin, 1928.

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Pinkall, U. *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, pp. 64–65, 1986.

Real Quadratic Field

A QUADRATIC FIELD $\mathbb{Q}(\sqrt{D})$ with $D > 0$.

see also QUADRATIC FIELD

Realizer

A SET of R of LINEAR EXTENSIONS of a POSET $P = (X, \leq)$ is a realizer of P (and is said to realize P) provided that for all $x, y \in X$, $x \leq y$ IFF x is below y in every member of R.

see also DOMINANCE, LINEAR EXTENSION, PARTIALLY ORDERED SET, POSET DIMENSION

Rearrangement Theorem

Each row and each column in the GROUP multiplication table lists each of the GROUP elements once and only once. From this, it follows that no two elements may be in the identical location in two rows or two columns. Thus, each row and each column is a rearranged list of the GROUP elements. Stated otherwise, given a GROUP of n distinct elements $(I, a, b, c, \ldots, n)$, the set of products $(aI, a^2, ab, ac, \ldots, an)$ reproduces the n original distinct elements in a new order.

see also GROUP

Reciprocal

The reciprocal of a REAL or COMPLEX NUMBER z is its MULTIPLICATIVE INVERSE $1/z$. The reciprocal of a COMPLEX NUMBER $z = x + iy$ is given by

$$\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i.$$

Reciprocal Difference

The reciprocal differences are closely related to the DIVIDED DIFFERENCE. The first few are explicitly given by

$$\rho(x_0, x_1) = \frac{x_0 - x_1}{f_0 - f_1} \tag{1}$$

$$\rho_2(x_0, x_1, x_2) = \frac{x_0 - x_2}{\rho(x_0, x_1) - \rho(x_1, x_2)} + f_1 \tag{2}$$

$$\rho_3(x_0, x_1, x_2, x_3) = \frac{x_0 - x_3}{\rho_2(x_0, x_1, x_2) - \rho_2(x_1, x_2, x_3)} + \rho(x_1, x_2) \tag{3}$$

$$\rho_n(x_0, x_1, \ldots, x_n) = \frac{x_0 - x_n}{\rho_{n-1}(x_0, \ldots, x_{n-1}) - \rho_{n-1}(x_1, \ldots, x_n)} + \rho_{n-2}(x_1, \ldots, x_{n-1}). \tag{4}$$

see also BACKWARD DIFFERENCE, CENTRAL DIFFERENCE, DIVIDED DIFFERENCE, FINITE DIFFERENCE, FORWARD DIFFERENCE

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 878, 1972.

Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 443, 1987.

Reciprocal Polyhedron

see DUAL POLYHEDRON

Reciprocating Sphere

see MIDSPHERE

Reciprocation

An incidence-preserving transformation in which points and lines are transformed into their poles and polars. A PROJECTIVE GEOMETRY-like DUALITY PRINCIPLE holds for reciprocation.

References

Coxeter, H. S. M. and Greitzer, S. L. "Reciprocation." §6.1 in *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 132–136, 1967.

Reciprocity Theorem

If there exists a RATIONAL INTEGER x such that, when n, p, and q are POSITIVE INTEGERS,

$$x^n \equiv q \pmod{p},$$

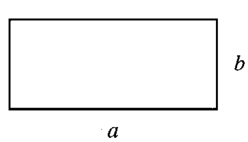
then q is the n-adic reside of p, i.e., q is an n-adic residue of p IFF $x^n \equiv q \pmod{p}$ is solvable for x.

The first case to be considered was $n = 2$ (the QUADRATIC RECIPROCITY THEOREM), of which Gauss gave the first correct proof. Gauss also solved the case $n = 3$ (CUBIC RECIPROCITY THEOREM) using INTEGERS of the form $a + b\rho$, when ρ is a root if $x^2 + x + 1 = 0$ and a, b are rational INTEGERS. Gauss stated the case $n = 4$ (QUARTIC RECIPROCITY THEOREM) using the GAUSSIAN INTEGERS.

Proof of n-adic reciprocity for PRIME n was given by Eisenstein in 1844–50 and by Kummer in 1850–61. In the 1920s, Artin formulated ARTIN'S RECIPROCITY THEOREM, a general reciprocity law for all orders.

see also ARTIN RECIPROCITY, CUBIC RECIPROCITY THEOREM, LANGLANDS RECIPROCITY, QUADRATIC RECIPROCITY THEOREM, QUARTIC RECIPROCITY THEOREM, ROOK RECIPROCITY THEOREM

Rectangle



A closed planar QUADRILATERAL with opposite sides of equal lengths a and b, and with four RIGHT ANGLES. The AREA of the rectangle is

$$A = ab,$$

and its DIAGONALS are of length

$$p, q = \sqrt{a^2 + b^2}.$$

A SQUARE is a degenerate rectangle with $a = b$.

see also GOLDEN RECTANGLE, PERFECT RECTANGLE, SQUARE

References

Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 122, 1987.

Eppstein, D. "Rectilinear Geometry." `http://www.ics.uci.edu/~eppstein/junkyard/rect.html`.

Rectangle Function

The rectangle function $\Pi(x)$ is a function which is 0 outside the interval $[-1, 1]$ and unity inside it. It is also called the GATE FUNCTION, PULSE FUNCTION, or WINDOW FUNCTION, and is defined by

$$\Pi(x) \equiv \begin{cases} 0 & \text{for } |x| > \frac{1}{2} \\ \frac{1}{2} & \text{for } |x| = \frac{1}{2} \\ 1 & \text{for } |x| < \frac{1}{2}. \end{cases} \tag{1}$$

The function $f(x) = h\Pi((x-c)/b)$ has height h, center c, and full-width b. Identities satisfied by the rectangle function include

$$\Pi(x) = H(x + \tfrac{1}{2}) - H(x - \tfrac{1}{2}) \tag{2}$$

$$= H(\tfrac{1}{2} + x) + H(\tfrac{1}{2} - x) - 1 \tag{3}$$

$$= H(\tfrac{1}{4} - x^2) \tag{4}$$

$$= \tfrac{1}{2}[\operatorname{sgn}(x + \tfrac{1}{2}) - \operatorname{sgn}(x - \tfrac{1}{2})], \tag{5}$$

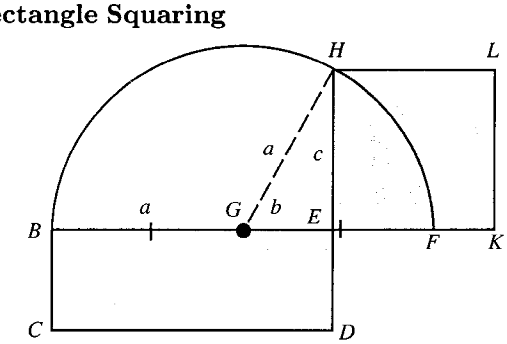
where $H(x)$ is the HEAVISIDE STEP FUNCTION. The FOURIER TRANSFORM of the rectangle function is given by

$$\mathcal{F}[\Pi(x)] = \int_{-\infty}^{\infty} e^{-2\pi i k x}\Pi(x)\,dx = \operatorname{sinc}(\pi k), \tag{6}$$

where $\operatorname{sinc}(x)$ is the SINC FUNCTION.

see also FOURIER TRANSFORM—RECTANGLE FUNCTION, HEAVISIDE STEP FUNCTION, RAMP FUNCTION

Rectangle Squaring



Given a RECTANGLE $\square BCDE$, draw $EF = DE$ on an extension of BE. Bisect BF and call the MIDPOINT G. Now draw a SEMICIRCLE centered at G, and construct the extension of ED which passes through the SEMICIRCLE at H. Then $\square EKLH$ has the same AREA as $\square BCDE$. This can be shown as follows:

$$A(\square BCDE) = BE \cdot ED = BE \cdot EF$$
$$= (a+b)(a-b) = a^2 - b^2 = c^2.$$

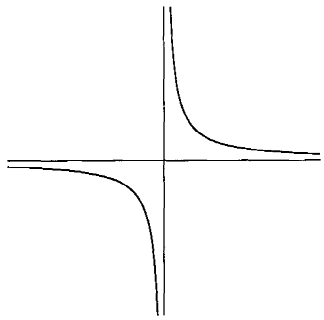
References

Dunham, W. "Hippocrates' Quadrature of the Lune." Ch. 1 in *Journey Through Genius: The Great Theorems of Mathematics.* New York: Wiley, pp. 13–14, 1990.

Rectangular Coordinates
see CARTESIAN COORDINATES

Rectangular Distribution
see UNIFORM DISTRIBUTION

Rectangular Hyperbola



A RIGHT HYPERBOLA of the special form

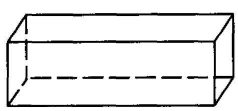
$$xy = ab,$$

so that the ASYMPTOTES are the lines $x = 0$ and $y = 0$. The rectangular hyperbola is sometimes also called an EQUILATERAL HYPERBOLA.

see also HYPERBOLA, RIGHT HYPERBOLA

References
Courant, R. and Robbins, H. *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, pp. 76–77, 1996.

Rectangular Parallelepiped



A closed box composed of 3 pairs of rectangular faces placed opposite each other and joined at RIGHT ANGLES to each other. This PARALLELEPIPED therefore corresponds to a rectangular "box." If the lengths of the sides are denoted a, b, and c, then the VOLUME is

$$V = abc, \tag{1}$$

the total SURFACE AREA is

$$A = 2(ab + bc + ca), \tag{2}$$

and the length of the "space" DIAGONAL is

$$d_{abc} = \sqrt{a^2 + b^2 + c^2}. \tag{3}$$

If $a = b = c$, then the rectangular parallelepiped is a CUBE.

see also CUBE, EULER BRICK, PARALLELEPIPED

References
Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 127, 1987.

Rectangular Projection
see EQUIRECTANGULAR PROJECTION

Rectifiable Current
The space of currents arising from rectifiable sets by integrating a differential form is called the space of 2-D rectifiable currents. For C a closed bounded rectifiable curve of a number of components in $\mathbb{R}^3$, C bounds a rectifiable current of least AREA. The theory of rectifiable currents generalizes to m-D surfaces in $\mathbb{R}^n$.

see also INTEGRAL CURRENT, REGULARITY THEOREM

References
Morgan, F. "What is a Surface?" *Amer. Math. Monthly* **103**, 369–376, 1996.

Rectifiable Set
The rectifiable sets include the image of any LIPSCHITZ FUNCTION f from planar domains into $\mathbb{R}^3$. The full set is obtained by allowing arbitrary measurable subsets of countable unions of such images of Lipschitz functions as long as the total AREA remains finite. Rectifiable sets have an "approximate" tangent plane at almost every point.

References
Morgan, F. "What is a Surface?" *Amer. Math. Monthly* **103**, 369–376, 1996.

Rectification
Rectification is the determination of the length of a curve.

see also QUADRABLE, SQUARING

Rectifying Latitude
An AUXILIARY LATITUDE which gives a sphere having correct distances along the meridians. It is denoted μ (or ω) and is given by

$$\mu = \frac{\pi M}{2M_p}. \tag{1}$$

M_p is evaluated for M at the north pole ($\phi = 90°$), and M is given by

$$\begin{aligned} M &= a(1 - e^2)\int_0^\phi \frac{d\phi}{(1 - e^2\sin^2\phi)^{3/2}} \\ &= a\left[\int_0^\phi \sqrt{1 - e^2\sin^2\phi}\,d\phi - \frac{e^2\sin\phi\cos\phi}{\sqrt{1 - e^2\sin^2\phi}}\right]. \end{aligned} \tag{2}$$

A series for M is

$$\begin{aligned} M = a[&(1 - \tfrac{1}{4}e^2 - \tfrac{3}{64}e^4 - \tfrac{5}{256}e^6 - \ldots)\phi \\ &- (\tfrac{3}{8}e^2 + \tfrac{3}{32}e^4 + \tfrac{45}{1024}e^6 + \ldots)\sin(2\phi) \\ &+ (\tfrac{15}{256}e^4 + \tfrac{45}{1024}e^6 + \ldots)\sin(4\phi) \\ &- (\tfrac{35}{3072}e^6 + \ldots)\sin(6\phi) + \ldots], \end{aligned} \tag{3}$$

and a series for μ is

$$\begin{aligned}\mu = \phi &- (\tfrac{3}{2}e_1 - \tfrac{9}{16}{e_1}^3 + \ldots)\sin(2\phi)\\ &+ (\tfrac{15}{16}{e_1}^2 - \tfrac{15}{32}{e_1}^4 + \ldots)\sin(4\phi)\\ &- (\tfrac{35}{48}{e_1}^3 - \ldots)\sin(6\phi) + (\tfrac{315}{512}{e_1}^4 - \ldots)\sin(8\phi) + \ldots,\end{aligned} \tag{4}$$

where

$$e_1 \equiv \frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}}. \tag{5}$$

The inverse formula is

$$\begin{aligned}\phi = \mu &+ (\tfrac{3}{2}e_1 - \tfrac{27}{32}{e_1}^3 + \ldots)\sin(2\mu)\\ &+ (\tfrac{21}{16}{e_1}^2 - \tfrac{55}{32}{e_1}^4 + \ldots)\sin(4\mu)\\ &+ (\tfrac{151}{96}{e_1}^3 - \ldots)\sin(6\mu)\\ &+ (\tfrac{1097}{512}{e_1}^4 - \ldots)\sin(8\mu) + \ldots.\end{aligned} \tag{6}$$

see also LATITUDE

References

Adams, O. S. "Latitude Developments Connected with Geodesy and Cartography with Tables, Including a Table for Lambert Equal-Area Meridional Projections." Spec. Pub. No. 67. U. S. Coast and Geodetic Survey, pp. 125–128, 1921.

Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, pp. 16–17, 1987.

Rectifying Plane

The PLANE spanned by the TANGENT VECTOR **T** and BINORMAL VECTOR **B** .

see also BINORMAL VECTOR, TANGENT VECTOR

Recurrence Relation

A mathematical relationship expressing f_n as some combination of f_i with $i < n$. The solutions to linear recurrence can be computed straightforwardly, but QUADRATIC RECURRENCES are not so well understood. The sequence generated by a recurrence relation is called a RECURRENCE SEQUENCE. Perhaps the most famous example of a recurrence relation is the one defining the FIBONACCI NUMBERS,

$$F_n = F_{n-2} + F_{n-1}$$

for $n \geq 3$ and with $F_1 = F_2 = 1$.

see also ARGUMENT ADDITION RELATION, ARGUMENT MULTIPLICATION RELATION, CLENSHAW RECURRENCE FORMULA, QUADRATIC RECURRENCE, RECURRENCE SEQUENCE, REFLECTION RELATION, TRANSLATION RELATION

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Recurrence Relations and Clenshaw's Recurrence Formula." §5.5 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 172–178, 1992.

Recurrence Sequence

A sequence of numbers generated by a RECURRENCE RELATION is called a recurrence sequence. Perhaps the most famous recurrence sequence is the FIBONACCI NUMBERS.

If a sequence $\{x_n\}$ with $x_1 = x_2 = 1$ is described by a two-term linear recurrence relation of the form

$$x_n = Ax_{n-1} + Bx_{n-2} \tag{1}$$

for $n \geq 3$ and A and B constants, then the closed form for x_n is given by

$$x_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{2}$$

where α and β are the ROOTS of the QUADRATIC EQUATION

$$x^2 - Ax - B = 0, \tag{3}$$

$$\alpha = \tfrac{1}{2}(A + \sqrt{A^2+4B}\,) \tag{4}$$

$$\beta = \tfrac{1}{2}(A - \sqrt{A^2+4B}\,). \tag{5}$$

The general second-order linear recurrence

$$x_n = Ax_{n-1} + Bx_{n-2} \tag{6}$$

for constants A and B with arbitrary x_1 and x_2 has terms

$$\begin{aligned}x_1 &= x_1\\ x_2 &= x_2\\ x_3 &= Bx_1 + Ax_2\\ x_4 &= Bx_2 + ABx_1 + A^2x_2\\ x_5 &= B^2x_1 + 2ABx_2 + A^2Bx_1 + A^3x_2\\ x_6 &= B^2x_2 + 2AB^2x_1 + 3A^2Bx_2 + A^3Bx_1 + A^4x_2.\end{aligned}$$

Dropping x_1, x_2, and A, this can be written

$$\begin{array}{lllll}1\\ 1\\ B & 1\\ B & B & 1\\ B^2 & 2B & B & 1\\ B^2 & 2B^2 & 3B & B & 1,\end{array}$$

which is simply PASCAL'S TRIANGLE on its side. An arbitrary term can therefore be written as

$$\begin{aligned}x_n &= \sum_{k=0}^{n-2} \binom{\lfloor \frac{1}{2}(n+k-2)\rfloor}{k} A^k B^{\lfloor (n-k-1)/2\rfloor}\\ &\qquad \times {x_1}^{[n+k \pmod 2]} {x_2}^{[n+k+1 \pmod 2]} \qquad (7)\\ &= -(Ax_1 - x_2)\sum_{k=0}^{n-2} A^{2k-n+2}B^{-k+n-2}\binom{k}{n-k-2}\\ &\qquad + x_1 \sum_{k=0}^{n-1} A^{2k-n+1}B^{-k+n-1}\binom{k}{n-k-1}. \qquad (8)\end{aligned}$$

The general linear third-order recurrence

$$x_n = Ax_{n-1} + Bx_{n-2} + Cx_{n-3} \tag{9}$$

has solution

$$\begin{aligned} x_n = x_1 &\left(\frac{\alpha^{-n}}{A+2\alpha B+3\alpha^2 C} + \frac{\beta^{-n}}{A+2\beta B+3\beta^2 C} + \frac{\gamma^{-n}}{A+2\gamma B+3\gamma^2 C} \right) \\ -(Ax_1 - x_2) &\left(\frac{\alpha^{1-n}}{A+2\alpha B+3\alpha^2 C} + \frac{\beta^{1-n}}{A+2\beta B+3\beta^2 B} + \frac{\gamma^{1-n}}{A+2\gamma C+3\gamma^2 C} \right) \\ -(Bx_1 + Ax_2 - x_3) &\left(\frac{\alpha^{2-n}}{A+2\alpha B+3\alpha^2 C} + \frac{\beta^{2-n}}{A+2\beta B+3\beta^2 C} + \frac{\gamma^{2-n}}{A+2\gamma B+3\gamma^2 C} \right), \end{aligned} \tag{10}$$

where α, β, and γ are the roots of the polynomial

$$Cx^3 + Bx^2 + Ax = 1. \tag{11}$$

A QUOTIENT-DIFFERENCE TABLE eventually yields a line of 0s IFF the starting sequence is defined by a linear recurrence relation.

A linear second-order recurrence

$$f_{n+1} = xf_n + yf_{n-1} \tag{12}$$

can be solved rapidly using a "rate doubling,"

$$f_{n+2} = (x^2 + 2y)f_n - y^2 f_{n-2}, \tag{13}$$

"rate tripling"

$$f_{n+3} = (x^3 + 3xy)f_n + y^3 f_{n-3}, \tag{14}$$

or in general, "rate k-tupling" formula

$$f_{n+k} = p_k f_n + q_k f_{n-k}, \tag{15}$$

where

$$p_0 = 2 \tag{16}$$

$$p_1 = x \tag{17}$$

$$p_k = 2(-y)^{k/2} T_k(x/(2i\sqrt{y})) \tag{18}$$

$$p_{k+1} = xp_k + yp_{k-1} \tag{19}$$

(here, $T_k(x)$ is a CHEBYSHEV POLYNOMIAL OF THE FIRST KIND) and

$$q_0 = -1 \tag{20}$$

$$q_1 = y \tag{21}$$

$$q_k = -(-y)^k \tag{22}$$

$$q_{k+1} = -yq_k \tag{23}$$

(Beeler *et al.* 1972, Item 14).

Let

$$s(X) = \prod_{i=1}^{m} (1 - \alpha_i X)^{n_i} = 1 - s_1 X - \ldots - s_n X^n, \tag{24}$$

where the generalized POWER sum $a(h)$ for $h = 0, 1, \ldots$ is given by

$$a(h) = \sum_{i=1}^{m} A_i(h) {\alpha_i}^h, \tag{25}$$

with distinct NONZERO roots α_i, COEFFICIENTS $A_i(h)$ which are POLYNOMIALS of degree $n_i - 1$ for POSITIVE INTEGERS n_i, and $i \in [1, m]$. Then the sequence $\{a_h\}$ with $a_h = a(h)$ satisfies the RECURRENCE RELATION

$$a_{h+n} = s_i a_{h+n-1} + \ldots + s_n a_h \tag{26}$$

(Meyerson and van der Poorten 1995).

The terms in a general recurrence sequence belong to a finitely generated RING over the INTEGERS, so it is impossible for every RATIONAL NUMBER to occur in any finitely generated recurrence sequence. If a recurrence sequence vanishes infinitely often, then it vanishes on an arithmetic progression with a common difference 1 that depends only on the roots. The number of values that a recurrence sequence can take on infinitely often is bounded by some INTEGER l that depends only on the roots. There is no recurrence sequence in which each INTEGER occurs infinitely often, or in which every GAUSSIAN INTEGER occurs (Myerson and van der Poorten 1995).

Let $\mu(n)$ be a bound so that a nondegenerate INTEGER recurrence sequence of order n takes the value zero at least $\mu(n)$ times. Then $\mu(2) = 1$, $\mu(3) = 6$, and $\mu(4) \geq 9$ (Myerson and van der Poorten 1995). The maximal case for $\mu(3)$ is

$$a_{n+3} = 2a_{n+2} - 4a_{n+1} + 4a_n \tag{27}$$

with

$$a_0 = a_1 = 0 \tag{28}$$

$$a_2 = 1. \tag{29}$$

The zeros are

$$a_0 = a_1 = a_4 = a_6 = a_{13} = a_{52} = 0 \tag{30}$$

(Beukers 1991).

see also BINET FORMS, BINET'S FORMULA, FAST FIBONACCI TRANSFORM, FIBONACCI SEQUENCE, LUCAS SEQUENCE, QUOTIENT-DIFFERENCE TABLE, SKOLEM-MAHLER-LERCH THEOREM

References

Beeler, M.; Gosper, R. W.; and Schroeppel, R. *HAKMEM.* Cambridge, MA: MIT Artificial Intelligence Laboratory, Memo AIM-239, Feb. 1972.

Beukers, F. "The Zero-Multiplicity of Ternary Recurrences." *Composito Math.* **77**, 165–177, 1991.

Myerson, G. and van der Poorten, A. J. "Some Problems Concerning Recurrence Sequences." *Amer. Math. Monthly* **102**, 698–705, 1995.

Recurring Digital Invariant

To define a recurring digital invariant of order k, compute the sum of the kth powers of the digits of a number n. If this number n' is equal to the original number n, then $n = n'$ is called a k-NARCISSISTIC NUMBER. If not, compute the sums of the kth powers of the digits of n', and so on. If this process eventually leads back to the original number n, the *smallest number* in the sequence $\{n, n', n'', \ldots\}$ is said to be a k-recurring digital invariant. For example,

$$\begin{aligned} 55 &: 5^3 + 5^3 = 250 \\ 250 &: 2^3 + 5^3 + 0^3 = 133 \\ 133 &: 1^3 + 3^3 + 3^3 = 55, \end{aligned}$$

so 55 is an order 3 recurring digital invariant. The following table gives recurring digital invariants of orders 2 to 10 (Madachy 1979).

Order	RDIs	Cycle Lengths
2	4	8
3	55, 136, 160, 919	3, 2, 3, 2
4	1138, 2178	7, 2
5	244, 8294, 8299, 9044, 9045, 10933,24584, 58618, 89883	28, 10, 6, 10, 22, 4, 12, 2, 2
6	17148, 63804, 93531, 239459, 282595	30, 2, 4, 10, 3
7	80441, 86874, 253074, 376762, 922428, 982108, five more	92, 56, 27, 30, 14, 21
8	6822, 7973187, 8616804	
9	322219, 2274831, 20700388, eleven more	
10	20818070, five more	

see also 196-ALGORITHM, ADDITIVE PERSISTENCE, DIGITAL ROOT, DIGITADITION, HAPPY NUMBER, KAPREKAR NUMBER, NARCISSISTIC NUMBER, VAMPIRE NUMBER

References
Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, pp. 163–165, 1979.

Recursion

A recursive process is one in which objects are defined in terms of other objects of the same type. Using some sort of RECURRENCE RELATION, the entire class of objects can then be built up from a few initial values and a small number of rules. The FIBONACCI NUMBERS are most commonly defined recursively. Care, however, must be taken to avoid SELF-RECURSION, in which an object is defined in terms of itself, leading to an infinite nesting.

see also ACKERMANN FUNCTION, PRIMITIVE RECURSIVE FUNCTION, RECURRENCE RELATION, RECURRENCE SEQUENCE, RICHARDSON'S THEOREM, SELF-RECURSION, SELF-SIMILARITY, TAK FUNCTION

References
Buck, R. C. "Mathematical Induction and Recursive Definitions." *Amer. Math. Monthly* **70**, 128–135, 1963.
Knuth, D. E. "Textbook Examples of Recursion." In *Artificial Intelligence and Mathematical Theory of Computation, Papers in Honor of John McCarthy* (Ed. V. Lifschitz). Boston, MA: Academic Press, pp. 207–229, 1991.
Péter, R. *Rekursive Funktionen.* Budapest: Akad. Kiado, 1951.

Recursive Function

A recursive function is a function generated by (1) ADDITION, (2) MULTIPLICATION, (3) selection of an element from a list, and (4) determination of the truth or falsity of the INEQUALITY $a < b$ according to the technical rules:

1. If F and the sequence of functions $G_1, \ldots, G_n$ are recursive, then so is $F(G_1, \ldots, G_n)$.
2. If F is a recursive function such that there is an x for each a with $H(a, x) = 0$, then the smallest x can be obtained recursively.

A TURING MACHINE is capable of computing recursive functions.

see also TURING MACHINE

References
Kleene, S. C. *Introduction to Metamathematics.* Princeton, NJ: Van Nostrand, 1952.

Recursive Monotone Stable Quadrature

A QUADRATURE (NUMERICAL INTEGRATION) algorithm which has a number of desirable properties.

References
Favati, P.; Lotti, G.; and Romani, F. "Interpolary Integration Formulas for Optimal Composition." *ACM Trans. Math. Software* **17**, 207–217, 1991.
Favati, P.; Lotti, G.; and Romani, F. "Algorithm 691: Improving QUADPACK Automatic Integration Routines." *ACM Trans. Math. Software* **17**, 218–232, 1991.

Red-Black Tree

An extended BINARY TREE satisfying the following conditions:

1. Every node has two CHILDREN, each colored either red or black.
2. Every LEAF node is colored black.
3. Every red node has both of its CHILDREN colored black.
4. Every path from the ROOT to a LEAF contains the same number (the "black-height") of black nodes.

Let n be the number of internal nodes of a red-black tree. Then the number of red-black trees for $n = 1$, 2, ... is 2, 2, 3, 8, 14, 20, 35, 64, 122, ... (Sloane's A001131). The number of trees with black roots and red roots are given by Sloane's A001137 and Sloane's A001138, respectively.

Let T_h be the GENERATING FUNCTION for the number of red-black trees of black-height h indexed by the number of LEAVES. Then

$$T_{h+1}(x) = [T_h(x)]^2 + [T_h(x)]^4, \quad (1)$$

where $T_1(x) = x+x^2$. If $T(x)$ is the GENERATING FUNCTION for the number of red-black trees, then

$$T(x) = x + x^2 + T(x^2(1+x)^2) \qquad (2)$$

(Ruskey). Let $rb(n)$ be the number of red-black trees with n LEAVES, $r(n)$ the number of red-rooted trees, and $b(n)$ the number of black-rooted trees. All three of the quantities satisfy the RECURRENCE RELATION

$$R(n) = \sum_{n/4 \leq n \leq n/2} \binom{2m}{n-2m} R(m), \qquad (3)$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT, $\text{rb}(1) = 1$, $\text{rb}(2) = 2$ for $R(n) = \text{rb}(n)$, $r(1) = r(3) = 0$, $r(2) = 1$ for $R(n) = r(n)$, and $b(1) = 1$ for $R(n) = b(n)$ (Ruskey).

References

Beyer, R. "Symmetric Binary B-Trees: Data Structures and Maintenance Algorithms." *Acta Informat.* **1**, 290–306, 1972.

Rivest, R. L.; Leiserson, C. E.; and Cormen, R. H. *Introduction to Algorithms.* New York: McGraw-Hill, 1990.

Ruskey, F. "Information on Red-Black Trees." `http://sue.csc.uvic.ca/~cos/inf/tree/RedBlackTree.html`.

Sloane, N. J. A. Sequences A001131, A001137, and A001138 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Red Net

The coloring red of two COMPLETE SUBGRAPHS of $n/2$ points (for EVEN n) in order to generate a BLUE-EMPTY GRAPH.

see also BLUE-EMPTY GRAPH, COMPLETE GRAPH

Reduced Amicable Pair

see QUASIAMICABLE PAIR

Reduced Fraction

A FRACTION a/b written in lowest terms, i.e., by dividing NUMERATOR and DENOMINATOR through by their GREATEST COMMON DIVISOR (a, b). For example, 2/3 is the reduced fraction of 8/12.

see also FRACTION, PROPER FRACTION

Reduced Latitude

see PARAMETRIC LATITUDE

Reducible Crossing

A crossing in a LINK projection which can be removed by rotating part of the LINK, also called REMOVABLE CROSSING.

see also ALTERNATING KNOT

Reducible Representation

see IRREDUCIBLE REPRESENTATION

Reducible Matrix

A SQUARE $n \times n$ matrix $\mathsf{A} = a_{ij}$ is called reducible if the indices 1, 2, ..., n can be divided into two disjoint nonempty sets i_1, i_2, ..., i_μ and j_1, j_2, ..., j_ν (with $\mu + \nu = n$) such that

$$a_{i_\alpha j_\beta} = 0$$

for $\alpha = 1, 2, \ldots, \mu$ and $\beta = 1, 2, \ldots, \nu$. A SQUARE MATRIX which is not reducible is said to be IRREDUCIBLE.

see also SQUARE MATRIX

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1103, 1979.

Reduction of Order

see ORDINARY DIFFERENTIAL EQUATION—SECOND-ORDER

Reduction Theorem

If a fixed point is added to each group of a special complete series, then the resulting series is complete.

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 253, 1959.

Redundancy

$$R(X_1, \ldots X_n) \equiv \sum_{i=1}^{n} H(X_i) - H(X_1, \ldots, X_n),$$

where $H(x_i)$ is the ENTROPY and $H(X_1, \ldots, X_n)$ is the joint ENTROPY. Linear redundancy is defined as

$$L(X_1, \ldots, X_n) \equiv -\tfrac{1}{2} \sum_{i=1}^{n} \ln \sigma_i,$$

where σ_i are EIGENVALUES of the correlation matrix.

see also PREDICTABILITY

References

Fraser, A. M. "Reconstructing Attractors from Scalar Time Series: A Comparison of Singular System and Redundancy Criteria." *Phys. D* **34**, 391–404, 1989.

Paluš, M. "Identifying and Quantifying Chaos by Using Information-Theoretic Functionals." In *Time Series Prediction: Forecasting the Future and Understanding the Past* (Ed. A. S. Weigend and N. A. Gerschenfeld). Proc. NATO Advanced Research Workshop on Comparative Time Series Analysis held in Sante Fe, NM, May 14–17, 1992. Reading, MA: Addison-Wesley, pp. 387–413, 1994.

Reeb Foliation

The Reeb foliation of the HYPERSPHERE $\mathbb{S}^3$ is a FOLIATION constructed as the UNION of two solid TORI with common boundary.

see also FOLIATION

References

Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, pp. 287–288, 1976.

Reef Knot

see SQUARE KNOT

Refinement

A refinement X of a COVER Y is a COVER such that every element $x \in X$ is a SUBSET of an element $y \in Y$.

see also COVER

Reflection

The operation of exchanging all points of a mathematical object with their MIRROR IMAGES (i.e., reflections in a mirror). Objects which do not change HANDEDNESS under reflection are said to be AMPHICHIRAL; those that do are said to be CHIRAL.

If the PLANE of reflection is taken as the yz-PLANE, the reflection in 2- or 3-D SPACE consists of making the transformation $x \to -x$ for each point. Consider an arbitrary point $\mathbf{x}_0$ and a PLANE specified by the equation

$$ax + by + xz + d = 0. \tag{1}$$

This PLANE has NORMAL VECTOR

$$\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \tag{2}$$

and the POINT-PLANE DISTANCE is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}. \tag{3}$$

The position of the point reflected in the given plane is therefore given by

$$\begin{aligned} \mathbf{x}_0' &= \mathbf{x}_0 - 2D\hat{\mathbf{n}} \\ &= \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} - 2|ax_0 + by_0 + cz_0 + d| \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \end{aligned} \tag{4}$$

see also AMPHICHIRAL, CHIRAL, DILATION, ENANTIOMER, EXPANSION, GLIDE, HANDEDNESS, IMPROPER ROTATION, INVERSION OPERATION, MIRROR IMAGE, PROJECTION, REFLECTION PROPERTY, REFLECTION RELATION, REFLEXIBLE, ROTATION, ROTOINVERSION, TRANSLATION

Reflection Property

In the plane, the reflection property can be stated as three theorems (Ogilvy 1990, pp. 73–77):

1. The LOCUS of the center of a variable CIRCLE, tangent to a fixed CIRCLE and passing through a fixed point inside that CIRCLE, is an ELLIPSE.
2. If a variable CIRCLE is tangent to a fixed CIRCLE and also passes through a fixed point outside the CIRCLE, then the LOCUS of its moving center is a HYPERBOLA.
3. If a variable CIRCLE is tangent to a fixed straight line and also passes through a fixed point not on the line, then the LOCUS of its moving center is a PARABOLA.

Let $\alpha : I \to \mathbb{R}^2$ be a smooth regular parameterized curve in $\mathbb{R}^2$ defined on an OPEN INTERVAL I, and let F_1 and F_2 be points in $\mathbb{P}^2 \backslash \alpha(I)$, where $\mathbb{P}^n$ is an n-D PROJECTIVE SPACE. Then α has a reflection property with FOCI F_1 and F_2 if, for each point $P \in \alpha(I)$,

1. Any vector normal to the curve α at P lies in the SPAN of the vectors $\overrightarrow{F_1P}$ and $\overrightarrow{F_2P}$.
2. The line normal to α at P bisects one of the pairs of opposite ANGLES formed by the intersection of the lines joining F_1 and F_2 to P.

A smooth connected plane curve has a reflection property IFF it is part of an ELLIPSE, HYPERBOLA, PARABOLA, CIRCLE, or straight LINE.

Foci	Sign	Both foci finite	One focus finite	Both foci ∞
distinct	+	confocal ellipses	confocal parabolas	‖ lines
distinct	−	confocal hyperbola and $\perp$ bisector of interfoci line segment	confocal parabolas	‖ lines
equal		concentric circles		‖ lines

Let $S \in \mathbb{R}^3$ be a smooth connected surface, and let F_1 and F_2 be points in $\mathbb{P}^3 \backslash S$, where $\mathbb{P}^n$ is an n-D PROJECTIVE SPACE. Then S has a reflection property with FOCI F_1 and F_2 if, for each point $P \in S$,

1. Any vector normal to S at P lies in the SPAN of the vectors $\overrightarrow{F_1P}$ and $\overrightarrow{F_2P}$.
2. The line normal to S at P bisects one of the pairs of opposite angles formed by the intersection of the lines joining F_1 and F_2 to P.

A smooth connected surface has a reflection property IFF it is part of an ELLIPSOID of revolution, a HYPERBOLOID of revolution, a PARABOLOID of revolution, a SPHERE, or PLANE.

Foci	Sign	Both foci finite	One focus finite	Both foci ∞
distinct	+	confocal ellipsoids	confocal paraboloids	‖ planes
distinct	−	confocal hyperboloids and plane $\perp$ bisector of interfoci line segment	confocal paraboloids	‖ planes
equal		concentric spheres		‖ planes

see also BILLIARDS

References

Drucker, D. "Euclidean Hypersurfaces with Reflective Properties." *Geometrica Dedicata* **33**, 325–329, 1990.

Drucker, D. "Reflective Euclidean Hypersurfaces." *Geometrica Dedicata* **39**, 361–362, 1991.

Drucker, D. "Reflection Properties of Curves and Surfaces." *Math. Mag.* **65**, 147–157, 1992.
Drucker, D. and Locke, P. "A Natural Classification of Curves and Surfaces with Reflection Properties." *Math. Mag.* **69**, 249–256, 1996.
Ogilvy, C. S. *Excursions in Geometry.* New York: Dover, pp. 73–77, 1990.
Wegner, B. "Comment on 'Euclidean Hypersurfaces with Reflective Properties'." *Geometrica Dedicata* **39**, 357–359, 1991.

Reflection Relation

A mathematical relationship relating $f(-x)$ to $f(x)$.

see also ARGUMENT ADDITION RELATION, ARGUMENT MULTIPLICATION RELATION, RECURRENCE RELATION, TRANSLATION RELATION

Reflexible

An object is reflexible if it is superposable with its image in a plane mirror. Also called AMPHICHIRAL.

see also AMPHICHIRAL, CHIRAL, ENANTIOMER, HANDEDNESS, MIRROR IMAGE, REFLECTION

References

Ball, W. W. R. and Coxeter, H. S. M. "Polyhedra." Ch. 5 in *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 130, 1987.

Reflexible Map

An AUTOMORPHISM which interchanges the two vertices of a regular map at each edge without interchanging the vertices.

see also EDMONDS' MAP

Reflexive Closure

The reflexive closure of a binary RELATION R on a SET X is the minimal REFLEXIVE RELATION R' on X that contains R. Thus $aR'a$ for every element a of X and $aR'b$ for distinct elements a and b, provided that aRb.

see also REFLEXIVE REDUCTION, REFLEXIVE RELATION, RELATION, TRANSITIVE CLOSURE

Reflexive Graph

see DIRECTED GRAPH

Reflexive Reduction

The reflexive reduction of a binary RELATION R on a SET X is the minimum relation R' on X with the same REFLEXIVE CLOSURE as R. Thus $aR'b$ for any elements a and b of X, provided that a and b are distinct and aRb.

see also REFLEXIVE CLOSURE, RELATION, TRANSITIVE REDUCTION

Reflexive Relation

A RELATION R on a SET S is reflexive provided that xRx for every x in S.

see also RELATION

Reflexivity

A REFLEXIVE RELATION.

Region

An open connected set is called a region (sometimes also called a DOMAIN).

Regression

A method for fitting a curve (not necessarily a straight line) through a set of points using some goodness-of-fit criterion. The most common type of regression is LINEAR REGRESSION.

see also LEAST SQUARES FITTING, LINEAR REGRESSION, MULTIPLE REGRESSION, NONLINEAR LEAST SQUARES FITTING, REGRESSION COEFFICIENT

References

Kleinbaum, D. G. and Kupper, L. L. *Applied Regression Analysis and Other Multivariable Methods.* North Scituate, MA: Duxbury Press, 1978.

Regression Coefficient

The slope b of a line obtained using linear LEAST SQUARES FITTING is called the regression coefficient.

see also CORRELATION COEFFICIENT, LEAST SQUARES FITTING

References

Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics, Pt. 2, 2nd ed.* Princeton, NJ: Van Nostrand, p. 254, 1951.

Regula Falsi

see FALSE POSITION METHOD

Regular Function

see HOLOMORPHIC FUNCTION

Regular Graph

A GRAPH is said to be regular of degree r if all LOCAL DEGREES are the same number r. Then

$$E = \tfrac{1}{2}nr,$$

where E is the number of EDGES. The connected 3-regular graphs have been determined by G. Brinkman up to 24 VERTICES.

see also COMPLETE GRAPH, COMPLETELY REGULAR GRAPH, LOCAL DEGREE, SUPERREGULAR GRAPH

References

Chartrand, G. *Introductory Graph Theory.* New York: Dover, p. 29, 1985.

Regular Isotopy

The equivalence of MANIFOLDS under continuous deformation within the embedding space. KNOTS of opposite CHIRALITY have AMBIENT ISOTOPY, but not regular isotopy.

see also AMBIENT ISOTOPY

Regular Isotopy Invariant

see BRACKET POLYNOMIAL

Regular Local Ring

A regular local ring is a LOCAL RING R with MAXIMAL IDEAL m so that m can be generated with exactly d elements where d is the KRULL DIMENSION of the RING R. Equivalently, R is regular if the VECTOR SPACE m/m^2 has dimension d.

see also KRULL DIMENSION, LOCAL RING, REGULAR RING, RING

References

Eisenbud, D. *Commutative Algebra with a View Toward Algebraic Geometry.* New York: Springer-Verlag, p. 242, 1995.

Regular Number

A number which has a finite DECIMAL expansion. A number which is not regular is said to be nonregular.

see also DECIMAL EXPANSION, REPEATING DECIMAL

Regular Parameterization

A parameterization of a SURFACE $\mathbf{x}(u,v)$ in u and v is regular if the TANGENT VECTORS

$$\frac{\partial \mathbf{x}}{\partial u} \quad \text{and} \quad \frac{\partial \mathbf{x}}{\partial v}$$

are always LINEARLY INDEPENDENT.

Regular Patch

A regular patch is a PATCH $\mathbf{x} : U \to \mathbb{R}^n$ for which the JACOBIAN $J(\mathbf{x})(u,v)$ has rank 2 for all $(u,v) \in U$. A PATCH is said to be regular at a point $(u_0, v_0) \in U$ providing that its JACOBIAN has rank 2 at (u_0, v_0). For example, the points at $\phi = \pm\pi/2$ in the standard parameterization of the SPHERE $(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$ are not regular.

An example of a PATCH which is regular but not INJECTIVE is the CYLINDER defined parametrically by $(\cos u, \sin u, v)$ with $u \in (-\infty, \infty)$ and $v \in (-2, 2)$. However, if $\mathbf{x} : U \to \mathbb{R}^n$ is an injective regular patch, then $\mathbf{x}$ maps U diffeomorphically onto $\mathbf{x}(U)$.

see also INJECTIVE PATCH, PATCH, REGULAR SURFACE

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 187, 1993.

Regular Point

see ORDINARY POINT

Regular Polygon

An n-sided POLYGON in which the sides are all the same length and are symmetrically placed about a common center. The sum of PERPENDICULARS from any point to the sides of a regular polygon of n sides is n times the APOTHEM. Only certain regular polygons are "CONSTRUCTIBLE" with RULER and STRAIGHTEDGE.

n	Regular Polygon
3	equilateral triangle
4	square
5	pentagon
6	hexagon
7	heptagon
8	octagon
9	nonagon
10	decagon
12	dodecagon
15	pentadecagon
16	hexadecagon
17	heptadecagon
18	octadecagon
20	icosagon
30	triacontagon

see also CONSTRUCTIBLE POLYGON, GEOMETROGRAPHY, HEPTADECAGON, POLYGON

References

Bishop, W. "How to Construct a Regular Polygon." *Amer. Math. Monthly* **85**, 186–188, 1978.

Regular Polyhedron

A polyhedron is said to be regular if its FACES and VERTEX FIGURES are REGULAR (not necessarily CONVEX) polygons (Coxeter 1973, p. 16). Using this definition, there are a total of nine regular polyhedra, five being the CONVEX PLATONIC SOLIDS and four being the CONCAVE (stellated) KEPLER-POINSOT SOLIDS. However, the term "regular polyhedra" is sometimes used to refer exclusively to the CONVEX PLATONIC SOLIDS.

It can be proven that only nine regular solids (in the Coxeter sense) exist by noting that a possible regular polyhedron must satisfy

$$\cos^2\left(\frac{\pi}{p}\right) + \cos^2\left(\frac{\pi}{q}\right) + \cos^2\left(\frac{\pi}{r}\right) = 1.$$

Gordon showed that the only solutions to

$$1 + \cos\phi_1 + \cos\phi_2 + \cos\phi_3 = 0$$

of the form $\phi_i = \pi m_i/n_i$ are the permutations of $(\frac{2}{3}\pi, \frac{2}{3}\pi, \frac{1}{3}\pi)$ and $(\frac{2}{3}\pi, \frac{2}{5}\pi, \frac{4}{5}\pi)$. This gives three permutations of (3, 3, 4) and six of (3, 5, $\frac{5}{3}$) as possible solutions to the first equation. Plugging back in gives the SCHLÄFLI SYMBOLS of possible regular polyhedra as $\{3,3\}$, $\{3,4\}$, $\{4,3\}$, $\{3,5\}$, $\{5,3\}$, $\{3,\frac{5}{2}\}$, $\{\frac{5}{2},3\}$, $\{5,\frac{5}{2}\}$, and $\{\frac{5}{2},5\}$ (Coxeter 1973, pp. 107–109). The first five of these are the PLATONIC SOLIDS and the remaining four the KEPLER-POINSOT SOLIDS.

Every regular polyhedron has $e+1$ axes of symmetry, where e is the number of EDGES, and $3h/2$ PLANES of symmetry, where h is the number of sides of the corresponding PETRIE POLYGON.

see also CONVEX POLYHEDRON, KEPLER-POINSOT SOLID, PETRIE POLYGON, PLATONIC SOLID, POLYHEDRON, POLYHEDRON COMPOUND, SPONGE, VERTEX FIGURE

References

Coxeter, H. S. M. "Regular and Semi-Regular Polytopes I." *Math. Z.* **46**, 380–407, 1940.

Coxeter, H. S. M. *Regular Polytopes, 3rd ed.* New York: Dover, pp. 1–17, 93, and 107–112, 1973.

Cromwell, P. R. *Polyhedra.* New York: Cambridge University Press, pp. 85–86, 1997.

Regular Prime

A PRIME which does not DIVIDE the CLASS NUMBER $h(p)$ of the CYCLOTOMIC FIELD obtained by adjoining a PRIMITIVE pTH ROOT of unity to the rational FIELD. A PRIME p is regular IFF p does not divide the NUMERATORS of the BERNOULLI NUMBERS B_{10}, B_{12}, ..., B_{2p-2}. A PRIME which is not regular is said to be an IRREGULAR PRIME.

In 1915, Jensen proved that there are infinitely many IRREGULAR PRIMES. It has not yet been proven that there are an INFINITE number of regular primes (Guy 1994, p. 145). Of the 283,145 PRIMES $< 4\times 10^6$, 171,548 (or 60.59%) are regular (the conjectured FRACTION is $e^{-1/2} \approx 60.65\%$). The first few are 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 43, 47, ... (Sloane's A007703).

see also BERNOULLI NUMBER, FERMAT'S THEOREM, IRREGULAR PRIME

References

Buhler, J.; Crandall, R. Ernvall, R.; and Metsankyla, T. "Irregular Primes and Cyclotomic Invariants to Four Million." *Math. Comput.* **61**, 151–153, 1993.

Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 145, 1994.

Ribenboim, P. "Regular Primes." §5.1 in *The New Book of Prime Number Records.* New York: Springer-Verlag, pp. 323–329, 1996.

Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, p. 153, 1993.

Sloane, N. J. A. Sequence A007703/M2411 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Regular Ring

In the sense of von Neumann, a regular ring is a RING R such that for all $a \in R$, there exists a $b \in R$ satisfying $a = aba$.

see also REGULAR LOCAL RING, RING

References

Jacobson, N. *Basic Algebra II, 2nd ed.* New York: W. H. Freeman, p. 196, 1989.

Regular Sequence

Let there be two PARTICULARLY WELL-BEHAVED FUNCTIONS $F(x)$ and $p_\tau(x)$. If the limit

$$\lim_{\tau\to 0}\int_{-\infty}^{\infty} p_\tau(x)F(x)\,dx$$

exists, then $p_\tau(x)$ is a regular sequence of PARTICULARLY WELL-BEHAVED FUNCTIONS.

Regular Singular Point

Consider a second-order ORDINARY DIFFERENTIAL EQUATION

$$y'' + P(x)y' + Q(x)y = 0.$$

If $P(x)$ and $Q(x)$ remain FINITE at $x = x_0$, then x_0 is called an ORDINARY POINT. If either $P(x)$ or $Q(x)$ diverges as $x \to x_0$, then x_0 is called a singular point. If either $P(x)$ or $Q(x)$ diverges as $x \to x_0$ but $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ remain FINITE as $x \to x_0$, then $x = x_0$ is called a regular singular point (or NONESSENTIAL SINGULARITY).

see also IRREGULAR SINGULARITY, SINGULAR POINT (DIFFERENTIAL EQUATION)

References

Arfken, G. "Singular Points." §8.4 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 451–453 and 461–463, 1985.

Regular Singularity

see REGULAR SINGULAR POINT

Regular Surface

A SUBSET $M \subset \mathbb{R}^n$ is called a regular surface if for each point $p \in M$, there exists a NEIGHBORHOOD V of p in $\mathbb{R}^n$ and a MAP $x: U \to \mathbb{R}^n$ of a OPEN SET $U \subset \mathbb{R}^2$ onto $V \cap M$ such that

1. x is differentiable,
2. $x: U \to V \cap M$ is a HOMEOMORPHISM,
3. Each map $x: U \to M$ is a REGULAR PATCH.

Any open subset of a regular surface is also a regular surface.

see also REGULAR PATCH

References

Gray, A. "The Definition of a Regular Surface in $\mathbb{R}^n$." §10.4 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 195–200, 1993.

Regular Triangle Center

A TRIANGLE CENTER is regular IFF there is a TRIANGLE CENTER FUNCTION which is a POLYNOMIAL in Δ, a, b, and c (where Δ is the AREA of the TRIANGLE) such that the TRILINEAR COORDINATES of the center are

$$f(a,b,c) : f(b,c,a) : f(c,a,b).$$

The ISOGONAL CONJUGATE of a regular center is a regular center. Furthermore, given two regular centers, any two of their HARMONIC CONJUGATE POINTS are also regular centers.

see also ISOGONAL CONJUGATE, TRIANGLE CENTER, TRIANGLE CENTER FUNCTION

Regularity Theorem

An AREA-minimizing surface (RECTIFIABLE CURRENT) bounded by a smooth curve in $\mathbb{R}^3$ is a smooth submanifold with boundary.

see also MINIMAL SURFACE, RECTIFIABLE CURRENT

References
Morgan, F. "What is a Surface?" *Amer. Math. Monthly* **103**, 369–376, 1996.

Regularized Beta Function

The regularized beta function is defined by

$$I(z;a,b) = \frac{B(z;a,b)}{B(a,b)},$$

where $B(z;a,b)$ is the incomplete BETA FUNCTION and $B(a,b)$ is the complete BETA FUNCTION.

see also BETA FUNCTION, REGULARIZED GAMMA FUNCTION

Regularized Gamma Function

The regularized gamma functions are defined by

$$P(a,z) = 1 - Q(a,z) \equiv \frac{\gamma(a,z)}{\Gamma(a)}$$

and

$$Q(a,z) = 1 - P(a,z) \equiv \frac{\Gamma(a,z)}{\Gamma(a)},$$

where $\gamma(a,z)$ and $\Gamma(a,z)$ are incomplete GAMMA FUNCTIONS and $\Gamma(a)$ is a complete GAMMA FUNCTION. Their derivatives are

$$\frac{d}{dz}P(a,z) = e^{-z}z^{a-1}$$
$$\frac{d}{dz}Q(a,z) = -e^{-z}z^{a-1}.$$

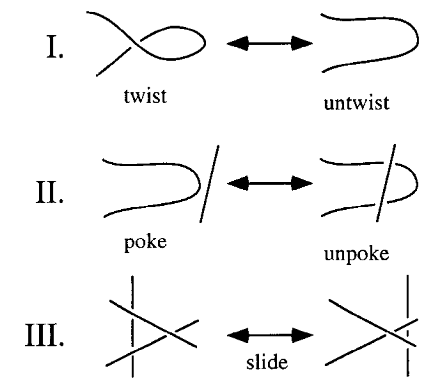
see also GAMMA FUNCTION, REGULARIZED BETA FUNCTION

References
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 160–161, 1992.

Regulus

The locus of lines meeting three given SKEW LINES. ("Regulus" is also the name of the brightest star in the constellation Leo.)

Reidemeister Moves



In the 1930s, Reidemeister first rigorously proved that KNOTS exist which are distinct from the UNKNOT. He did this by showing that all KNOT deformations can be reduced to a sequence of three types of "moves," called the (I) TWIST MOVE, (II) POKE MOVE, and (III) SLIDE MOVE.

REIDEMEISTER'S THEOREM guarantees that moves I, II, and III correspond to AMBIENT ISOTOPY (moves II and III alone correspond to REGULAR ISOTOPY). He then defined the concept of COLORABILITY, which is invariant under Reidemeister moves.

see also AMBIENT ISOTOPY, COLORABLE, MARKOV MOVES, REGULAR ISOTOPY, UNKNOT

Reidemeister's Theorem

Two LINKS can be continuously deformed into each other IFF any diagram of one can be transformed into a diagram of the other by a sequence of REIDEMEISTER MOVES.

see also REIDEMEISTER MOVES

Reinhardt Domain

A Reinhardt domain with center **c** is a DOMAIN D in C^n such that whenever D contains z_0, the DOMAIN D also contains the closed POLYDISK.

References
Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 101, 1980.

Relation

A relation is any SUBSET of a CARTESIAN PRODUCT. For instance, a SUBSET of $A \times B$, called a (binary) "relation from A to B," is a collection of ORDERED PAIRS (a,b) with first components from A and second components from B, and, in particular, a SUBSET of $A \times A$ is called a "relation on A." For a binary relation R, one often writes aRb to mean that (a,b) is in R.

see also ADJACENCY RELATION, ANTISYMMETRIC RELATION, ARGUMENT ADDITION RELATION, ARGUMENT MULTIPLICATION RELATION, COVER RELATION, EQUIVALENCE RELATION, IRREFLEXIVE, PARTIAL ORDER, RECURRENCE RELATION, REFLECTION RELATION, REFLEXIVE RELATION, SYMMETRIC RELATION, TRANSITIVE, TRANSLATION RELATION

Relative Error

Let the true value of a quantity be x and the measured or inferred value x_0. Then the relative error is defined by

$$\delta x = \frac{\Delta x}{x} = \frac{x_0 - x}{x} = \frac{x_0}{x} - 1,$$

where Δx is the ABSOLUTE ERROR. The relative error of the QUOTIENT or PRODUCT of a number of quantities is less than or equal to the SUM of their relative errors. The PERCENTAGE ERROR is 100% times the relative error.

see also ABSOLUTE ERROR, ERROR PROPAGATION, PERCENTAGE ERROR

References
Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 14, 1972.

Relative Extremum

A RELATIVE MAXIMUM or RELATIVE MINIMUM, also called a LOCAL EXTREMUM.

see also EXTREMUM, GLOBAL EXTREMUM, RELATIVE MAXIMUM, RELATIVE MINIMUM

Relative Maximum

A MAXIMUM within some NEIGHBORHOOD which need not be a GLOBAL MAXIMUM.

see also GLOBAL MAXIMUM, MAXIMUM, RELATIVE MINIMUM

Relative Minimum

A MINIMUM within some NEIGHBORHOOD which need not be a GLOBAL MINIMUM.

see also GLOBAL MINIMUM, MINIMUM, RELATIVE MAXIMUM

Relatively Prime

Two integers are relatively prime if they share no common factors (divisors) except 1. Using the notation (m,n) to denote the GREATEST COMMON DIVISOR, two integers m and n are relatively prime if $(m,n) = 1$. Relatively prime integers are sometimes also called STRANGERS or COPRIME and are denoted $m \perp n$.

The probability that two INTEGERS picked at random are relatively prime is $[\zeta(2)]^{-1} = 6/\pi^2$, where $\zeta(z)$ is the RIEMANN ZETA FUNCTION. This result is related to the fact that the GREATEST COMMON DIVISOR of m and n, $(m,n) = k$, can be interpreted as the number of LATTICE POINTS in the PLANE which lie on the straight LINE connecting the VECTORS $(0,0)$ and (m,n) (excluding (m,n) itself). In fact $6/\pi^2$ the fractional number of LATTICE POINTS VISIBLE from the ORIGIN (Castellanos 1988, pp. 155–156).

Given three INTEGERS chosen at random, the probability that no common factor will divide them all is

$$[\zeta(3)]^{-1} \approx 1.202^{-1} = 0.832\ldots,$$

where $\zeta(3)$ is APÉRY'S CONSTANT. This generalizes to k random integers (Schoenfeld 1976).

see also DIVISOR, GREATEST COMMON DIVISOR, VISIBILITY

References
Castellanos, D. "The Ubiquitous Pi." *Math. Mag.* **61**, 67–98, 1988.
Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 3–4, 1994.
Schoenfeld, L. "Sharper Bounds for the Chebyshev Functions $\theta(x)$ and $\psi(x)$, II." *Math. Comput.* **30**, 337–360, 1976.

Relaxation Methods

Methods of solving an ORDINARY DIFFERENTIAL EQUATION by replacing it with a FINITE DIFFERENCE equation on a regular grid spanning the domain of interest. The finite difference equations are then solved using an n-D NEWTON'S METHOD or other similar algorithm.

References
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Richardson Extrapolation and the Bulirsch-Stoer Method." §17.3 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 753–763, 1992.

Remainder

In general, a remainder is a quantity "left over" after performing a particular algorithm. The term is most commonly used to refer to the number left over when two integers are divided by each other in INTEGER DIVISION. For example, $55\backslash 7 = 7$, with a remainder of 6. Of course in real division, there is no such thing as a remainder since, for example, $55/7 = 7 + 6/7$.

The term remainder is also sometimes to the RESIDUE of a CONGRUENCE.

see also DIVISION, INTEGER DIVISION, RESIDUE (CONGRUENCE)

Remainder Theorem

see POLYNOMIAL REMAINDER THEOREM

Rembs' Surfaces

A special class of ENNEPER'S SURFACES which can be given parametrically by

$$x = a(U\cos u - U'\sin u) \tag{1}$$
$$y = -a(U\sin u + U'\cos u) \tag{2}$$
$$z = v - aV', \tag{3}$$

where

$$U \equiv \frac{\cosh(u\sqrt{C})}{\sqrt{C}} \tag{4}$$
$$V \equiv \frac{\cos(v\sqrt{C+1})}{\sqrt{C+1}} \tag{5}$$
$$a \equiv \frac{2V}{(C+1)(U^2-V^2)}. \tag{6}$$

The value of v is restricted to

$$|v| \leq v_0 \equiv \frac{\pi}{2\sqrt{C+1}} \tag{7}$$

(Reckziegel 1986), and the values $v = \pm v_0$ correspond to the ends of the cleft in the surface.

see also ENNEPER'S SURFACES, KUEN SURFACE, SIEVERT'S SURFACE

References

Fischer, G. (Ed.). Plate 88 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, p. 84, 1986.

Reckziegel, H. "Sievert's Surface." §3.4.4.3 in *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, pp. 39–40, 1986.

Rembs, E. "Enneper'sche Flächen konstanter positiver Krümmung und Hazzidakissche Transformationen." *Jahrber. DMV* **39**, 278–283, 1930.

Removable Crossing

see REDUCIBLE CROSSING

Removable Singularity

A SINGULAR POINT z_0 of a FUNCTION $f(z)$ for which it is possible to assign a COMPLEX NUMBER in such a way that $f(z)$ becomes ANALYTIC. A more precise way of defining a removable singularity is as a SINGULARITY z_0 of a function $f(z)$ about which the function $f(z)$ is bounded. For example, the point $x_0 = 0$ is a removable singularity in the SINC FUNCTION $\operatorname{sinc} x = \sin x/x$, since this function satisfies $\operatorname{sinc} 0 = 1$.

Rencontres Number

see DERANGEMENT, SUBFACTORIAL

Rendezvous Values

see MAGIC GEOMETRIC CONSTANTS

Rényi's Parking Constants

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Given the CLOSED INTERVAL $[0, x]$ with $x > 1$, let 1-D "cars" of unit length be parked randomly on the interval. The MEAN number $M(x)$ of cars which can fit (without overlapping!) satisfies

$$M(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1 \\ 1 + \frac{2}{x-1}\int_0^{x-1} M(y)\,dy & \text{for } x \geq 1. \end{cases} \tag{1}$$

The mean density of the cars for large x is

$$m \equiv \lim_{x\to\infty} \frac{M(x)}{x} = \int_0^\infty \exp\left(-2\int_0^x \frac{1-e^{-y}}{y}\,dy\right)dx$$
$$= 0.7475979203\ldots. \tag{2}$$

Furthermore,

$$M(x) = mx + m - 1 + \mathcal{O}(x^{-n}) \tag{3}$$

for all n (Rényi 1958), which was strengthened by Dvoretzky and Robbins (1964) to

$$M(x) = mx + m - 1 + \mathcal{O}\left[\left(\frac{2e}{x}\right)^{x-3/2}\right]. \tag{4}$$

Dvoretzky and Robbins (1964) also proved that

$$\inf_{x\leq t\leq x+1} \frac{M(t)+1}{t+1} \leq m \leq \sup_{x\leq t\leq x+1} \frac{M(t)+1}{t+1}. \tag{5}$$

Let $V(x)$ be the variance of the number of cars, then Dvoretzky and Robbins (1964) and Mannion (1964) showed that

$$v \equiv \lim_{x\to\infty} \frac{V(x)}{x}$$
$$= 2\int_0^\infty \left\{x\int_0^1 e^{-xy}R_2(y)\,dy + x^2\left[\int_0^\infty e^{-xy}R_1(y)\,dy\right]^2\right\}$$
$$\times \exp\left(-2\int_0^x \frac{1-e^{-y}}{y}\,dy\right)dx = 0.038156\ldots, \tag{6}$$

where

$$R_1(x) = M(x) - mx - m + 1 \tag{7}$$

$$R_2(x) = \begin{cases} (1-m-mx)^2 & \text{for } 0 \leq x \leq 1 \\ 4(1-m)^2 & \text{for } x = 1 \\ \frac{2}{x-1}\left[\int_0^{x-1} R_2(y)\,dy\right. & \text{for } x > 1 \\ \quad \left.+ \int_0^{x-1} R_1(y)R_1(x-y-1)\,dy\right], & \end{cases} \tag{8}$$

and the numerical value is due to Blaisdell and Solomon (1970). Dvoretzky and Robbins (1964) also proved that

$$\inf_{x\leq t\leq x+1}\frac{V(t)}{t+1}\leq v\leq \sup_{x\leq t\leq x+1}\frac{V(t)}{t+1}, \tag{9}$$

and that

$$V(x)=vx+v+\mathcal{O}\left[\left(\frac{4e}{x}\right)^{x-4}\right]. \tag{10}$$

Palasti (1960) conjectured that in 2-D,

$$\lim_{x,y\to\infty}\frac{M(x,y)}{xy}=m^2, \tag{11}$$

but this has not yet been proven or disproven (Finch).

References

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Solomon, H. and Weiner, H. J. "A Review of the Packing Problem." *Comm. Statist. Th. Meth.* **15**, 2571–2607, 1986.

Rep-Tile

A POLYGON which can be divided into smaller copies of itself.

see also DISSECTION

References

Gardner, M. Ch. 19 in *The Unexpected Hanging and Other Mathematical Diversions.* Chicago, IL: Chicago University Press, 1991.

Repartition

see ADÉLE

Repdigit

A number composed of a single digit is called a repdigit. If the digits are all 1s, the repdigit is called a REPUNIT. The BEAST NUMBER 666 is a repdigit.

see also KEITH NUMBER, REPUNIT

Repeating Decimal

A number whose decimal representation eventually becomes periodic (i.e., the same sequence of digits repeats indefinitely) is called a repeating decimal. Numbers such as 0.5 can be regarded as repeating decimals since $0.5 = 0.5000\ldots = 0.4999\ldots$. All RATIONAL NUMBERS have repeating decimals, e.g., $1/11 = 0.\overline{09}$. However, TRANSCENDENTAL NUMBERS, such as $\pi = 3.141592\ldots$ do not.

see also CYCLIC NUMBER, DECIMAL EXPANSION, FULL REPTEND PRIME, IRRATIONAL NUMBER, MIDY'S THEOREM, RATIONAL NUMBER, REGULAR NUMBER

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 53–54, 1987.

Courant, R. and Robbins, H. "Rational Numbers and Periodic Decimals." §2.2.4 in *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, pp. 66–68, 1996.

Repfigit Number

see KEITH NUMBER

Replicate

One out of a set of identical observations in a given experiment under identical conditions.

Reptend Prime

see FULL REPTEND PRIME

Representation

The representation of a GROUP G on a COMPLEX VECTOR SPACE V is a group action of G on V by linear transformations. Two finite dimensional representations π on V and π' on V' are equivalent if there is an invertible linear map $E : V \mapsto V'$ such that $\pi'(g)E = E\pi(g)$ for all $g \in G$. π is said to be irreducible if it has no proper NONZERO invariant SUBSPACES.

see also CHARACTER (MULTIPLICATIVE), PETER-WEYL THEOREM, PRIMARY REPRESENTATION, SCHUR'S LEMMA

References

Knapp, A. W. "Group Representations and Harmonic Analysis, Part II." *Not. Amer. Math. Soc.* **43**, 537–549, 1996.

Repunit

A (generalized) repunit to the base b is a number of the form

$$M_n^b = \frac{b^n-1}{b-1}.$$

The term "repunit" was coined by Beiler (1966), who also gave the first tabulation of known factors. Repunits $M_n = M_n^2 = 2^n - 1$ with $b = 2$ are called MERSENNE

NUMBERS. If $b = 10$, the number is called a repunit (since the digits are all 1s). A number of the form

$$R_n = \frac{10^n - 1}{10 - 1} = R_n = \frac{10^n - 1}{9}$$

is therefore a (decimal) repunit of order n.

b	Sloane	b-Repunits
2	000225	1, 3, 7, 15, 31, 63, 127, ...
3	003462	1, 4, 13, 40, 121, 364, ...
4	002450	1, 5, 21, 85, 341, 1365, ...
5	003463	1, 6, 31, 156, 781, 3906, ...
6	003464	1, 7, 43, 259, 1555, 9331, ...
7	023000	1, 8, 57, 400, 2801, 19608, ...
8	023001	1, 9, 73, 585, 4681, 37449, ...
9	002452	1, 10, 91, 820, 7381, 66430, ...
10	002275	1, 11, 111, 1111, 11111, ...
11	016123	1, 12, 133, 1464, 16105, 177156, ...
12	016125	1, 13, 157, 1885, 22621, 271453, ...

Williams and Seah (1979) factored generalized repunits for $3 \le b \le 12$ and $2 \le n \le 1000$. A (base-10) repunit can be PRIME only if n is PRIME, since otherwise $10^{ab} - 1$ is a BINOMIAL NUMBER which can be factored algebraically. In fact, if $n = 2a$ is EVEN, then $10^{2a} - 1 = (10^a - 1)(10^a + 1)$. The only base-10 repunit PRIMES R_n for $n \le 16{,}500$ are $n = 2$, 19, 23, 317, and 1031 (Sloane's A004023; Madachy 1979, Williams and Dubner 1986, Ball and Coxeter 1987). The number of factors for the base-10 repunits for $n = 1, 2, \ldots$ are 1, 1, 2, 2, 2, 5, 2, 4, 4, 4, 2, 7, 3, ... (Sloane's A046053).

b	Sloane	n of Prime b-Repunits
2	000043	2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, ...
3	028491	3, 7, 13, 71, 103, 541, 1091, 1367, ...
5	004061	3, 7, 11, 13, 47, 127, 149, 181, 619, ...
6	004062	2, 3, 7, 29, 71, 127, 271, 509, 1049, ...
7	004063	5, 13, 131, 149, 1699, ...
10	004023	2, 19, 23, 317, 1031, ...
11	005808	17, 19, 73, 139, 907, 1907, 2029, 4801, ...
12	004064	2, 3, 5, 19, 97, 109, 317, 353, 701, ...

A table of the factors not obtainable algebraically for generalized repunits (a continuously updated version of Brillhart *et al.* 1988) is maintained online. These tables include factors for $10^n - 1$ (with $n \le 209$ odd) and $10^n + 1$ (for $n \le 210$ EVEN and ODD) in the files `ftp://sable.ox.ac.uk/pub/math/cunningham/10-` and `ftp://sable.ox.ac.uk/pub/math/cunningham/10+`. After algebraically factoring R_n, these are sufficient for complete factorizations. Yates (1982) published all the repunit factors for $n \le 1000$, a portion of which are reproduced in the *Mathematica*® notebook by Weisstein.

A SMITH NUMBER can be constructed from every factored repunit.

see also CUNNINGHAM NUMBER, FERMAT NUMBER, MERSENNE NUMBER, REPDIGIT, SMITH NUMBER

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 66, 1987.

Beiler, A. H. "11111...111." Ch. 11 in *Recreations in the Theory of Numbers: The Queen of Mathematics Entertains.* New York: Dover, 1966.

Brillhart, J.; Lehmer, D. H.; Selfridge, J.; Wagstaff, S. S. Jr.; and Tuckerman, B. *Factorizations of $b^n \pm 1$, $b = 2, 3, 5, 6, 7, 10, 11, 12$ Up to High Powers, rev. ed.* Providence, RI: Amer. Math. Soc., 1988. Updates are available electronically from `ftp://sable.ox.ac.uk/pub/math/cunningham`.

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Ribenboim, P. "Repunits and Similar Numbers." §5.5 in *The New Book of Prime Number Records.* New York: Springer-Verlag, pp. 350–354, 1996.

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Residual

The residual is the sum of deviations from a best-fit curve of arbitrary form.

$$R \equiv \sum [y_i - f(x_i, a_1, \ldots, a_n)]^2.$$

The residual should not be confused with the CORRELATION COEFFICIENT.

Residual vs. Predictor Plot

A plot of y_i vs. $e_i \equiv \hat{y}_i - y_i$. Random scatter indicates the model is probably good. A pattern indicates a problem with the model. If the spread in e_i increases as y_i increases, the errors are called HETEROSCEDASTIC.

Residue Class

The residue classes of a function $f(x)$ mod n are all possible values of the RESIDUE $f(x) \pmod{n}$. For example, the residue classes of $x^2 \pmod 6$ are $\{0, 1, 3, 4\}$, since

$$\begin{aligned} 0^2 &\equiv 0 \pmod 6 \\ 1^2 &\equiv 1 \pmod 6 \\ 3^2 &\equiv 3 \pmod 6 \\ 4^2 &\equiv 4 \pmod 6 \end{aligned}$$

are all the possible residues. A COMPLETE RESIDUE SYSTEM is a set of integers containing one element from each class, so in this case, $\{0, 1, 9, 4\}$ would be a COMPLETE RESIDUE SYSTEM.

The $\phi(m)$ residue classes prime to m form a GROUP under the binary multiplication operation (mod m), where $\phi(m)$ is the TOTIENT FUNCTION (Shanks 1993) and the GROUP is classed a MODULO MULTIPLICATION GROUP.

see also COMPLETE RESIDUE SYSTEM, CONGRUENCE, CUBIC NUMBER, QUADRATIC RECIPROCITY THEOREM, QUADRATIC RESIDUE, RESIDUE (CONGRUENCE), SQUARE NUMBER

References

Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, p. 56 and 59–63, 1993.

Residue (Complex Analysis)

The constant a_{-1} in the LAURENT SERIES

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

of $f(z)$ is called the residue of $f(z)$. The residue is a very important property of a complex function and appears in the amazing RESIDUE THEOREM of CONTOUR INTEGRATION.

see also CONTOUR INTEGRATION, LAURENT SERIES, RESIDUE THEOREM

References

Arfken, G. "Calculus of Residues." §7.2 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 400–421, 1985.

Residue (Congruence)

The number b in the CONGRUENCE $a \equiv b \pmod m$ is called the residue of $a \pmod m$. The residue of large numbers can be computed quickly using CONGRUENCES. For example, to find $37^{13} \pmod{17}$, note that

$$\begin{aligned} 37 &\equiv 3 \\ 37^2 &\equiv 3^2 \equiv 9 \equiv -8 \\ 37^4 &\equiv 81 \equiv -4 \\ 37^8 &\equiv 16 \equiv -1, \end{aligned}$$

so

$$37^{13} \equiv 37^{1+4+8} \equiv 3(-4)(-1) \equiv 12 \pmod{17}.$$

see also COMMON RESIDUE, CONGRUENCE, MINIMAL RESIDUE

References

Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, pp. 55–56, 1993.

Residue Index

$p-1$ divided by the HAUPT-EXPONENT of a base b mod p for a given PRIME p.

see also HAUPT-EXPONENT

Residue Theorem (Complex Analysis)

Given a complex function $f(z)$, consider the LAURENT SERIES

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n. \tag{1}$$

Integrate term by term using a closed contour γ encircling z_0,

$$\begin{aligned} \int_\gamma f(z)\,dz &= \sum_{n=-\infty}^{\infty} a_n \int_\gamma (z - z_0)^n\,dz \\ &= \sum_{n=-\infty}^{-2} a_n \int_\gamma (z - z_0)^n\,dz \\ &\quad + a_{-1} \int_\gamma \frac{dz}{z - z_0} + \sum_{n=0}^{\infty} a_n \int_\gamma (z - z_0)^n\,dz. \end{aligned} \tag{2}$$

The CAUCHY INTEGRAL THEOREM requires that the first and last terms vanish, so we have

$$\int_\gamma f(z)\,dz = a_{-1} \int_\gamma \frac{dz}{z - z_0}. \tag{3}$$

But we can evaluate this function (which has a POLE at z_0) using the CAUCHY INTEGRAL FORMULA,

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)\,dz}{z - z_0}. \tag{4}$$

This equation must also hold for the constant function $f(z) = 1$, in which case it is also true that $f(z_0) = 1$, so

$$1 = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}, \tag{5}$$

$$\int_\gamma \frac{dz}{z - z_0} = 2\pi i, \tag{6}$$

and (3) becomes

$$\int_\gamma f(z)\,dz = 2\pi i a_{-1}. \tag{7}$$

The quantity a_{-1} is known as the RESIDUE of $f(z)$ at z_0. Generalizing to a curve passing through multiple poles, (7) becomes

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{i=1}^{\text{poles in }\gamma} n(\gamma, z_0^{(i)}) a_{-1}^{(i)}, \tag{8}$$

where n is the WINDING NUMBER and the (i) superscript denotes the quantity corresponding to POLE i.

If the path does not completely encircle the RESIDUE, take the CAUCHY PRINCIPAL VALUE to obtain

$$\int f(z)\,dz = (\theta_2 - \theta_1) i a_{-1}. \tag{9}$$

If f has only ISOLATED SINGULARITIES, then

$$\sum_{z_0^{(i)} \in \mathbb{C}^*} a_{-1}{}^{(i)} = 0. \tag{10}$$

The residues may be found without explicitly expanding into a LAURENT SERIES as follows.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n. \tag{11}$$

If $f(z)$ has a POLE of order m at z_0, then $a_n = 0$ for $n < -m$ and $a_{-m} \neq 0$. Therefore,

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} a_{-m+n} (z - z_0)^{-m+n} \tag{12}$$

$$(z - z_0)^m f(z) = \sum_{n=0}^{\infty} a_{-m+n} (z - z_0)^n \tag{13}$$

$$\frac{d}{dz}[(z - z_0)^m f(z)] = \sum_{n=0}^{\infty} n a_{-m+n} (z - z_0)^{n-1}$$
$$= \sum_{n=1}^{\infty} n a_{-m+n} (z - z_0)^{n-1}$$
$$= \sum_{n=0}^{\infty} (n+1) a_{-m+n+1} (z - z_0)^n \tag{14}$$

$$\frac{d^2}{dz^2}[(z - z_0)^m f(z)] = \sum_{n=0}^{\infty} n(n+1) a_{-m+n+1} (z - z_0)^{n-1}$$
$$= \sum_{n=1}^{\infty} n(n+1) a_{-m+n+1} (z - z_0)^{n-1}$$
$$= \sum_{n=0}^{\infty} (n+1)(n+2) a_{-m+n+2} (z - z_0)^n. \tag{15}$$

Iterating,

$$\frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)]$$
$$= \sum_{n=0}^{\infty} (n+1)(n+2)(n+m-1) a_{n-1} (z - z_0)^n$$
$$= (m-1)! a_{-1}$$
$$+ \sum_{n=1}^{\infty} (n+1)(n+2)(n+m-1) a_{n-1} (z - z_0)^{n-1}. \tag{16}$$

So

$$\lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)]$$
$$= \lim_{z \to z_0} (m-1)! a_{-1} + 0 = (m-1)! a_{-1}, \tag{17}$$

and the RESIDUE is

$$a_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)]_{z=z_0}. \tag{18}$$

This amazing theorem says that the value of a CONTOUR INTEGRAL in the COMPLEX PLANE depends *only* on the properties of a few special points *inside* the contour.

see also CAUCHY INTEGRAL FORMULA, CAUCHY INTEGRAL THEOREM, CONTOUR INTEGRAL, LAURENT SERIES, POLE, RESIDUE (COMPLEX ANALYSIS)

Residue Theorem (Group)

If two groups are residual to a third, every group residual to one is residual to the other. The Gambier extension of this theorem states that if two groups are pseudoresidual to a third, then every group pseudoresidual to the first with an excess greater than or equal to the excess of the first minus the excess of the second is pseudoresidual to the second, with an excess ≥ 0.

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, pp. 30–31, 1959.

Resolution

Resolution is a widely used word with many different meanings. It can refer to resolution of equations, resolution of singularities (in ALGEBRAIC GEOMETRY), resolution of modules or more sophisticated structures, etc. In a BLOCK DESIGN, a PARTITION R of a BIBD's set of blocks B into PARALLEL CLASSES, each of which in turn partitions the set V, is called a resolution (Abel and Furino 1996).

A resolution of the MODULE M over the RING R is a complex of R-modules C_i and morphisms d_i and a MORPHISM ϵ such that

$$\cdots \to C_i \to^{d_i} C_{i-1} \to \cdots \to C_0 \to^{\epsilon} M \to 0$$

satisfying the following conditions:

1. The composition of any two consecutive morphisms is the zero map,
2. For all i, $(\ker d_i)/(\operatorname{im} d_{i+1}) = 0$,
3. $C_0/(\ker \epsilon) \simeq M$,

where ker is the kernel and im is the image. Here, the quotient

$$\frac{(\ker d_i)}{(\operatorname{im} d_{i+1})}$$

is the ith HOMOLOGY GROUP.

If all modules C_i are projective (free), then the resolution is called projective (free). There is a similar concept for resolutions "to the right" of M, which are called injective resolutions.

see also HOMOLOGY GROUP, MODULE, MORPHISM, RING

References
Abel, R. J. R. and Furino, S. C. "Resolvable and Near Resolvable Designs." §I.6 in *The CRC Handbook of Combinatorial Designs* (Ed. C. J. Colbourn and J. H. Dinitz). Boca Raton, FL: CRC Press, p. 4 and 87–94, 1996.
Jacobson, N. *Basic Algebra II, 2nd ed.* New York: W. H. Freeman, p. 339, 1989.

Resolution Class

see PARALLEL CLASS

Resolution Modulus

The least POSITIVE INTEGER m^* with the property that $\chi(y) = 1$ whenever $y \equiv 1 \pmod{m^*}$ and $(y, m) = 1$.

Resolvable

A balanced incomplete BLOCK DESIGN (B, V) is called resolvable if there exists a PARTITION R of its set of blocks B into PARALLEL CLASSES, each of which in turn partitions the set V. The partition R is called a RESOLUTION.

see also BLOCK DESIGN, PARALLEL CLASS

References
Abel, R. J. R. and Furino, S. C. "Resolvable and Near Resolvable Designs." §I.6 in *The CRC Handbook of Combinatorial Designs* (Ed. C. J. Colbourn and J. H. Dinitz). Boca Raton, FL: CRC Press, p. 4 and 87–94, 1996.

Resolving Tree

A tree of LINKS obtained by repeatedly choosing a crossing, applying the SKEIN RELATIONSHIP to obtain two simpler LINKS, and repeating the process. The DEPTH of a resolving tree is the number of levels of links, not including the top. The DEPTH of the LINK is the minimal depth for any resolving tree of that LINK.

Resonance Overlap

Isolated resonances in a DYNAMICAL SYSTEM can cause considerable distortion of preserved TORI in their NEIGHBORHOOD, but they do not introduce any CHAOS into a system. However, when two or more resonances are simultaneously present, they will render a system nonintegrable. Furthermore, if they are sufficiently "close" to each other, they will result in the appearance of widespread (large-scale) CHAOS.

To investigate this problem, Walker and Ford (1969) took the integrable Hamiltonian

$$H_0(I_1, I_2) = I_1 + I_2 - I_1^2 - 3I_1I_2 + {I_2}^2$$

and investigated the effect of adding a 2:2 resonance and a 3:2 resonance

$$H(\mathbf{I}, \theta) = H_0(\mathbf{I}) + \alpha I_1 I_2 \cos(2\theta_1 - 2\theta_2) + \beta {I_1}^{3/2} I_2 \cos(2\theta_1 - 3\theta_2).$$

At low energies, the resonant zones are well-separated. As the energy increases, the zones overlap and a "macroscopic zone of instability" appears. When the overlap starts, many higher-order resonances are also involved so fairly large areas of PHASE SPACE have their TORI destroyed and the ensuing CHAOS is "widespread" since trajectories are now free to wander between regions that previously were separated by nonresonant TORI.

Walker and Ford (1969) were able to numerically predict the energy at which the overlap of the resonances first occurred. They plotted the θ_2-axis intercepts of the inner 2:2 and the outer 2:3 separatrices as a function of total energy. The energy at which they crossed was found to be identical to that at which 2:2 and 2:3 resonance zones began to overlap.

see also CHAOS, RESONANCE OVERLAP METHOD

References
Walker, G. H. and Ford, J. "Amplitude Instability and Ergodic Behavior for Conservative Nonlinear Oscillator Systems." *Phys. Rev.* **188**, 416–432, 1969.

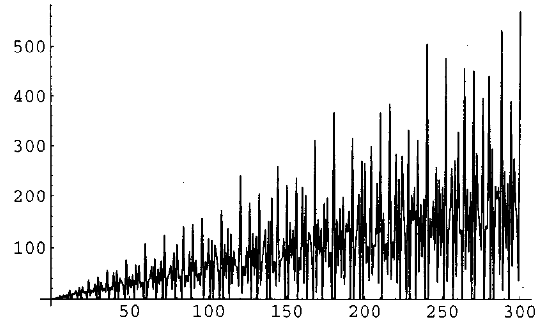
Resonance Overlap Method

A method for predicting the onset of widespread CHAOS.

see also GREENE'S METHOD

References
Chirikov, B. V. "A Universal Instability of Many-Dimensional Oscillator Systems." *Phys. Rep.* **52**, 264–379, 1979.
Tabor, M. *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, pp. 154–163, 1989.

Restricted Divisor Function



The sum of the ALIQUOT DIVISORS of n, given by

$$s(n) \equiv \sigma(n) - n,$$

where $\sigma(n)$ is the DIVISOR FUNCTION. The first few values are 0, 1, 1, 3, 1, 6, 1, 7, 4, 8, 1, 16, ... (Sloane's A001065).

see also DIVISOR FUNCTION

References
Sloane, N. J. A. Sequence A001065/M2226 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Restricted Growth Function

see RESTRICTED GROWTH STRING

Restricted Growth String

For a SET PARTITION of n elements, the n-character string $a_1a_2\ldots a_n$ in which each character gives the BLOCK ($\mathbf{B}_0$, $\mathbf{B}_1$, ...) into which the corresponding element belongs is called the restricted growth string (or sometimes the RESTRICTED GROWTH FUNCTION). For example, for the SET PARTITION $\{\{1\},\{2\},\{3,4\}\}$, the restricted growth string would be 0122. If the BLOCKS are "sorted" so that $a_1 = 0$, then the restricted growth string satisfies the INEQUALITY

$$a_{i+1} \leq 1 + \max\{a_1, a_2, \ldots, a_i\}$$

for $i = 1, 2, \ldots, n-1$.

References
Ruskey, F. "Info About Set Partitions." http://sue.csc.uvic.ca/~cos/inf/setp/SetPartitions.html.

Resultant

Given a POLYNOMIAL $p(x)$ of degree n with roots α_i, $i = 1, \ldots, n$ and a POLYNOMIAL $q(x)$ of degree m with roots β_j, $j = 1, \ldots, m$, the resultant is defined by

$$R(p,q) = \prod_{i=1}^{n}\prod_{j=1}^{m}(\beta_j - \alpha_i).$$

There exists an ALGORITHM similar to the EUCLIDEAN ALGORITHM for computing resultants (Pohst and Zassenhaus 1989). The resultant is the DETERMINANT of the corresponding SYLVESTER MATRIX. Given p and q, then

$$h(x) = R(q(t), p(x-t))$$

is a POLYNOMIAL of degree mn, having as its roots all sums of the form $\alpha_i + \beta_j$.

see also DISCRIMINANT (POLYNOMIAL), SYLVESTER MATRIX

References
Pohst, M. and Zassenhaus, H. *Algorithmic Algebraic Number Theory.* Cambridge, England: Cambridge University Press, 1989.
Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, p. 348, 1991.

Retardance

A shift in PHASE.

see also PHASE

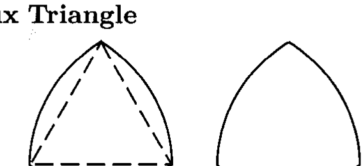
Reuleaux Polygon

A curvilinear polygon built up of circular ARCS. The Reuleaux polygon is a generalization of the REULEAUX TRIANGLE.

see also CURVE OF CONSTANT WIDTH, REULEAUX TRIANGLE

References
Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, pp. 52–54, 1991.

Reuleaux Triangle



A CURVE OF CONSTANT WIDTH constructed by drawing arcs from each VERTEX of an EQUILATERAL TRIANGLE between the other two VERTICES. It is the basis for the Harry Watt square drill bit. It has the smallest AREA for a given width of any CURVE OF CONSTANT WIDTH.

The AREA of each meniscus-shaped portion is

$$A = \tfrac{1}{6}\pi r^2 - \tfrac{1}{2}r\left(\frac{\sqrt{3}}{2}r\right) = \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)r^2, \qquad (1)$$

where we have subtracted the AREA of the wedge from that of the EQUILATERAL TRIANGLE. The total AREA is then

$$A = 3\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)r^2 + \frac{\sqrt{3}}{4}r^2 = \frac{\pi - \sqrt{3}}{2}r^2. \qquad (2)$$

When rotated in a square, the fractional AREA covered is

$$A_{\text{covered}} = 2\sqrt{3} + \tfrac{1}{6}\pi = 0.9877700392\ldots. \qquad (3)$$

The center does *not* stay fixed as the TRIANGLE is rotated, but moves along a curve composed of four arcs of an ELLIPSE (Wagon 1991).

see also CURVE OF CONSTANT WIDTH, FLOWER OF LIFE, PIECEWISE CIRCULAR CURVE, REULEAUX POLYGON

References
Bogomolny, A. "Shapes of Constant Width." http://www.cut-the-knot.com/do_you_know/cwidth.html.
Eppstein, D. "Reuleaux Triangles." http://www.ics.uci.edu/~eppstein/junkyard/reuleaux.html.
Reuleaux, F. *The Kinematics of Machinery.* New York: Dover, 1963.
Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, pp. 52–54 and 381–383, 1991.
Yaglom, I. M. and Boltyansky, B. G. *Convex Shapes.* Moscow: Nauka, 1951.

Reversal

The reversal of a decimal number $abc\cdots$ is $\cdots cba$. Ball and Coxeter (1987) consider numbers whose reversals are integral multiples of themselves. PALINDROMIC NUMBER and numbers ending with a ZERO are trivial examples. The first few nontrivial examples are 8712, 9801, 87912, 98901, 879912, 989901, 8799912, 9899901, 87128712, 87999912, 98019801, 98999901, ... (Sloane's A031877). The pattern continues for large numbers, with numbers of the form $87\underbrace{9\cdots9}12$ equal to 4 times their reversals and numbers of the form $98\underbrace{9\cdots9}01$ equal to 9 times their reversals. In addition, runs of numbers of either of these forms can be concatenated to yield numbers of the form $87\underbrace{9\cdots9}12\cdots87\underbrace{9\cdots9}12$, equal to 4 times their reversals, and $98\underbrace{9\cdots9}01\cdots98\underbrace{9\cdots9}01$, equal to 9 times their reversals.

The product of a 2-digit number and its reversal is never a SQUARE NUMBER except when the digits are the same (Ogilvy 1988). Numbers whose product is the reversal of the products of their reversals include

$$312\times 221 = 68952$$

$$213\times 122 = 25986$$

(Ball and Coxeter 1987, p. 14).

see also RATS SEQUENCE

References
Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 14–15, 1987.
Ogilvy, C. S. and Anderson, J. T. *Excursions in Number Theory.* New York: Dover, pp. 88–89, 1988.
Sloane, N. J. A. Sequence A031877 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Reverse Greedy Algorithm

An algorithm for computing a UNIT FRACTION.

see also GREEDY ALGORITHM, UNIT FRACTION

Reversion of Series

see SERIES REVERSION

Reverse-Then-Add Sequence

An integer sequence produced by the 196-ALGORITHM.

see also 196-ALGORITHM, SORT-THEN-ADD SEQUENCE

Reznik's Identity

For P and Q POLYNOMIALS in n variables,

$$|P\cdot Q|_2{}^2 = \sum_{i_1,\ldots,i_n\geq 0} \frac{|P^{i_1,\ldots,i_n)}(D_1,\ldots,D_n)Q(x_1,\ldots,x_n)|_2{}^2}{i_1!\cdots i_n!},$$

where $D_i \equiv \partial/\partial x_i$, $|X|_2$ is the BOMBIERI NORM, and

$$P^{(i_1,\ldots,i_n)} = D_1^{i_1}\cdots D_n^{i_n}P.$$

BOMBIERI'S INEQUALITY follows from this identity.

see also BEAUZAMY AND DÉGOT'S IDENTITY

Rhodonea

see ROSE

Rhomb

see RHOMBUS

Rhombic Dodecahedral Number

A FIGURATE NUMBER which is constructed as a centered CUBE with a SQUARE PYRAMID appended to each face,

$$RhoDod_n = CCub_n + 6P_{n-1} = (2n-1)(2n^2-2n+1),$$

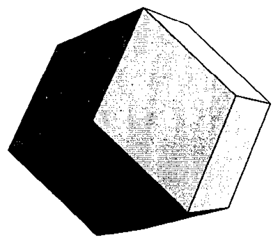
where $CCub_n$ is a CENTERED CUBE NUMBER and P_n is a PYRAMIDAL NUMBER. The first few are 1, 15, 65, 175, 369, 671, ... (Sloane's A005917). The GENERATING FUNCTION of the rhombic dodecahedral numbers is

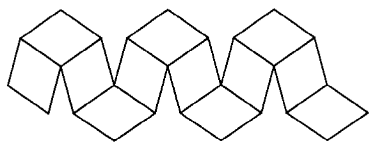
$$\frac{x(1+11x+11x^2+x^3)}{(x-1)^4} = x+15x^2+65x^3+175x^4+\ldots.$$

see also OCTAHEDRAL NUMBER

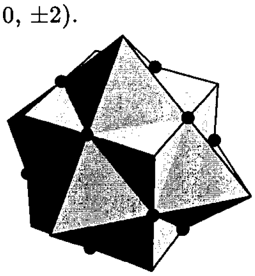
References
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 53–54, 1996.
Sloane, N. J. A. Sequence A005917/M4968 in "An On-Line Version of the Encyclopedia of Integer Sequences."

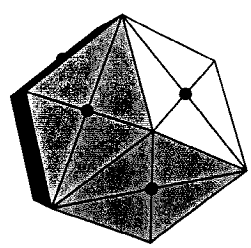
Rhombic Dodecahedron





The DUAL POLYHEDRON of the CUBOCTAHEDRON, also sometimes called the RHOMBOIDAL DODECAHEDRON (Cotton 1990). Its 14 vertices are joined by 12 RHOMBUSES, and one possible way to construct it is known as the BAUSPIEL. The rhombic dodecahedron is a ZONOHEDRON and a SPACE-FILLING POLYHEDRON. The vertices are given by (±1, ±1, ±1), (±2, 0, 0), (0, ±2, 0), (0, 0, ±2).





The edges of the CUBE-OCTAHEDRON COMPOUND intersecting in the points plotted above are the diagonals of RHOMBUSES, and the 12 RHOMBUSES form a rhombic dodecahedron (Ball and Coxeter 1987).

see also BAUSPIEL, CUBE-OCTAHEDRON COMPOUND, DODECAHEDRON, PYRITOHEDRON, RHOMBIC TRIACONTAHEDRON, RHOMBUS, TRIGONAL DODECAHEDRON, ZONOHEDRON

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 137, 1987.

Cotton, F. A. *Chemical Applications of Group Theory, 3rd ed.* New York: Wiley, p. 62, 1990.

Rhombic Icosahedron

A ZONOHEDRON which can be derived from the TRIACONTAHEDRON by removing any one of the zones and bringing together the two pieces into which the remainder of the surface is thereby divided.

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 143, 1987.

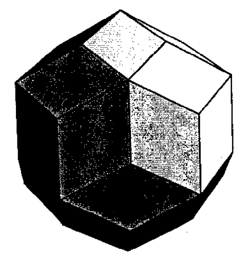
Bilinski, S. "Über die Rhombenisoeder." *Glasnik Mat.-Fiz. Astron. Društro Mat. Fiz. Hrvatske Ser. II* **15**, 251–263, 1960.

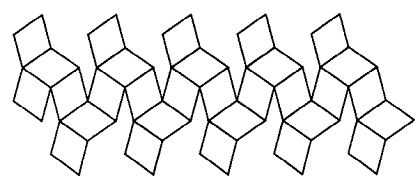
Rhombic Polyhedron

A POLYHEDRON with extra square faces, given by the SCHLÄFLI SYMBOL $r\left\{ {p \atop q} \right\}$.

see also RHOMBIC DODECAHEDRON, RHOMBIC ICOSAHEDRON, RHOMBIC TRIACONTAHEDRON, SNUB POLYHEDRON, TRUNCATED POLYHEDRON

Rhombic Triacontahedron





A ZONOHEDRON which is the DUAL POLYHEDRON of the ICOSIDODECAHEDRON. It is composed of 30 RHOMBUSES joined at 32 vertices. Ede (1958) enumerates 13 basic series of stellations of the rhombic triacontahedron, the total number of which is extremely large. Messer (1995) describes 226 stellations. The intersecting edges of the DODECAHEDRON-ICOSAHEDRON COMPOUND form the diagonals of 30 RHOMBUSES which comprise the TRIACONTAHEDRON. The CUBE 5-COMPOUND has the 30 facial planes of the rhombic triacontahedron (Ball and Coxeter 1987).

see also CUBE 5-COMPOUND, DODECAHEDRON-ICOSAHEDRON COMPOUND, ICOSIDODECAHEDRON, RHOMBIC DODECAHEDRON, RHOMBUS, ZONOHEDRON

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 137, 1987.

Bulatov, V.v "Stellations of Rhombic Triacontahedron." `http://www.physics.orst.edu/~bulatov/polyhedra/rtc/`.

Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., p. 127, 1989.

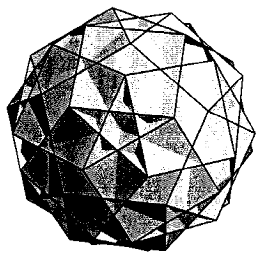
Ede, J. D. "Rhombic Triacontahedra." *Math. Gazette* **42**, 98–100, 1958.

Messer, P. W. "Les étoilements du rhombitricontaèdre et plus." *Structural Topology* No. 21, 25–46, 1995.

Rhombicosacron

The DUAL POLYHEDRON of the RHOMBICOSAHEDRON.

Rhombicosahedron



The UNIFORM POLYHEDRON U_{56} whose DUAL POLYHEDRON is the RHOMBICOSACRON. It has WYTHOFF SYMBOL $2\,3\,\frac{\frac{5}{4}}{\frac{5}{2}}$. Its faces are $20\{6\} + 30\{4\}$. The CIRCUMRADIUS for unit edge length is

$$R = \tfrac{1}{2}\sqrt{7}.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 149–150, 1971.

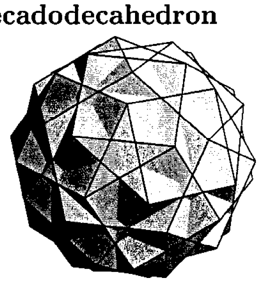
Rhombicosidodecahedron

see BIGYRATE DIMINISHED RHOMBICOSIDODECAHEDRON, DIMINISHED RHOMBICOSIDODECAHEDRON, GREAT RHOMBICOSIDODECAHEDRON (ARCHIMEDEAN), GREAT RHOMBICOSIDODECAHEDRON (UNIFORM), GYRATE BIDIMINISHED RHOMBICOSIDODECAHEDRON, GYRATE RHOMBICOSIDODECAHEDRON, METABIDIMINISHED RHOMBICOSIDODECAHEDRON, METABIGYRATE RHOMBICOSIDODECAHEDRON, METAGYRATE DIMINISHED RHOMBICOSIDODECAHEDRON, PARABIDIMINISHED RHOMBICOSIDODECAHEDRON, PARABIGYRATE RHOMBICOSIDODECAHEDRON, PARAGYRATE DIMINISHED RHOMBICOSIDODECAHEDRON, SMALL RHOMBICOSIDODECAHEDRON, TRIDIMINISHED RHOMBICOSIDODECAHEDRON, TRIGYRATE RHOMBICOSIDODECAHEDRON

Rhombicuboctahedron

see GREAT RHOMBICUBOCTAHEDRON (ARCHIMEDEAN), GREAT RHOMBICUBOCTAHEDRON (UNIFORM), SMALL RHOMBICUBOCTAHEDRON

Rhombidodecadodecahedron



The UNIFORM POLYHEDRON U_{38} whose DUAL POLYHEDRON is the MEDIAL DELTOIDAL HEXECONTAHEDRON. It has SCHLÄFLI SYMBOL $r\left\{\begin{matrix}\frac{5}{2}\\ 5\end{matrix}\right\}$ and WYTHOFF SYMBOL $\frac{5}{2}\,5\,|\,2$. Its faces are $12\{\frac{5}{2}\} + 30\{4\} + 12\{5\}$. The CIRCUMRADIUS for unit edge length is

$$R = \tfrac{1}{2}\sqrt{7}.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 116–117, 1989.

Rhombihexacron

see GREAT RHOMBIHEXACRON, SMALL RHOMBIHEXACRON

Rhombihexahedron

see GREAT RHOMBIHEXAHEDRON, SMALL RHOMBIHEXAHEDRON

Rhombitruncated Cuboctahedron

see GREAT RHOMBICUBOCTAHEDRON (ARCHIMEDEAN)

Rhombitruncated Icosidodecahedron

see GREAT RHOMBICOSIDODECAHEDRON (ARCHIMEDEAN)

Rhombohedron

A PARALLELEPIPED bounded by six congruent RHOMBS.

see also PARALLELEPIPED, RHOMB

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 142 and 161, 1987.

Rhomboid

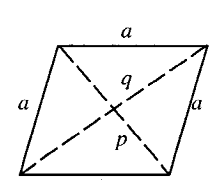
A PARALLELOGRAM in which angles are oblique and adjacent sides are of unequal length.

see also DIAMOND, LOZENGE, PARALLELOGRAM, QUADRILATERAL, RHOMBUS, SKEW QUADRILATERAL, TRAPEZIUM, TRAPEZOID

Rhomboidal Dodecahedron

see RHOMBIC DODECAHEDRON

Rhombus



A QUADRILATERAL with both pairs of opposite sides PARALLEL and all sides the same length, i.e., an equilateral PARALLELOGRAM. The word RHOMB is sometimes used instead of rhombus. The DIAGONALS p and q of a rhombus satisfy

$$p^2 + q^2 = 4a^2,$$

and the AREA is

$$A = \tfrac{1}{2}pq.$$

A rhombus whose ACUTE ANGLES are 45° is called a LOZENGE.

see also DIAMOND, LOZENGE, PARALLELOGRAM, QUADRILATERAL, RHOMBIC DODECAHEDRON, RHOMBIC ICOSAHEDRON, RHOMBIC TRIACONTAHEDRON, RHOMBOID, SKEW QUADRILATERAL, TRAPEZIUM, TRAPEZOID

References

Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 123, 1987.

Rhumb Line

see LOXODROME

Ribbon Knot

If the KNOT K is the boundary $K = f(\mathbb{S}^1)$ of a singular disk $f: \mathbb{D} \to \mathbb{S}^3$ which has the property that each self-intersecting component is an arc $A \subset f(\mathbb{D}^2)$ for which $f^{-1}(A)$ consists of two arcs in $\mathbb{D}^2$, one of which is interior, then K is said to be a ribbon knot. Every ribbon knot is a SLICE KNOT, and it is conjectured that every SLICE KNOT is a ribbon knot.

see also SLICE KNOT

References

Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, p. 225, 1976.

Ribet's Theorem

If the TANIYAMA-SHIMURA CONJECTURE holds for all semistable ELLIPTIC CURVES, then FERMAT'S LAST THEOREM is true. Before its proof by Ribet in 1986, the theorem had been called the epsilon conjecture. It had its roots in a surprising result of G. Frey.

see also ELLIPTIC CURVE, FERMAT'S LAST THEOREM, MODULAR FORM, MODULAR FUNCTION, TANIYAMA-SHIMURA CONJECTURE

Riccati-Bessel Functions

$$S_n(z) \equiv zj_n(z) = \sqrt{\frac{\pi z}{2}}J_{n+1/2}(z)$$

$$C_n(z) \equiv -zn_n(z) = -\sqrt{\frac{\pi z}{2}}N_{n+1/2}(z),$$

where $j_n(z)$ and $n_n(z)$ are SPHERICAL BESSEL FUNCTIONS OF THE FIRST and SECOND KIND.

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Riccati-Bessel Functions." §10.3 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 445, 1972.

Riccati Differential Equation

$$y' = P(z) + Q(z)y + R(z)y^2, \tag{1}$$

where $y' \equiv dy/dz$. The transformation

$$w \equiv -\frac{y'}{yR(z)} \tag{2}$$

leads to the second-order linear homogeneous equation

$$R(z)y'' - [R'(z) + Q(z)R(z)]y' + [R(z)]^2P(z)y = 0. \tag{3}$$

Another equation sometimes called the Riccati differential equation is

$$z^2w'' + [z^2 - n(n+1)]w = 0, \tag{4}$$

which has solutions

$$w = Azj_n(z) + Bzy_n(z). \tag{5}$$

Yet another form of "the" Riccati differential equation is

$$\frac{dy}{dz} = az^n + by^2, \tag{6}$$

which is solvable by algebraic, exponential, and logarithmic functions only when $n = -4m/(2m \pm 1)$, for $m = 0$, 1, 2, ...

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Riccati-Bessel Functions." §10.3 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 445, 1972.
Glaisher, J. W. L. "On Riccati's Equation." *Quart. J. Pure Appl. Math.* **11**, 267–273, 1871.

Ricci Curvature

The mathematical object which controls the growth rate of the volume of metric balls in a MANIFOLD.

see also BISHOP'S INEQUALITY, MILNOR'S THEOREM

Ricci Tensor

$$R_{\mu\kappa} \equiv R^\lambda{}_{\mu\lambda\kappa},$$

where $R^\lambda{}_{\mu\lambda\kappa}$ is the RIEMANN TENSOR.

see also CURVATURE SCALAR, RIEMANN TENSOR

Rice Distribution

$$P(Z) = \frac{Z}{\sigma^2}\exp\left(-\frac{Z^2 + |V|^2}{2\sigma^2}\right) I_0\left(\frac{Z|V|}{\sigma^2}\right),$$

where $I_0(z)$ is a MODIFIED BESSEL FUNCTION OF THE FIRST KIND and $Z > 0$. For a derivation, see Papoulis (1962). For $|V| = 0$, this reduces to the RAYLEIGH DISTRIBUTION.

see also RAYLEIGH DISTRIBUTION

References

Papoulis, A. *The Fourier Integral and Its Applications.* New York: McGraw-Hill, 1962.

Richard's Paradox

It is possible to describe a set of POSITIVE INTEGERS that cannot be listed in a book containing a set of counting numbers on each consecutively numbered page.

Richardson Extrapolation

The consideration of the result of a numerical calculation as a function of an adjustable parameter (usually the step size). The function can then be fitted and evaluated at $h = 0$ to yield very accurate results. Press *et al.* (1992) describe this process as turning lead into gold. Richardson extrapolation is one of the key ideas used in the popular and robust BULIRSCH-STOER ALGORITHM of solving ORDINARY DIFFERENTIAL EQUATIONS.

see also BULIRSCH-STOER ALGORITHM

References

Acton, F. S. *Numerical Methods That Work, 2nd printing.* Washington, DC: Math. Assoc. Amer., p. 106, 1990.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Richardson Extrapolation and the Bulirsch-Stoer Method." §16.4 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 718–725, 1992.

Richardson's Theorem

Let R be the class of expressions generated by

1. The RATIONAL NUMBERS and the two REAL NUMBERS π and $\ln 2$,
2. The variable x,
3. The operations of ADDITION, MULTIPLICATION, and composition, and
4. The SINE, EXPONENTIAL, and ABSOLUTE VALUE functions.

Then if $E \in R$, the predicate "$E = 0$" is recursively UNDECIDABLE.

see also RECURSION, UNDECIDABLE

References

Caviness, B. F. "On Canonical Forms and Simplification." *J. Assoc. Comp. Mach.* **17**, 385–396, 1970.

Petkovšek, M.; Wilf, H. S.; and Zeilberger, D. *A=B.* Wellesley, MA: A. K. Peters, 1996.

Richardson, D. "Some Unsolvable Problems Involving Elementary Functions of a Real Variable." *J. Symbolic Logic* **33**, 514–520, 1968.

Ridders' Method

A variation of the FALSE POSITION METHOD for finding ROOTS which fits the function in question with an exponential.

see also FALSE POSITION METHOD

References

Ostrowski, A. M. Ch. 12 in *Solutions of Equations and Systems of Equations, 2nd ed.* New York: Academic Press, 1966.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Secant Method, False Position Method, and Ridders' Method." §9.2 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 347–352, 1992.

Ralston, A. and Rabinowitz, P. §8.3 in *A First Course in Numerical Analysis, 2nd ed.* New York: McGraw-Hill, 1978.

Ridders, C. F. J. "A New Algorithm for Computing a Single Root of a Real Continuous Function." *IEEE Trans. Circuits Systems* **26**, 979–980, 1979.

Ridge

An $(n-2)$-D FACE of an n-D POLYTOPE.

see also POLYTOPE

Riemann-Christoffel Tensor

see RIEMANN TENSOR

Riemann Curve Theorem

If two algebraic plane curves with only ordinary singular points and CUSPS are related such that the coordinates of a point on either are RATIONAL FUNCTIONS of a corresponding point on the other, then the curves have the same GENUS (CURVE). This can be stated equivalently as the GENUS of a curve is unaltered by a BIRATIONAL TRANSFORMATION.

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 120, 1959.

Riemann Differential Equation

see RIEMANN P-DIFFERENTIAL EQUATION

Riemann's Formula

$$J(x) = \mathrm{Li}(x) - \sum \mathrm{Li}(x^\rho) + \ln 2 \int_x^\infty \frac{dt}{t(t^2-1)\ln t},$$

where $\mathrm{Li}(x)$ is the LOGARITHMIC INTEGRAL, the sum is taken over all nontrivial zeros ρ (i.e., those other than $-2, -4, \ldots$) of the RIEMANN ZETA FUNCTION $\zeta(s)$, and $J(x)$ is RIEMANN WEIGHTED PRIME-POWER COUNTING FUNCTION.

see also LOGARITHMIC INTEGRAL, PRIME NUMBER THEOREM, RIEMANN WEIGHTED PRIME-POWER COUNTING FUNCTION, RIEMANN ZETA FUNCTION

Riemann Function

The function obtained by approximating the RIEMANN WEIGHTED PRIME-POWER COUNTING FUNCTION J_2 in

$$\pi(x) = \sum_{n=1}^\infty \frac{\mu(n)}{n} J_2(x^{1/n}) \tag{1}$$

by the LOGARITHMIC INTEGRAL $\mathrm{Li}(x)$. This gives

$$R(n) \equiv 1 + \sum_{k=1}^{\infty} \frac{1}{k\zeta(k+1)} \frac{(\ln n)^k}{k!} \tag{2}$$

$$= \sum_{m=1}^{\infty} \frac{\mu(m)}{m} \mathrm{Li}(n^{1/m}), \tag{3}$$

where $\zeta(z)$ is the RIEMANN ZETA FUNCTION, $\mu(n)$ is the MÖBIUS FUNCTION, and $\mathrm{Li}(x)$ is the LOGARITHMIC INTEGRAL. Then

$$\pi(x) = R(x) - \sum_{\rho} R(x^{\rho}), \tag{4}$$

where π is the PRIME COUNTING FUNCTION. Ramanujan independently derived the formula for $R(n)$, but nonrigorously (Berndt 1994, p. 123).

see also MANGOLDT FUNCTION, PRIME NUMBER THEOREM, RIEMANN-MANGOLDT FUNCTION, RIEMANN ZETA FUNCTION

References

Berndt, B. C. *Ramanujan's Notebooks, Part IV.* New York: Springer-Verlag, 1994.
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 144–145, 1996.
Riesel, H. and Göhl, G. "Some Calculations Related to Riemann's Prime Number Formula." *Math. Comput.* **24**, 969–983, 1970.
Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, pp. 28–29 and 362–372, 1991.

Riemann Hypothesis

First published in Riemann (1859), the Riemann hypothesis states that the nontrivial ROOTS of the RIEMANN ZETA FUNCTION

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

where $s \in \mathbb{C}$ (the COMPLEX NUMBERS), all lie on the "CRITICAL LINE" $\Re[s] = 1/2$, where $\Re[z]$ denotes the REAL PART of z. The Riemann hypothesis is also known as ARTIN'S CONJECTURE.

In 1914, Hardy proved that an INFINITE number of values for s can be found for which $\zeta(s) = 0$ and $\Re[s] = 1/2$. However, it is not known if *all* nontrivial roots s satisfy $\Re[s] = 1/2$, so the conjecture remains open. André Weil proved the Riemann hypothesis to be true for field functions (Weil 1948, Eichler 1966, Ball and Coxeter 1987). In 1974, Levin showed that at least 1/3 of the ROOTS must lie on the CRITICAL LINE (Le Lionnais 1983), a result which has since been sharpened to 40% (Vardi 1991, p. 142). It is known that the zeros are symmetrical placed about the line $\Im[s] = 0$.

The Riemann hypothesis is equivalent to $\Lambda \leq 0$, where Λ is the DE BRUIJN-NEWMAN CONSTANT (Csordas *et al.* 1994). It is also equivalent to the assertion that for some constant c,

$$|\mathrm{Li}(x) - \pi(x)| \leq c\sqrt{x}\ln x, \tag{2}$$

where $\mathrm{Li}(x)$ is the LOGARITHMIC INTEGRAL and π is the PRIME COUNTING FUNCTION (Wagon 1991).

The hypothesis was computationally tested and found to be true for the first 2×10^8 zeros by Brent *et al.* (1979), a limit subsequently extended to the first $1.5 \times 10^9 + 1$ zeros by Brent *et al.* (1979). Brent's calculation covered zeros $\sigma + it$ in the region $0 < t < 81,702,130.19$.

There is also a finite analog of the Riemann hypothesis concerning the location of zeros for function fields defined by equations such as

$$ay^l + bz^m + c = 0. \tag{3}$$

This hypothesis, developed by Weil, is analogous to the usual Riemann hypothesis. The number of solutions for the particular cases $(l, m) = (2, 2)$, (3,3), (4,4), and (2,4) were known to Gauss.

see also CRITICAL LINE, EXTENDED RIEMANN HYPOTHESIS, GRONWALL'S THEOREM, MERTENS CONJECTURE, MILLS' CONSTANT, RIEMANN ZETA FUNCTION

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 75, 1987.
Brent, R. P.; Vandelune, J.; te Riele, H. J. J.; and Winter, D. T. "On the Zeros of the Riemann Zeta Function in the Critical Strip. I." *Math. Comput.* **33**, 1361–1372, 1979.
Brent, R. P.; Vandelune, J.; te Riele, H. J. J.; and Winter, D. T. "On the Zeros of the Riemann Zeta Function in the Critical Strip. II." *Math. Comput.* **39**, 681–688, 1982. Abstract available at `ftp://nimbus.anu.edu.au/pub/Brent/rpb070a.dvi.Z`.
Csordas, G.; Smith, W.; and Varga, R. S. "Lehmer Pairs of Zeros, the de Bruijn-Newman Constant and the Riemann Hypothesis." *Constr. Approx.* **10**, 107–129, 1994.
Eichler, M. *Introduction to the Theory of Algebraic Numbers and Functions.* New York: Academic Press, 1966.
Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 25, 1983.
Odlyzko, A. "The 10^{20}th Zero of the Riemann Zeta Function and 70 Million of Its Neighbors."
Riemann, B. "Über die Anzahl der Primzahlen unter einer gegebenen Grösse," *Mon. Not. Berlin Akad.*, pp. 671–680, Nov. 1859.
Sloane, N. J. A. Sequence A002410/M4924 in "An On-Line Version of the Encyclopedia of Integer Sequences."
Vandelune, J. and te Riele, H. J. J. "On The Zeros of the Riemann Zeta-Function in the Critical Strip. III." *Math. Comput.* **41**, 759–767, 1983.
Vandelune, J.; te Riele, H. J. J.; and Winter, D. T. "On the Zeros of the Riemann Zeta Function in the Critical Strip. IV." *Math. Comput.* **46**, 667–681, 1986.
Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, p. 33, 1991.
Weil, A. *Sur les courbes algébriques et les variét'es qui s'en déduisent.* Paris, 1948.

Riemann Integral

The Riemann integral is the INTEGRAL normally encountered in CALCULUS texts and used by physicists and engineers. Other types of integrals exist (e.g., the LEBESGUE INTEGRAL), but are unlikely to be encountered outside the confines of advanced mathematics texts.

The Riemann integral is based on the JORDAN MEASURE, and defined by taking a limit of a RIEMANN SUM,

$$\int_b^a f(x)\,dx \equiv \lim_{\max \Delta x_k \to 0} \sum_{k=1}^n f(x_k^*)\Delta x_k \tag{1}$$

$$\iint f(x,y)\,dA \equiv \lim_{\max \Delta A_k \to 0} \sum_{k=1}^n f(x_k^*, y_k^*)\Delta A_k \tag{2}$$

$$\iiint f(x,y,z)\,dV \equiv \lim_{\max \Delta V_k \to 0} \sum_{k=1}^n f(x_k^*, y_k^*, z_k^*)\Delta V_k, \tag{3}$$

where $a \leq x \leq b$ and x_k^*, y_k^*, and z_k^* are arbitrary points in the intervals Δx_k, Δy_k, and Δz_k, respectively. The value $\max \Delta x_k$ is called the MESH SIZE of a partition of the interval $[a,b]$ into subintervals Δx_k.

As an example of the application of the Riemann integral definition, find the AREA under the curve $y = x^r$ from 0 to a. Divide (a,b) into n segments, so $\Delta x_k = \frac{b-a}{n} \equiv h$, then

$$f(x_1) = f(0) = 0 \tag{4}$$

$$f(x_2) = f(\Delta x_k) = h^r \tag{5}$$

$$f(x_3) = f(2\Delta x_k) = (2h)^r. \tag{6}$$

By induction

$$f(x_k) = f([k-1]\Delta x_k) = [(k-1)h]^r = h^r(k-1)^r, \tag{7}$$

so

$$f(x_k)\Delta x_k = h^{r+1}(k-1)^r \tag{8}$$

$$\sum_{k=1}^n f(x_k)\Delta x_k = h^{r+1}\sum_{k=1}^n (k-1)^r. \tag{9}$$

For example, take $r = 2$.

$$\begin{aligned}\sum_{k=1}^n f(x_k)\Delta x_k &= h^3 \sum_{k=1}^n (k-1)^2 \\ &= h^3 \left(\sum_{k=1}^n k^2 - 2\sum_{k=1}^n k + \sum_{k=1}^n 1 \right) \\ &= h^3 \left[\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + n \right],\end{aligned} \tag{10}$$

so

$$\begin{aligned} I &\equiv \lim_{n\to\infty} \sum_{k=1}^n f(x_k{}^*)\Delta x_k = \lim_{n\to\infty}\sum_{k=1}^n f(x_k)\Delta x_k \\ &= \lim_{n\to\infty} h^3 \left[\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + n \right] \\ &= a^3 \lim_{n\to\infty} \left[\frac{n(n+1)(2n+1)}{6n^3} - \frac{n(n+1)}{n^3} + \frac{n}{n^3} \right] \\ &= \tfrac{1}{3}a^3. \end{aligned} \tag{11}$$

see also INTEGRAL, RIEMANN SUM

References
Kestelman, H. "Riemann Integration." Ch. 2 in *Modern Theories of Integration, 2nd rev. ed.* New York: Dover, pp. 33–66, 1960.

Riemann's Integral Theorem

Associated with an irreducible curve of GENUS (CURVE) p, there are p LINEARLY INDEPENDENT integrals of the first sort. The ROOTS of the integrands are groups of the canonical series, and every such group will give rise to exactly one integral of the first sort.

References
Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 274, 1959.

Riemann-Lebesgue Lemma

Sometimes also called MERCER'S THEOREM.

$$\lim_{n\to\infty} \int_a^b K(\lambda, z) C \sin(nz)\,dz = 0$$

for arbitrarily large C and "nice" $K(\lambda, z)$. Gradshteyn and Ryzhik (1979) state the lemma as follows. If $f(x)$ is integrable on $[\pi, \pi]$, then

$$\lim_{t\to\infty} \int_{-\pi}^{\pi} f(x)\sin(tx)\,dx \to 0$$

and

$$\lim_{t\to\infty} \int_{-\pi}^{\pi} f(x)\cos(tx)\,dx \to 0.$$

References
Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1101, 1979.

Riemann-Mangoldt Function

$$\begin{aligned} f(x) &= \sum_{n>1}^{\infty} \frac{\pi_0(x^{1/n})}{n} \\ &= \mathrm{Li}(x) - \sum_{\substack{\text{nontrivial } \rho \\ \zeta(\rho)=0}} \mathrm{ei}(\rho \ln x) - \ln 2 \\ &\quad + \int_x^\infty \frac{dt}{t(t^2-1)\ln t}, \end{aligned} \tag{1}$$

where $\zeta(z)$ is the RIEMANN ZETA FUNCTION, $\mathrm{Li}(x)$ is the LOGARITHMIC INTEGRAL and $\mathrm{ei}(x)$ is the EXPONENTIAL INTEGRAL. The MANGOLDT FUNCTION is given by

$$\Lambda(n) = \begin{cases} \ln p & \text{if } n = p^m \text{ for } (m \geq 1) \text{ and } p \text{ prime} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$-\frac{\zeta'(x)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \tag{3}$$

for $\Re[s] > 1$.

$$J(x) = \sum_{n \leq x} \frac{\Lambda(n)}{\ln n}. \tag{4}$$

The SUMMATORY Riemann-Mangoldt function is defined by

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \theta(x) + \theta(x^{1/2}) + \ldots. \tag{5}$$

see also PRIME NUMBER THEOREM, RIEMANN FUNCTION

References

Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, pp. 364–365, 1991.

Riemann Mapping Theorem

Let z_0 be a point in a simply connected region $R \neq \mathbb{C}$. Then there is a unique ANALYTIC FUNCTION $w = f(z)$ mapping R one-to-one onto the DISK $|w| < 1$ such that $f(z_0) = 0$ and $f'(z_0) = 0$. The COROLLARY guarantees that any two simply connected regions except $\mathbb{R}^2$ can be mapped CONFORMALLY onto each other.

Riemann's Moduli Problem

Find an ANALYTIC parameterization of the compact RIEMANN SURFACES in a fixed HOMOMORPHISM class. The AHLFORS-BERS THEOREM proved that RIEMANN'S MODULI SPACE gives the solution.

see also AHLFORS-BERS THEOREM, RIEMANN'S MODULI SPACE

Riemann's Moduli Space

Riemann's moduli space R_p is the space of ANALYTIC EQUIVALENCE CLASSES of RIEMANN SURFACES of fixed GENUS p.

see also AHLFORS-BERS THEOREM, RIEMANN'S MODULI PROBLEM, RIEMANN SURFACE

Riemann P-Differential Equation

$$\frac{d^2u}{dz^2} + \left[\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c}\right]\frac{du}{dz}$$
$$+ \left[\frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-c)(b-a)}{z-b} + \frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right]\frac{u}{(z-a)(z-b)(z-c)} = 0,$$

where

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1.$$

Solutions are RIEMANN P-SERIES (Abramowitz and Stegun 1972, pp. 564–565).

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Riemann's Differential Equation." §15.6 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 564–565, 1972.

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 541–543, 1953.

Riemann P-Series

The solutions to the RIEMANN P-DIFFERENTIAL EQUATION

$$z = P\begin{Bmatrix} a & b & c \\ \alpha & \beta & \gamma\ ;\ z \\ \alpha' & \beta' & \gamma' \end{Bmatrix}.$$

Solutions are given in terms of the HYPERGEOMETRIC FUNCTION by

$$u_1 = \left(\frac{z-a}{z-b}\right)^\alpha \left(\frac{z-c}{z-b}\right)^\gamma \times {}_2F_1(\alpha+\beta+\gamma, \alpha+\beta'+\gamma; 1+\alpha-\alpha'; \lambda)$$

$$u_2 = \left(\frac{z-a}{z-b}\right)^{\alpha'} \left(\frac{z-c}{z-b}\right)^\gamma \times {}_2F_1(\alpha'+\beta+\gamma, \alpha'+\beta'+\gamma; 1+\alpha'-\alpha; \lambda)$$

$$u_3 = \left(\frac{z-a}{z-b}\right)^\alpha \left(\frac{z-c}{z-b}\right)^{\gamma'} \times {}_2F_1(\alpha+\beta+\gamma', \alpha+\beta'+\gamma'; 1+\alpha-\alpha'; \lambda)$$

$$u_4 = \left(\frac{z-a}{z-b}\right)^{\alpha'} \left(\frac{z-c}{z-b}\right)^{\gamma'} \times {}_2F_1(\alpha'+\beta+\gamma', \alpha'+\beta'+\gamma'; 1+\alpha'-\alpha; \lambda),$$

where

$$\lambda \equiv \frac{(z-a)(c-b)}{(z-b)(c-a)}.$$

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Riemann's Differential Equation." §15.6 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 564–565, 1972.

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Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis, 4th ed.* Cambridge, England: Cambridge University Press, pp. 283–284, 1990.

Riemann-Roch Theorem

The dimension of a complete series is equal to the sum of the order and index of specialization of any group, less the GENUS of the base curve

$$r = N + i + p.$$

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 261, 1959.

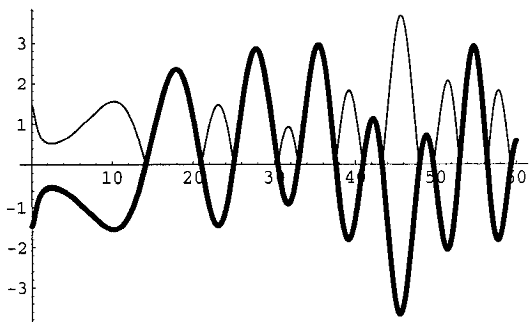
Riemann Series Theorem

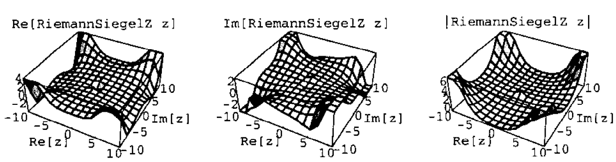
By a suitable rearrangement of terms, a conditionally convergent SERIES may be made to converge to any desired value, or to DIVERGE.

References

Bromwich, T. J. I'a. and MacRobert, T. M. *An Introduction to the Theory of Infinite Series, 3rd ed.* New York: Chelsea, p. 74, 1991.

Riemann-Siegel Functions

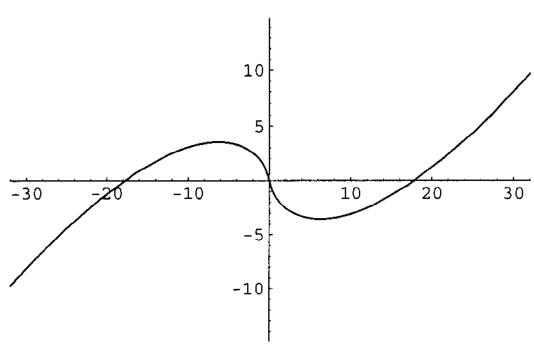


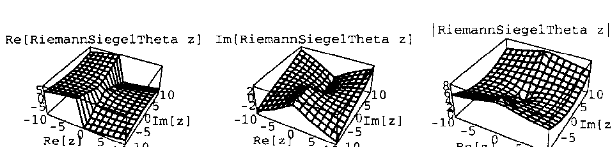


For a REAL POSITIVE t, the Riemann-Siegel Z function is defined by

$$Z(t) \equiv e^{i\vartheta(t)}\zeta(\tfrac{1}{2}+it).$$

The top plot superposes $Z(t)$ (thick line) on $|\zeta(\frac{1}{2}+it)|$, where $\zeta(z)$ is the RIEMANN ZETA FUNCTION.





The Riemann-Siegel theta function appearing above is defined by

$$\begin{aligned}\vartheta &\equiv \Im[\ln\Gamma(\tfrac{1}{4}+\tfrac{1}{2}it)-\tfrac{1}{2}t\ln\pi]\\ &= \arg[\Gamma(\tfrac{1}{4}+\tfrac{1}{2}it)]-\tfrac{1}{2}t\ln\pi.\end{aligned}$$

These functions are implemented in *Mathematica®* (Wolfram Research, Champaign, IL) as `RiemannSiegelZ[z]` and `RiemannSiegelTheta[z]`, illustrated above.

see also RIEMANN ZETA FUNCTION

References

Vardi, I. *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, p. 143, 1991.

Riemann Space

see METRIC SPACE

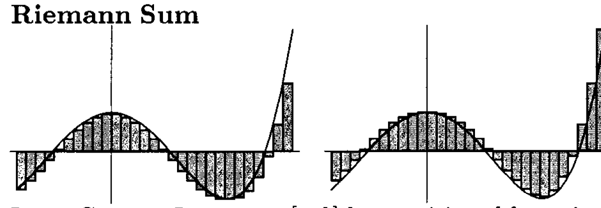
Riemann Sphere

A 1-D COMPLEX MANIFOLD $\mathbb{C}^*$, which is the one-point compactification of the COMPLEX numbers $\mathbb{C}\cup\{\infty\}$, together with two charts. For all points in the COMPLEX PLANE, the chart is the IDENTITY MAP from the SPHERE (with infinity removed) to the COMPLEX PLANE. For the point at infinity, the chart neighborhood is the sphere (with the ORIGIN removed), and the chart is given by sending infinity to 0 and all other points z to $1/z$.

Riemann-Stieltjes Integral

see STIELTJES INTEGRAL

Riemann Sum



Let a CLOSED INTERVAL $[a, b]$ be partitioned by points $a < x_1 < x_2 < \ldots < x_{n-1} < b$, the lengths of the resulting intervals between the points are denoted Δx_1, Δx_2, ..., Δx_n. Then the quantity

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

is called a Riemann sum for a given function $f(x)$ and partition. The value $\max \Delta x_k$ is called the MESH SIZE of the partition. If the LIMIT $\max \Delta x_k \to 0$ exists, this limit is known as the Riemann INTEGRAL of $f(x)$ over the interval $[a, b]$. The shaded areas in the above plots show the LOWER and UPPER SUMS for a constant MESH SIZE.

see also LOWER SUM, RIEMANN INTEGRAL, UPPER SUM

Riemann Surface

The Riemann surface S of the ALGEBRAIC FUNCTION FIELD K is the set of nontrivial discrete valuations on K. Here, the set S corresponds to the IDEALS of the RING A of INTEGERS of K over $\mathbb{C}(z)$. (A consists of the elements of K that are ROOTS of MONIC POLYNOMIALS over $\mathbb{C}[z]$.)

see also ALGEBRAIC FUNCTION FIELD, IDEAL, RING

References
Fischer, G. (Ed.). Plates 123–126 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, pp. 120–123, 1986.

Riemann Tensor

A TENSOR sometimes known as the RIEMANN-CHRISTOFFEL TENSOR. Let

$$\tilde{D}_s \equiv \frac{\partial}{\partial x^s} - \sum_l \left\{ \begin{matrix} s & & u \\ & l & \end{matrix} \right\}, \qquad (1)$$

where the quantity inside the $\left\{ \begin{matrix} s & & u \\ & l & \end{matrix} \right\}$ is a CHRISTOFFEL SYMBOL OF THE SECOND KIND. Then

$$R_{pqrs} \equiv \tilde{D}_q \left\{ \begin{matrix} p & & r \\ & s & \end{matrix} \right\} - \tilde{D}_r \left\{ \begin{matrix} r & & q \\ & s & \end{matrix} \right\}. \qquad (2)$$

Broken down into its simplest decomposition in N-D,

$$\begin{aligned} R_{\lambda\mu\nu\kappa} &= \frac{1}{N-2}(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}) \\ &\quad - \frac{R}{(N-1)(N-2)}(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}) + C_{\lambda\mu\nu\kappa}. \qquad (3) \end{aligned}$$

Here, $R_{\mu\nu}$ is the RICCI TENSOR, R is the CURVATURE SCALAR, and $C_{\lambda\mu\nu\kappa}$ is the WEYL TENSOR. In terms of the JACOBI TENSOR $J^\mu{}_{\nu\alpha\beta}$,

$$R^\mu{}_{\alpha\nu\beta} = \tfrac{2}{3}(J^\mu_{\nu\alpha\beta} - J^\mu_{\beta\alpha\nu}). \qquad (4)$$

The Riemann tensor is the only tensor that can be constructed from the METRIC TENSOR and its first and second derivatives,

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + \Gamma^\alpha_{\mu\gamma}\Gamma^\mu_{\beta\delta} - \Gamma^\alpha_{\mu\delta} - \Gamma^\alpha_{\beta\mu}c_{\gamma\delta}{}^\mu, \qquad (5)$$

where Γ are CONNECTION COEFFICIENTS and c are COMMUTATION COEFFICIENTS. The number of independent coordinates in n-D is

$$C_n \equiv \tfrac{1}{12}n^2(n^2-1), \qquad (6)$$

and the number of SCALARS which can be constructed from $R_{\lambda\mu\nu\kappa}$ and $g_{\mu\nu}$ is

$$S_n \equiv \begin{cases} 1 & \text{for } n = 2 \\ \frac{1}{12}n(n-1)(n-2)(n+3) & \text{for } n = 1, n > 2. \end{cases} \qquad (7)$$

In 1-D, $R_{1111} = 0$.

n	C_n	S_n
1	0	0
2	1	1
3	6	3
4	20	14

see also BIANCHI IDENTITIES, CHRISTOFFEL SYMBOL OF THE SECOND KIND, COMMUTATION COEFFICIENT, CONNECTION COEFFICIENT, CURVATURE SCALAR, GAUSSIAN CURVATURE, JACOBI TENSOR, PETROV NOTATION, RICCI TENSOR, WEYL TENSOR

Riemann Theta Function

Let the IMAGINARY PART of a $g \times g$ MATRIX F be POSITIVE DEFINITE, and $\mathbf{m} = (m_1, \ldots, m_g)$ be a row VECTOR with coefficients in $\mathbb{Z}$. Then the Riemann theta function is defined by

$$\vartheta(u) = \sum_{\mathbf{m}} \exp[2\pi i(\mathbf{m}^{\mathrm{T}}u + \tfrac{1}{2}\mathsf{F}^{\mathrm{T}}\mathbf{m})].$$

see also RAMANUJAN THETA FUNCTIONS, THETA FUNCTION

References
Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 9, 1980.

Riemann Weighted Prime-Power Counting Function

The Riemann weighted prime-power counting function is defined by

$$J_2(x) \equiv \begin{cases} \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \ldots - \frac{1}{2m} & \text{for } p^m \text{ with } p \text{ a prime} \\ \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \ldots & \text{otherwise} \end{cases} \qquad (1)$$

$$= \lim_{t\to\infty} \frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{x^s}{s} \ln\zeta(s)\,ds. \qquad (2)$$

The PRIME COUNTING FUNCTION is given in terms of $J_2(x)$ by

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J_2(x^{1/n}). \qquad (3)$$

The function also satisfies the identity

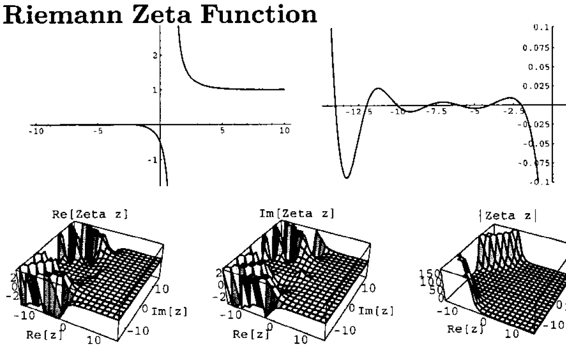
$$\frac{\ln\zeta(s)}{s} = \int_1^\infty J_2(x)x^{-s-1}\,dx. \qquad (4)$$

see also MANGOLDT FUNCTION, PRIME COUNTING FUNCTION, RIEMANN'S FORMULA

Riemann Xi Function

see XI FUNCTION

Riemann Zeta Function



The Riemann zeta function can be defined by the integral

$$\zeta(x) \equiv \frac{1}{\Gamma(x)} \int_0^\infty \frac{u^{x-1}}{e^u - 1}\, du, \tag{1}$$

where $x > 1$. If x is an INTEGER n, then

$$\frac{u^{n-1}}{e^u - 1} = \frac{e^{-u}u^{n-1}}{1 - e^{-u}} = e^{-u}u^{n-1} \sum_{k=1}^\infty e^{-ku}u^{n-1}, \tag{2}$$

so

$$\int_0^\infty \frac{u^{n-1}}{e^u - 1}\, du = \sum_{k=1}^\infty \int_0^\infty e^{-ku}u^{n-1}\, du. \tag{3}$$

Let $y \equiv ku$, then $dy = k\, du$ and

$$\begin{aligned} \zeta(n) &= \frac{1}{\Gamma(n)} \sum_{k=1}^\infty \int_0^\infty e^{-ku}u^{n-1}\, du \\ &= \frac{1}{\Gamma(n)} \sum_{k=1}^\infty \int_0^\infty e^{-y} \left(\frac{y}{k}\right)^{n-1} \frac{dy}{k} \\ &= \frac{1}{\Gamma(n)} \sum_{k=1}^\infty \frac{1}{k^n} \int_0^\infty e^{-y}y^{n-1}\, dy, \end{aligned} \tag{4}$$

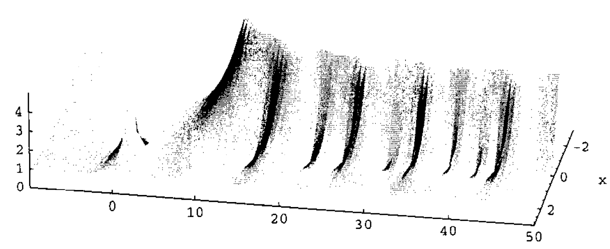
where $\Gamma(n)$ is the GAMMA FUNCTION. Integrating the final expression in (4) gives $\Gamma(n)$, which cancels the factor $1/\Gamma(n)$ and gives the most common form of the Riemann zeta function,

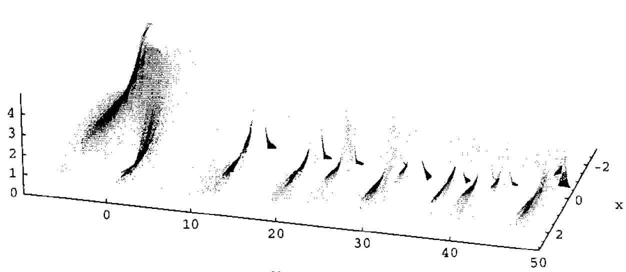
$$\zeta(n) = \sum_{k=1}^\infty \frac{1}{k^n}. \tag{5}$$

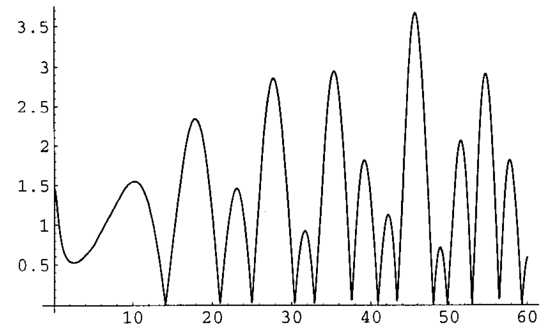
At $n = 1$, the zeta function reduces to the HARMONIC SERIES (which diverges), and therefore has a singularity. In the COMPLEX PLANE, trivial zeros occur at -2, -4, -6, ..., and nontrivial zeros at

$$s \equiv \sigma + it \tag{6}$$

for $0 \leq \sigma \leq 1$. The figures below show the structure of $\zeta(z)$ by plotting $|\zeta(z)|$ and $1/|\zeta(z)|$.







The RIEMANN HYPOTHESIS asserts that the nontrivial ROOTS of $\zeta(s)$ all have REAL PART $\sigma = \Re[s] = 1/2$, a line called the "CRITICAL STRIP." This is known to be true for the first 1.5×10^{12} roots (Brent *et al.* 1979). The above plot shows $|\zeta(1/2+it)|$ for t between 0 and 60. As can be seen, the first few nontrivial zeros occur at $t =$ 14.134725, 21.022040, 25.010858, 30.424876, 32.935062, 37.586178, ... (Wagon 1991, pp. 361–362 and 367–368).

The Riemann zeta function can also be defined in terms of MULTIPLE INTEGRALS by

$$\zeta(n) = \underbrace{\int_0^1 \cdots \int_0^1}_{n} \frac{\prod_{i=1}^n dx_i}{1 - \prod_{i=1}^n x_i}. \tag{7}$$

The Riemann zeta function can be split up into

$$\zeta(\tfrac{1}{2} + it) = z(t)e^{-i\vartheta(t)}, \tag{8}$$

where $z(t)$ and $\vartheta(t)$ are the RIEMANN-SIEGEL FUNCTIONS. An additional identity is

$$\lim_{s \to 1} \zeta(s) - \frac{1}{s-1} = \gamma, \tag{9}$$

where γ is the EULER-MASCHERONI CONSTANT.

The Riemann zeta function is related to the DIRICHLET LAMBDA FUNCTION $\lambda(\nu)$ and DIRICHLET ETA FUNCTION $\eta(\nu)$ by

$$\frac{\zeta(\nu)}{2^\nu} = \frac{\lambda(\nu)}{2^\nu - 1} = \frac{\eta(\nu)}{2^\nu - 2} \tag{10}$$

and

$$\zeta(\nu) + \eta(\nu) = 2\lambda(\nu) \tag{11}$$

(Spanier and Oldham 1987). It is related to the LIOUVILLE FUNCTION $\lambda(n)$ by

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^\infty \frac{\lambda(n)}{n^s} \tag{12}$$

(Lehman 1960, Hardy and Wright 1979). Furthermore,

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n=1}^\infty \frac{2^{\omega(n)}}{n^s}, \tag{13}$$

where $\omega(n) = \sigma_0(n)$ is the number of different prime factors of n (Hardy and Wright 1979).

A generalized Riemann zeta function $\zeta(s,a)$ known as the HURWITZ ZETA FUNCTION can also be defined such that

$$\zeta(s) \equiv \zeta(s,0). \tag{14}$$

The Riemann zeta function may be computed analytically for EVEN n using either CONTOUR INTEGRATION or PARSEVAL'S THEOREM with the appropriate FOURIER SERIES. An interesting formula involving the product of PRIMES was first discovered by Euler in 1737,

$$\begin{aligned}\zeta(x)(1-2^{-x}) &= \left(1+\frac{1}{2^x}+\frac{1}{3^x}+\ldots\right)\left(1-\frac{1}{2^x}\right)\\ &= \left(1+\frac{1}{2^x}+\frac{1}{3^x}+\ldots\right)-\left(\frac{1}{2^x}+\frac{1}{4^x}+\frac{1}{6^x}+\ldots\right)\end{aligned} \tag{15}$$

$$\begin{aligned}&\zeta(x)(1-2^{-x})(1-3^{-x})\\ &= \left(1+\frac{1}{3^x}+\frac{1}{5^x}+\frac{1}{7^x}+\ldots\right)-\left(\frac{1}{3^x}+\frac{1}{9^x}+\frac{1}{15^x}+\ldots\right)\end{aligned} \tag{16}$$

$$\begin{aligned}&\zeta(x)(1-2^{-x})(1-3^{-x})\cdots(1-p^{-x})\cdots\\ &\qquad = \zeta(x)\prod_{n=2}^{\infty}(1-p^{-x}) = 1.\end{aligned} \tag{17}$$

Here, each subsequent multiplication by the next PRIME p leaves only terms which are POWERS of p^{-x}. Therefore,

$$\zeta(x) = \left[\prod_{p=2}^{\infty}(1-p^{-x})\right]^{-1}, \tag{18}$$

where p runs over all PRIMES. Euler's product formula can also be written

$$\zeta(s) = (1-2^{-s})^{-1} \prod_{\substack{q\equiv 1\\ (\mathrm{mod}\ 4)}} (1-q^{-s})^{-1} \prod_{\substack{r\equiv 3\\ (\mathrm{mod}\ 4)}} (1-r^{-s})^{-1}. \tag{19}$$

A few sum identities involving $\zeta(n)$ are

$$\sum_{n=2}^{\infty}[\zeta(n)-1] = 1 \tag{20}$$

$$\sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1] = \tfrac{1}{2}. \tag{21}$$

The Riemann zeta function is related to the GAMMA FUNCTION $\Gamma(z)$ by

$$\Gamma\left(\frac{s}{2}\right)\pi^{-z/2}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2}\zeta(1-s). \tag{22}$$

$\zeta(n)$ was proved to be transcendental for all even n by Euler. Apéry (1979) proved $\zeta(3)$ to be IRRATIONAL with the aid of the k^{-3} sum formula below. As a result, $\zeta(3)$ is sometimes called APÉRY'S CONSTANT.

$$\zeta(2) = 3\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}} \tag{23}$$

$$\zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}} \tag{24}$$

$$\zeta(4) = \frac{36}{17}\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}} \tag{25}$$

(Guy 1994, p. 257). A relation of the form

$$\zeta(5) = Z_5\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^5\binom{2k}{k}} \tag{26}$$

has been searched for with Z_5 a RATIONAL or ALGEBRAIC NUMBER, but if Z_5 is a ROOT of a POLYNOMIAL of degree 25 or less, then the Euclidean norm of the coefficients must be larger than 2×10^{37} (Bailey, Bailey and Plouffe). Therefore, no such sums are known for $\zeta(n)$ are known for $n \geq 5$.

The zeta function is defined for $\Re[s] > 1$, but can be analytically continued to $\Re[s] > 0$ as follows

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n n^{-s} + \sum_{n=1}^{\infty} n^{-s} &= 2\sum_{n=2,4,\ldots}^{\infty} n^{-s}\\ &= 2\sum_{k=1}^{\infty}(2k)^{-s} = 2^{1-s}\sum_{k=1}^{\infty}k^{-s}\end{aligned} \tag{27}$$

$$\sum_{n=1}^{\infty}(-1)^n n^{-s} + \zeta(s) = 2^{1-s}\zeta(s) \tag{28}$$

$$\zeta(s) = \frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}(-1)^n n^{-s}. \tag{29}$$

The DERIVATIVE of the Riemann zeta function is defined by

$$\zeta'(s) = -s\sum_{k=1}^{\infty}k^{-s}\ln k = -\sum_{k=2}^{\infty}\frac{\ln k}{k^s}. \tag{30}$$

As $s \to 0$,

$$\zeta'(0) = -\tfrac{1}{2}\ln(2\pi). \tag{31}$$

For EVEN $n \equiv 2k$,

$$\zeta(n) = \frac{2^{n-1}|B_n|\pi^n}{n!}, \tag{32}$$

where B_n is a BERNOULLI NUMBER. Another intimate connection with the BERNOULLI NUMBERS is provided by

$$B_n = (-1)^{n+1}n\zeta(1-n). \tag{33}$$

No analytic form for $\zeta(n)$ is known for ODD $n \equiv 2k+1$, but $\zeta(2k+1)$ can be expressed as the sum limit

$$\zeta(2k+1) = (\tfrac{1}{2}\pi)^{2k+1} \lim_{t\to\infty} \frac{1}{t^{2k+1}} \sum_{i=1}^{\infty} \left[\cot\left(\frac{i}{2t+1}\right)\right]^{2k+1} \quad (34)$$

(Stark 1974). The values for the first few integral arguments are

$$\begin{aligned}
\zeta(0) &\equiv -\tfrac{1}{2}\\
\zeta(1) &= \infty\\
\zeta(2) &= \frac{\pi^2}{6}\\
\zeta(3) &= 1.2020569032\ldots\\
\zeta(4) &= \frac{\pi^4}{90}\\
\zeta(5) &= 1.0369277551\ldots\\
\zeta(6) &= \frac{\pi^6}{945}\\
\zeta(7) &= 1.0083492774\ldots\\
\zeta(8) &= \frac{\pi^8}{9450}\\
\zeta(9) &= 1.0020083928\ldots\\
\zeta(10) &= \frac{\pi^{10}}{93{,}555}.
\end{aligned}$$

Euler gave $\zeta(2)$ to $\zeta(26)$ for EVEN n, and Stieltjes (1993) determined the values of $\zeta(2)$, ..., $\zeta(70)$ to 30 digits of accuracy in 1887. The denominators of $\zeta(2n)$ for $n =$ 1, 2, ... are 6, 90, 945, 9450, 93555, 638512875, ... (Sloane's A002432).

Using the LLL ALGORITHM, Plouffe (inspired by Zucker 1979, Zucker 1984, and Berndt 1988) has found some beautiful infinite sums for $\zeta(n)$ with ODD n. Let

$$S_\pm(n) \equiv \sum_{k=1}^{\infty} \frac{1}{k^n(e^{2\pi k} \pm 1)}, \quad (35)$$

then

$$\zeta(3) = \tfrac{7}{180}\pi^3 - 2S_-(3) \quad (36)$$

$$\zeta(5) = \tfrac{1}{294}\pi^5 - \tfrac{72}{35}S_-(5) - \tfrac{2}{35}S_+(5) \quad (37)$$

$$\zeta(7) = \tfrac{19}{56700}\pi^7 - 2S_-(7) \quad (38)$$

$$\zeta(9) = \tfrac{125}{3704778}\pi^9 - \tfrac{992}{495}S_-(9) - \tfrac{2}{495}S_+(9) \quad (39)$$

$$\zeta(11) = \tfrac{1453}{425675250}\pi^{11} - 2S_-(11) \quad (40)$$

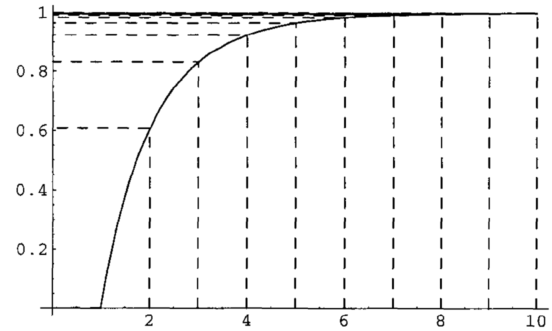
$$\zeta(13) = \tfrac{89}{257432175}\pi^{13} - \tfrac{16512}{8255}S_-(13) - \tfrac{2}{8255}S_+(13) \quad (41)$$

$$\zeta(15) = \tfrac{13687}{390769879500}\pi^{15} - 2S_-(15) \quad (42)$$

$$\zeta(17) = \tfrac{397549}{112024529867250}\pi^{17} - \tfrac{261632}{130815}S_-(17) - \tfrac{2}{130815}S_+(17) \quad (43)$$

$$\zeta(19) = \tfrac{7708537}{21438612514068750}\pi^{19} - 2S_-(19) \quad (44)$$

$$\zeta(21) = \tfrac{68529640373}{1881063815762259253125}\pi^{21} - \tfrac{4196352}{2098175}S_-(21) - \tfrac{2}{2098175}S_+(21) \quad (45)$$



The inverse of the RIEMANN ZETA FUNCTION $1/\zeta(p)$ is the asymptotic density of pth-powerfree numbers (i.e., SQUAREFREE numbers, CUBEFREE numbers, etc.). The following table gives the number $Q_p(n)$ of pth-powerfree numbers $\leq n$ for several values of n.

p	$1/\zeta(p)$	10	100	10^3	10^4	10^5	10^6
2	0.607927	7	61	608	6083	60794	607926
3	0.831907	9	85	833	8319	83190	831910
4	0.923938	10	93	925	9240	92395	923939
5	0.964387	10	97	965	9645	96440	964388
6	0.982953	10	99	984	9831	98297	982954

The value for $\zeta(2)$ can be found using a number of different techniques (Apostol 1983, Choe 1987, Giesy 1972, Holme 1970, Kimble 1987, Knopp and Schur 1918, Kortram 1996, Matsuoka 1961, Papadimitriou 1973, Simmons 1992, Stark 1969, Stark 1970, Yaglom and Yaglom 1987). The problem of finding this value analytically is sometimes known as the BASLER PROBLEM (Castellanos 1988). Yaglom and Yaglom (1987), Holme (1970), and Papadimitrou (1973) all derive the result from DE MOIVRE'S IDENTITY or related identities.

Consider the FOURIER SERIES of $f(x) = x^{2n}$

$$f(x) = \tfrac{1}{2}a_0 + \sum_{m=1}^{\infty} a_m \cos(mx) + \sum_{m=1}^{\infty} b_m \sin(mx), \quad (46)$$

which has coefficients given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^{2n}\,dx = \frac{2}{\pi}\left[\frac{x^{2n+1}}{2n+1}\right]_0^{\pi} = \frac{2\pi^{2n}}{2n+1} \quad (47)$$

$$a_m = \frac{1}{\pi}\int_{\pi}^{\pi} x^{2n}\cos(mx)\,dx = \frac{2}{\pi}\int_0^{\pi} x^{2n}\cos(mx)\,dx \quad (48)$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} x^{2n}\sin(mx)\,dx = 0, \quad (49)$$

where the latter is true since the integrand is ODD. Therefore, the FOURIER SERIES is given explicitly by

$$x^{2n} = \frac{\pi^{2n}}{2n+1} + \sum_{m=1}^{\infty} a_m \cos(mx). \quad (50)$$

Now, a_m is given by the COSINE INTEGRAL

$$a_m = \frac{2}{\pi}(-1)^{n+1}(2n)!\left[\sin(mx)\sum_{k=0}^{n}\frac{(-1)^k}{(2k)!m^{2n-2k+1}}x^{2k} + \cos(mx)\sum_{k=1}^{n}\frac{(-1)^{k+1}}{(2k-3)!m^{2n-2k+2}}x^{2k-1}\right]_0^\pi. \quad (51)$$

But $\cos(m\pi) = (-1)^m$, and $\sin(m\pi) = \sin 0 = 0$, so

$$a_m = \frac{2}{\pi}(-1)^{n+1}(2n)!(-1)^m\sum_{k=1}^{n}\frac{(-1)^{k+1}}{(2k-3)!m^{2n-2k+2}}\pi^{2k-1}$$
$$= (-1)^{m+n}2(2n)!\sum_{k=1}^{n}\frac{(-1)^k}{(2k-3)!m^{2n-2k+2}}\pi^{2k-2}. \quad (52)$$

Now, if $n = 1$,

$$a_m = (-1)^{m+1}2(2!)\sum_{k=1}^{1}\frac{(-1)^k}{(2k-3)!m^{4-2k}}\pi^{2k-2}$$
$$= 4(-1)^{m+1}\frac{(-1)}{(-1)!m^2}\pi^0 = \frac{4(-1)^m}{m^2}, \quad (53)$$

so the FOURIER SERIES is

$$x^2 = \frac{\pi^2}{3} + 4\sum_{m=1}^{\infty}\frac{(-1)^m\cos(mx)}{m^2}. \quad (54)$$

Letting $x \equiv \pi$ gives $\cos(m\pi) = (-1)^m$, so

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{m=1}^{\infty}\frac{1}{m^2}, \quad (55)$$

and we have

$$\zeta(2) = \sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\pi^2}{6}. \quad (56)$$

Higher values of n can be obtained by finding a_m and proceeding as above.

The value $\zeta(2)$ can also be found simply using the ROOT LINEAR COEFFICIENT THEOREM. Consider the equation $\sin z = 0$ and expand sin in a MACLAURIN SERIES

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots = 0 \quad (57)$$

$$0 = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \ldots = 1 - \frac{w}{3!} + \frac{w^2}{5!} + \ldots, \quad (58)$$

where $w \equiv z^2$. But the zeros of $\sin(z)$ occur at π, 2π, 3π, ..., so the zeros of $\sin w = \sin\sqrt{z}$ occur at π^2, $(2\pi)^2$, ... Therefore, the sum of the roots equals the COEFFICIENT of the leading term

$$\frac{1}{\pi^2} + \frac{1}{(2\pi)^2} + \frac{1}{(3\pi^2)} + \ldots = \frac{1}{3!} = \frac{1}{6}, \quad (59)$$

which can be rearranged to yield

$$\zeta(2) = \frac{\pi^2}{6}. \quad (60)$$

Yet another derivation (Simmons 1992) evaluates the integral using the integral

$$I = \int_0^1\int_0^1\frac{dx\,dy}{1-xy} = \int_0^1\int_0^1(1 + xy + x^2y^2 + \ldots)\,dx\,dy$$
$$= \int_0^1[(x + \tfrac{1}{2}x^2y + \tfrac{1}{3}x^3y^2 + \ldots)]_0^1\,dy$$
$$= \int_0^1(1 + \tfrac{1}{2}y + \tfrac{1}{3}y^2 + \ldots)\,dy$$
$$= \left[y + \frac{y^2}{2^2} + \frac{y^3}{3^2} + \ldots\right]_0^1 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots. \quad (61)$$

To evaluate the integral, rotate the coordinate system by $\pi/4$ so

$$x = u\cos\theta - v\sin\theta = \tfrac{1}{2}\sqrt{2}\,(u-v) \quad (62)$$
$$y = u\sin\theta + v\cos\theta = \tfrac{1}{2}\sqrt{2}\,(u+v) \quad (63)$$

and

$$xy = \tfrac{1}{2}(u^2 - v^2) \quad (64)$$
$$1 - xy = \tfrac{1}{2}(2 - u^2 + v^2). \quad (65)$$

Then

$$I = 4\int_0^{\sqrt{2}/2}\int_0^u\frac{du\,dv}{2-u^2+v^2} + 4\int_{\sqrt{2}/2}^{\sqrt{2}}\int_0^{\sqrt{2}-u}\frac{du\,dv}{2-u^2+v^2} \equiv I_1 + I_2. \quad (66)$$

Now compute the integrals I_1 and I_2.

$$I_1 = 4\int_0^{\sqrt{2}/2}\left[\int_0^u\frac{dv}{2-u^2+v^2}\right]du$$
$$= 4\int_0^{\sqrt{2}2}\left[\frac{1}{\sqrt{2-u^2}}\tan^{-1}\left(\frac{v}{\sqrt{2-u^2}}\right)\right]_0^u du$$
$$= 4\int_0^{\sqrt{2}/2}\frac{1}{\sqrt{2-u^2}}\tan^{-1}\left(\frac{u}{\sqrt{2-u^2}}\right)du. \quad (67)$$

Make the substitution

$$u = \sqrt{2}\sin\theta \quad (68)$$
$$\sqrt{2-u^2} = \sqrt{2}\cos\theta \quad (69)$$
$$du = \sqrt{2}\cos\theta\,d\theta, \quad (70)$$

so

$$\tan^{-1}\left(\frac{u}{\sqrt{2-u^2}}\right) = \tan^{-1}\left(\frac{\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right) = \theta \qquad (71)$$

and

$$I_1 = 4\int_0^{\pi/6} \frac{1}{\sqrt{2}\cos\theta}\theta\sqrt{2}\cos\theta\,d\theta = 2[\theta^2]_0^{\pi/6} = \frac{\pi^2}{18}. \qquad (72)$$

I_2 can also be computed analytically,

$$\begin{aligned} I_2 &= 4\int_{\sqrt{2}/2}^{\sqrt{2}} \left[\int_0^{\sqrt{2}-u} \frac{dv}{2-u^2+v^2}\right] du \\ &= 4\int_{\sqrt{2}/2}^{\sqrt{2}} \left[\frac{1}{\sqrt{2-u^2}}\tan^{-1}\left(\frac{v}{\sqrt{2-u^2}}\right)\right]_0^{\sqrt{2}-u} du \\ &= 4\int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}}\tan^{-1}\left(\frac{\sqrt{2}-u}{\sqrt{2-u^2}}\right) du. \qquad (73) \end{aligned}$$

But

$$\begin{aligned} \tan^{-1}\left(\frac{\sqrt{2}-u}{\sqrt{2-u^2}}\right) &= \tan^{-1}\left(\frac{\sqrt{2}-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right) \\ &= \tan\left(\frac{1-\sin\theta}{\cos\theta}\right) = \tan^{-1}\left(\frac{\cos\theta}{1+\sin\theta}\right) \\ &= \tan^{-1}\left[\frac{\sin(\frac{1}{2}\pi-\theta)}{1+\cos(\frac{1}{2}\pi-\theta)}\right] \\ &= \tan^{-1}\left\{\frac{2\sin[\frac{1}{2}(\frac{1}{2}\pi-\theta)]\cos[\frac{1}{2}(\frac{1}{2}\pi-\theta)]}{2\cos^2[\frac{1}{2}(\frac{1}{2}\pi-\theta)]}\right\} \\ &= \tfrac{1}{2}(\tfrac{1}{2}\pi-\theta), \qquad (74) \end{aligned}$$

so

$$\begin{aligned} I_2 &= 4\int_{\pi/6}^{\pi/2} \frac{1}{\sqrt{2}\cos\theta}(\tfrac{1}{4}\pi-\tfrac{1}{2}\theta)\sqrt{2}\cos\theta\,d\theta \\ &= 4\left[\tfrac{1}{4}\pi\theta-\tfrac{1}{4}\theta^2\right]_{\pi/6}^{\pi/2} \\ &= 4\left[\left(\frac{\pi^2}{8}-\frac{\pi^2}{16}\right)-\left(\frac{\pi^2}{24}-\frac{\pi^2}{144}\right)\right] = \frac{\pi^2}{9}. \qquad (75) \end{aligned}$$

Combining I_1 and I_2 gives

$$\zeta(2) = I_1 + I_2 = \frac{\pi^2}{18} + \frac{\pi^2}{9} = \frac{\pi^2}{6}. \qquad (76)$$

see also ABEL'S FUNCTIONAL EQUATION, DEBYE FUNCTIONS, DIRICHLET BETA FUNCTION, DIRICHLET ETA FUNCTION, DIRICHLET LAMBDA FUNCTION, HARMONIC SERIES, HURWITZ ZETA FUNCTION, KHINTCHINE'S CONSTANT, LEHMER'S PHENOMENON, PSI FUNCTION, RIEMANN HYPOTHESIS, RIEMANN P-SERIES, RIEMANN-SIEGEL FUNCTIONS, STIELTJES CONSTANTS, XI FUNCTION

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Riemannian Geometry

The study of MANIFOLDS having a complete RIEMANNIAN METRIC. Riemannian geometry is a general space based on the LINE ELEMENT

$$ds = F(x^1, \ldots, x^n; dx^1, \ldots, dx^n),$$

with $F(x, y) > 0$ for $y \neq 0$ a function on the TANGENT BUNDLE TM. In addition, F is homogeneous of degree 1 in y and of the form

$$F^2 = g_{ij}(x)\, dx^i\, dx^j$$

(Chern 1996). If this restriction is dropped, the resulting geometry is called FINSLER GEOMETRY.

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Riemannian Geometry (Non-Euclidean)

see ELLIPTIC GEOMETRY

Riemannian Manifold

A MANIFOLD possessing a METRIC TENSOR. For a complete Riemannian manifold, the METRIC $d(x,y)$ is defined as the length of the shortest curve (GEODESIC) between x and y.

see also BISHOP'S INEQUALITY, CHEEGER'S FINITENESS THEOREM

Riemannian Metric

Suppose for every point x in a COMPACT MANIFOLD M, an INNER PRODUCT $\langle\cdot,\cdot\rangle_x$ is defined on a TANGENT SPACE T_xM of M at x. Then the collection of all these INNER PRODUCTS is called the Riemannian metric. In 1870, Christoffel and Lipschitz showed how to decide when two Riemannian metrics differ by only a coordinate transformation.

see also COMPACT MANIFOLD, LINE ELEMENT, METRIC TENSOR

Riesel Number

There exist infinitely many ODD INTEGERS k such that $k \cdot 2^n - 1$ is COMPOSITE for every $n \geq 1$. Numbers k with this property are called RIESEL NUMBERS, and analogous numbers with the minus sign replaced by a plus are called SIERPIŃSKI NUMBERS OF THE SECOND KIND. The smallest known Riesel number is $k = 509{,}203$, but there remain 963 smaller candidates (the smallest of which is 659) which generate only composite numbers for all n which have been checked (Ribenboim 1996, p. 358).

Let $a(k)$ be smallest n for which $(2k-1)\cdot 2^n - 1$ is PRIME, then the first few values are 2, 0, 2, 1, 1, 2, 3, 1, 2, 1, 1, 4, 3, 1, 4, 1, 2, 2, 1, 3, 2, 7, ... (Sloane's A046069), and second smallest n are 3, 1, 4, 5, 3, 26, 7, 2, 4, 3, 2, 6, 9, 2, 16, 5, 3, 6, 2553, ... (Sloane's A046070).

see also CUNNINGHAM NUMBER, MERSENNE NUMBER, SIERPIŃSKI'S COMPOSITE NUMBER THEOREM, SIERPIŃSKI NUMBER OF THE SECOND KIND

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Riesz-Fischer Theorem

A function is L_2- (square-) integrable IFF its FOURIER SERIES is L_2-convergent. The application of this theorem requires use of the LEBESGUE INTEGRAL.

see also LEBESGUE INTEGRAL

Riesz Representation Theorem

Let f be a bounded linear FUNCTIONAL on a HILBERT SPACE H. Then there exists exactly one $x_0 \in H$ such that $f(x) = \langle x, x_0 \rangle$ for all $x \in H$. Also, $\|f\| = \|x_0\|$.

see also FUNCTIONAL, HILBERT SPACE

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Riesz's Theorem

Every continuous linear functional $U[f]$ for $f \in C[a,b]$ can be expressed as a STIELTJES INTEGRAL

$$U[f] = \int_a^b f(x)\,dw(x),$$

where $w(x)$ is determined by U and is of bounded variation on $[a,b]$.

see also STIELTJES INTEGRAL

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Riffle Shuffle

A SHUFFLE, also called a FARO SHUFFLE, in which a deck of $2n$ cards is divided into two HALVES which are then alternatively interleaved from the left and right hands (an "in-shuffle") or from the right and left hands (an "out-shuffle"). Using an "in-shuffle," a deck originally arranged as 1 2 3 4 5 6 7 8 would become 5 1 6 2 7 3 8 4. Using an "out-shuffle," the deck order would become 1 5 2 6 3 7 4 8. Riffle shuffles are used in card tricks (Marlo 1958ab, Adler 1973), and also in the theory of parallel processing (Stone 1971, Chen *et al.* 1981).

In general, card k moves to the position originally occupied by the $2k$th card (mod $2n+1$). Therefore, in-shuffling $2n$ cards $2n$ times (where $2n+1$ is PRIME) results in the original card order. Similarly, out-shuffling $2n$ cards $2n-2$ times (where $2n-1$ is PRIME) results in the original order (Diaconis *et al.* 1983, Conway and Guy 1996). Amazingly, this means that an ordinary deck of 52 cards is returned to its original order after 8 out-shuffles.

Morris (1994) further discusses aspects of the perfect riffle shuffle (in which the deck is cut exactly in half and cards are perfectly interlaced). Ramnath and Scully (1996) give an algorithm for the shortest sequence of in- and out-shuffles to move a card from arbitrary position i to position j. This algorithm works for any deck with an EVEN number of cards and is $\mathcal{O}(\log n)$.

see also CARDS, SHUFFLE

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Rigby Points

The PERSPECTIVE CENTERS of the TANGENTIAL and CONTACT TRIANGLES of the inner and outer SODDY POINTS. The Rigby points are given by

$$Ri = I + \tfrac{4}{3}Ge$$

$$Ri' = I - \tfrac{4}{3}Ge,$$

where I is the INCENTER and Ge is the GERGONNE POINT.

see also CONTACT TRIANGLE, GERGONNE POINT, GRIFFITHS POINTS, INCENTER, OLDKNOW POINTS, SODDY POINTS, TANGENTIAL TRIANGLE

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Right Angle

An ANGLE equal to half the ANGLE from one end of a line segment to the other. A right angle is $\pi/2$ radians or $90°$. A TRIANGLE containing a right angle is called a RIGHT TRIANGLE. However, a TRIANGLE cannot contain more than one right angle, since the sum of the two right angles plus the third angle would exceed the $180°$ total possessed by a TRIANGLE.

see also ACUTE ANGLE, OBLIQUE ANGLE, OBTUSE ANGLE, RIGHT TRIANGLE, SEMICIRCLE, STRAIGHT ANGLE, THALES' THEOREM

Right Conoid

A RULED SURFACE is called a right conoid if it can be generated by moving a straight LINE intersecting a fixed straight LINE such that the LINES are always PERPENDICULAR (Kreyszig 1991, p. 87). Taking the PERPENDICULAR plane as the xy-plane and the line to be the x-AXIS gives the parametric equations

$$\begin{aligned} x(u,v) &= v\cos\vartheta(u) \\ y(u,v) &= v\sin\vartheta(u) \\ z(u,v) &= h(u) \end{aligned}$$

(Gray 1993). Taking $h(u) = 2u$ and $\vartheta(u) = u$ gives the HELICOID.

see also HELICOID, PLÜCKER'S CONOID, WALLIS'S CONICAL EDGE

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Right Hyperbola

A HYPERBOLA for which the ASYMPTOTES are PERPENDICULAR. This occurs when the SEMIMAJOR and SEMIMINOR AXES are equal. Taking $a = b$ in the equation of a HYPERBOLA with SEMIMAJOR AXIS parallel to the x-AXIS and SEMIMINOR AXIS parallel to the y-AXIS (i.e., vertical DIRECTRIX),

$$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1$$

therefore gives

$$(x-x_0)^2 - (y-y_0)^2 = a^2.$$

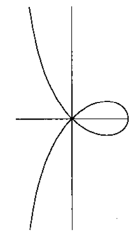
A special type of right hyperbola is the so-called RECTANGULAR HYPERBOLA, which has equation $xy = ab$.

see also HYPERBOLA, RECTANGULAR HYPERBOLA

Right Line

see LINE

Right Strophoid



The STROPHOID of a line L with pole O not on L and fixed point O' being the point where the PERPENDICULAR from O to L cuts L is called a right strophoid. It is therefore a general STROPHOID with $a = \pi/2$.

The right strophoid is given by the Cartesian equation

$$y^2 = \frac{c-x}{c+x}x^2, \tag{1}$$

or the polar equation

$$r = c\cos(2\theta)\sec\theta. \tag{2}$$

The parametric form of the strophoid is

$$x(t) = \frac{1-t^2}{t^2+1} \tag{3}$$

$$y(t) = \frac{t(t^2-1)}{t^2+1}. \tag{4}$$

The right strophoid has CURVATURE

$$\kappa(t) = -\frac{4(1+3t^2)}{(1+6t^2+t^4)^{3/2}} \tag{5}$$

and TANGENTIAL ANGLE

$$\phi(t) = -2\tan^{-1}t - \tan^{-1}\left(\frac{2t}{1+t^2}\right). \tag{6}$$

The right strophoid first appears in work by Isaac Barrow in 1670, although Torricelli describes the curve in his letters around 1645 and Roberval found it as the LOCUS of the focus of the conic obtained when the plane cutting the CONE rotates about the tangent at its vertex (MacTutor Archive). The AREA of the loop is

$$A_{\text{loop}} = \tfrac{1}{2}c^2(4-\pi) \tag{7}$$

(MacTutor Archive).

Let C be the CIRCLE with center at the point where the right strophoid crosses the x-axis and radius the distance of that point from the origin. Then the right strophoid is invariant under inversion in the CIRCLE C and is therefore an ANALLAGMATIC CURVE.

see also STROPHOID, TRISECTRIX

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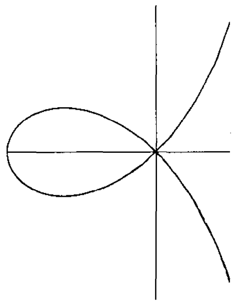
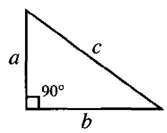
MacTutor History of Mathematics Archive. "Right Strophoid." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Right.html`.

Right Strophoid Inverse Curve



The INVERSE CURVE of a right strophoid is the same strophoid.

Right Triangle



A TRIANGLE with an ANGLE of 90° ($\pi/2$ radians). The sides a, b, and c of such a TRIANGLE satisfy the PYTHAGOREAN THEOREM. The largest side is conventionally denoted c and is called the HYPOTENUSE.

For any three similar shapes on the sides of a right triangle,

$$A_1 + A_2 = A_3, \qquad (1)$$

which is equivalent to the PYTHAGOREAN THEOREM. For a right triangle with sides a, b, and HYPOTENUSE c, let r be the INRADIUS. Then

$$\tfrac{1}{2}ab = \tfrac{1}{2}ra + \tfrac{1}{2}rb + \tfrac{1}{2}rc = \tfrac{1}{2}r(a+b+c). \qquad (2)$$

Solving for r gives

$$r = \frac{ab}{a+b+c}. \qquad (3)$$

But any PYTHAGOREAN TRIPLE can be written

$$a = m^2 - n^2 \qquad (4)$$

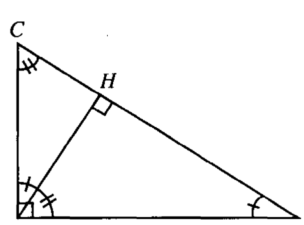
$$b = 2mn \qquad (5)$$

$$c = m^2 + n^2, \qquad (6)$$

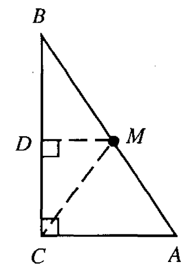
so (5) becomes

$$r = \frac{(m^2 - n^2)2mn}{m^2 - n^2 + 2mn + m^2 + n^2} = n(m-n), \qquad (7)$$

which is an INTEGER when m and n are integers.



Given a right triangle ΔABC, draw the ALTITUDE AH from the RIGHT ANGLE A. Then the triangles ΔAHC and ΔBHA are similar.



In a right triangle, the MIDPOINT of the HYPOTENUSE is equidistant from the three VERTICES (Dunham 1990). This can be proved as follows. Given ΔABC, let M be the MIDPOINT of AB (so that $AM = BM$). Draw $DM || CA$, then since ΔBDM is similar to ΔBCA, it follows that $BD = DC$. Since both ΔBDM and ΔCDM are right triangles and the corresponding legs are equal, the HYPOTENUSES are also equal, so we have $AM = BM = CM$ and the theorem is proved.

see also ACUTE TRIANGLE, ARCHIMEDES' MIDPOINT THEOREM, BROCARD MIDPOINT, CIRCLE-POINT MIDPOINT THEOREM, FERMAT'S RIGHT TRIANGLE THEOREM, ISOSCELES TRIANGLE, MALFATTI'S RIGHT TRIANGLE PROBLEM, OBTUSE TRIANGLE, PYTHAGOREAN TRIPLE, QUADRILATERAL, RAT-FREE SET, TRIANGLE

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Rigid

A FRAMEWORK is rigid IFF continuous motion of the points of the configuration maintaining the bar constraints comes from a family of motions of all EUCLIDEAN SPACE which are distance-preserving. A GRAPH G is (generically) d-rigid if, for almost all (i.e., an open dense set of) CONFIGURATIONS of p, the FRAMEWORK $G(p)$ is rigid in $\mathbb{R}^d$.

One of the first results in rigidity theory was the RIGIDITY THEOREM by Cauchy in 1813. Although rigidity problems were of immense interest to engineers, intensive mathematical study of these types of problems has occurred only relatively recently (Connelly 1993, Graver *et al.* 1993).

see also BAR (EDGE), FLEXIBLE POLYHEDRON, FRAMEWORK, LAMAN'S THEOREM, LIEBMANN'S THEOREM, RIGIDITY THEOREM

References

Connelly, R. "Rigidity." Ch. 1.7 in *Handbook of Convex Geometry, Vol. A* (Ed. P. M. Gruber and J. M. Wills). Amsterdam, Netherlands: North-Holland, pp. 223–271, 1993.

Crapo, H. and Whiteley, W. "Statics of Frameworks and Motions of Panel Structures, A Projective Geometry Introduction." *Structural Topology* **6**, 43–82, 1982.

Graver, J.; Servatius, B.; and Servatius, H. *Combinatorial Rigidity.* Providence, RI: Amer. Math. Soc., 1993.

Rigid Motion

A transformation consisting of ROTATIONS and TRANSLATIONS which leaves a given arrangement unchanged.

see also EUCLIDEAN MOTION, PLANE, ROTATION

References

Courant, R. and Robbins, H. *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, p. 141, 1996.

Rigidity Theorem

If the faces of a *convex* POLYHEDRON were made of metal plates and the EDGES were replaced by hinges, the POLYHEDRON would be RIGID. The theorem was stated by Cauchy (1813), although a mistake in this paper went unnoticed for more than 50 years. An example of a *concave* "FLEXIBLE POLYHEDRON" (with 18 triangular faces) for which this is not true was given by Connelly (1978), and a FLEXIBLE POLYHEDRON with only 14 triangular faces was subsequently found by Steffen (Mackenzie 1998).

see also FLEXIBLE POLYHEDRON, RIGID

References

Cauchy, A. L. "Sur les polygons et le polyhéders." *XVIe Cahier* **IX**, 87–89, 1813.
Connelly, R. "A Flexible Sphere." *Math. Intel.* **1**, 130–131, 1978.
Graver, J.; Servatius, B.; and Servatius, H. *Combinatorial Rigidity.* Providence, RI: Amer. Math. Soc., 1993.
Mackenzie, D. "Polyhedra Can Bend But Not Breathe." *Science* **279**, 1637, 1998.

Ring

A ring is a set together with two BINARY OPERATORS $S(+,*)$ satisfying the following conditions:

1. Additive associativity: For all $a, b, c \in S$, $(a+b)+c = a+(b+c)$,
2. Additive commutativity: For all $a, b \in S$, $a+b = b+a$,
3. Additive identity: There exists an element $0 \in S$ such that for all $a \in S$, $0+a = a+0 = a$,
4. Additive inverse: For every $a \in S$ there exists $-a in S$ such that $a+(-a) = (-a)+a = 0$,
5. Multiplicative associativity: For all $a, b, c \in S$, $(a*b)*c = a*(b*c)$,
6. Left and right distributivity: For all $a, b, c \in S$, $a*(b+c) = (a*b)+(a*c)$ and $(b+c)*a = (b*a)+(c*a)$.

A ring is therefore an ABELIAN GROUP under addition and a SEMIGROUP under multiplication. A ring must contain at least one element, but need not contain a multiplicative identity or be commutative. The number of finite rings of n elements for $n = 1, 2, \ldots$, are 1, 2, 2, 11, 2, 4, 2, 52, 11, 4, 2, 22, 2, 4, 4, ... (Sloane's A027623 and A037234; Fletcher 1980). In general, the number of rings of order p^3 for p an ODD PRIME is $3p+50$ and 52 for $p = 2$ (Ballieu 1947, Gilmer and Mott 1973).

A ring with a multiplicative identity is sometimes called a UNIT RING. Fraenkel (1914) gave the first abstract definition of the ring, although this work did not have much impact.

A ring that is COMMUTATIVE under multiplication, has a unit element, and has no divisors of zero is called an INTEGRAL DOMAIN. A ring which is also a COMMUTATIVE multiplication group is called a FIELD. The simplest rings are the INTEGERS $\mathbb{Z}$, POLYNOMIALS $\mathbb{R}[x]$ and $\mathbb{R}[x, y]$ in one and two variables, and SQUARE $n \times n$ REAL MATRICES.

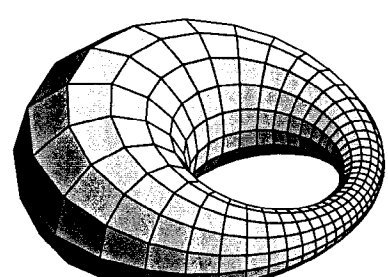
Rings which have been investigated and found to be of interest are usually named after one or more of their investigators. This practice unfortunately leads to names which give very little insight into the relevant properties of the associated rings.

see also ABELIAN GROUP, ARTINIAN RING, CHOW RING, DEDEKIND RING, DIVISION ALGEBRA, FIELD, GORENSTEIN RING, GROUP, GROUP RING, IDEAL, INTEGRAL DOMAIN, MODULE, NILPOTENT ELEMENT, NOETHERIAN RING, NUMBER FIELD, PRIME RING, PRÜFER RING, QUOTIENT RING, REGULAR RING, RINGOID, SEMIPRIME RING, SEMIRING, SEMISIMPLE RING, SIMPLE RING, UNIT RING, ZERO DIVISOR

References

Ballieu, R. "Anneaux finis; systèmes hypercomplexes de rang trois sur un corps commutatif." *Ann. Soc. Sci. Bruxelles. Sér. I* **61**, 222–227, 1947.
Fletcher, C. R. "Rings of Small Order." *Math. Gaz.* **64**, 9–22, 1980.
Fraenkel, A. "Über die Teiler der Null und die Zerlegung von Ringen." *J. Reine Angew. Math.* **145**, 139–176, 1914.
Gilmer, R. and Mott, J. "Associative Rings of Order p^3." *Proc. Japan Acad.* **49**, 795–799, 1973.
Kleiner, I. "The Genesis of the Abstract Ring Concept." *Amer. Math. Monthly* **103**, 417–424, 1996.
Sloane, N. J. A. Sequences A027623 and A037234 in "An On-Line Version of the Encyclopedia of Integer Sequences."
van der Waerden, B. L. *A History of Algebra.* New York: Springer-Verlag, 1985.

Ring Cyclide



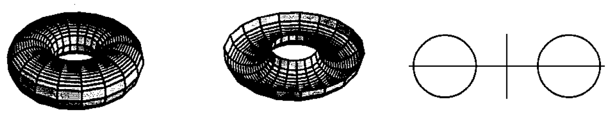
The inversion of a RING TORUS. If the inversion center lies on the torus, then the ring cyclide degenerates to a PARABOLIC RING CYCLIDE.

see also CYCLIDE, PARABOLIC CYCLIDE, RING CYCLIDE, RING TORUS, SPINDLE CYCLIDE, TORUS

Ring Function

see TOROIDAL FUNCTION

Ring Torus



One of the three STANDARD TORI given by the parametric equations

$$\begin{aligned} x &= (c + a\cos v)\cos u \\ y &= (c + a\cos v)\sin u \\ z &= a\sin v \end{aligned}$$

with $c > a$. This is the TORUS which is generally meant when the term "torus" is used without qualification. The inversion of a ring torus is a RING CYCLIDE if the INVERSION CENTER does not lie on the torus and a PARABOLIC RING CYCLIDE if it does. The above left figure shows a ring torus, the middle a cutaway, and the right figure shows a CROSS-SECTION of the ring torus through the xz-plane.

see also CYCLIDE, HORN TORUS, PARABOLIC RING CYCLIDE, RING CYCLIDE, SPINDLE TORUS, STANDARD TORI, TORUS

References

Gray, A. "Tori." §11.4 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 218–220, 1993.

Pinkall, U. "Cyclides of Dupin." §3.3 in *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, pp. 28–30, 1986.

Ringoid

A ringoid R is a set $(R, +, \times)$ with two binary operators, conventionally denoted addition $(+)$ and multiplication $(\times)$, where $\times$ distributes over $+$ left and right:

$$a(b+c) = ab + acand(b+c)a = ba + ca.$$

A ringoid can be empty.

see also BINARY OPERATOR, RING, SEMIRING

References

Rosenfeld, A. *An Introduction to Algebraic Structures.* New York: Holden-Day, 1968.

Risch Algorithm

An ALGORITHM for indefinite integration.

see also INDEFINITE INTEGRAL

Rising Factorial

see POCHHAMMER SYMBOL

Rivest-Shamir-Adleman Number

see RSA NUMBER

RMS

see ROOT-MEAN-SQUARE

Robbin Constant

$$R = \tfrac{4}{105} + \tfrac{17}{105}\sqrt{2} - \tfrac{2}{35}\sqrt{3} + \tfrac{1}{5}\ln(1+\sqrt{2}) + \tfrac{2}{5}\ln(2+\sqrt{3}) - \tfrac{1}{15}\pi = 0.661707182\ldots.$$

see also TRANSFINITE DIAMETER

References

Plouffe, S. "The Robbin Constant." `http://lacim.uqam.ca/piDATA/robbin.txt`.

Robbin's Inequality

If the fourth MOMENT $\mu_4 \neq 0$, then

$$P(|\bar{x} - \mu_4| \geq \lambda) \leq \frac{\mu_4 + 3(N-1)\sigma^4}{N^3\lambda^4},$$

where σ^2 is the VARIANCE.

Robbins Algebra

Building on work of Huntington (1933), Robbins conjectured that the equations for a Robbins algebra, commutivity, associativity, and the ROBBINS EQUATION

$$n(n(x+y) + n(x+n(y))) = x,$$

imply those for a BOOLEAN ALGEBRA. The conjecture was finally proven using a computer (McCune 1997).

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Huntington, E. V. "New Sets of Independent Postulates for the Algebra of Logic, with Special Reference to Whitehead and Russell's *Principia Mathematica.*" *Trans. Amer. Math. Soc.* **35**, 274–304, 1933.

Huntington, E. V. "Boolean Algebra. A Correction." *Trans. Amer. Math. Soc.* **35**, 557–558, 1933.

McCune, W. "Solution of the Robbins Problem." *J. Automat. Reason.* **19**, 263–276, 1997.

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Robbins Equation

$$n(n(x+y) + n(x+n(y))) = x.$$

see also ROBBINS ALGEBRA

Robertson Condition

For the HELMHOLTZ DIFFERENTIAL EQUATION to be SEPARABLE in a coordinate system, the SCALE FACTORS h_i in the LAPLACIAN

$$\nabla^2 = \sum_{i=1}^{3} \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_i} \left(\frac{h_1 h_2 h_3}{{h_i}^2} \frac{\partial}{\partial u_i} \right) \tag{1}$$

and the functions $f_i(u_i)$ and Φ_{ij} defined by

$$\frac{1}{f_n} \frac{\partial}{\partial u_n} \left(f_n \frac{\partial X_n}{\partial u_n} \right) + ({k_1}^2 \Phi_{n1} + {k_2}^2 \Phi_{n2} + {k_3}^2 \Phi_{n3}) X_n = 0 \tag{2}$$

must be of the form of a STÄCKEL DETERMINANT

$$S = |\Phi_{mn}| = \begin{vmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{vmatrix} = \frac{h_1 h_2 h_3}{f_1(u_1) f_2(u_2) f_3(u_3)}. \tag{3}$$

see also HELMHOLTZ DIFFERENTIAL EQUATION, LAPLACE'S EQUATION, SEPARATION OF VARIABLES, STÄCKEL DETERMINANT

References

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part 1.* New York: McGraw-Hill, p. 510, 1953.

Robertson Conjecture

A conjecture due to M. S. Robertson (1936) which treats a UNIVALENT POWER SERIES containing only ODD powers within the UNIT DISK. This conjecture IMPLIES the BIEBERBACH CONJECTURE and follows in turn from the MILIN CONJECTURE. de Branges' proof of the BIEBERBACH CONJECTURE proceeded by proving the MILIN CONJECTURE, thus establishing the Robertson conjecture and hence implying the truth of the BIEBERBACH CONJECTURE.

see also BIEBERBACH CONJECTURE, MILIN CONJECTURE

References

Stewart, I. *From Here to Infinity: A Guide to Today's Mathematics.* Oxford, England: Oxford University Press, p. 165, 1996.

Robertson-Seymour Theorem

A generalization of the KURATOWSKI REDUCTION THEOREM by Robertson and Seymour, which states that the collection of finite graphs is well-quasi-ordered by minor embeddability, from which it follows that Kuratowski's "forbidden minor" embedding obstruction generalizes to higher genus surfaces.

Formally, for a fixed INTEGER $g \geq 0$, there is a finite list of graphs $L(g)$ with the property that a graph C embeds on a surface of genus g IFF it does not contain, as a minor, any of the graphs on the list L.

References

Fellows, M. R. "The Robertson-Seymour Theorems: A Survey of Applications." *Comtemp. Math.* **89**, 1–18, 1987.

Robin Boundary Conditions

PARTIAL DIFFERENTIAL EQUATION BOUNDARY CONDITIONS which, for an elliptic partial differential equation in a region Ω, specify that the sum of αu and the normal derivative of $u = f$ at all points of the boundary of Ω, α and f being prescribed.

Robin's Constant

see TRANSFINITE DIAMETER

Robinson Projection

A PSEUDOCYLINDRICAL MAP PROJECTION which distorts shape, AREA, scale, and distance to create attraction average projection properties.

References

Dana, P. H. "Map Projections." `http://www.utexas.edu/depts/grg/gcraft/notes/mapproj/mapproj.html`.

Robust Estimation

An estimation technique which is insensitive to small departures from the idealized assumptions which have been used to optimize the algorithm. Classes of such techniques include M-ESTIMATES (which follow from maximum likelihood considerations), L-ESTIMATES (which are linear combinations of ORDER STATISTICS), and R-ESTIMATES (based on RANK tests).

see also L-ESTIMATE, M-ESTIMATE, R-ESTIMATE

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Robust Estimation." §15.7 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 694–700, 1992.

Rodrigues's Curvature Formula

$$d\hat{\mathbf{N}} + \kappa_i \, d\mathbf{r} = \mathbf{0},$$

where $\hat{\mathbf{N}}$ is the unit NORMAL VECTOR and κ_i is one of the two PRINCIPAL CURVATURES.

see also NORMAL VECTOR, PRINCIPAL CURVATURES

Rodrigues Formula

An operator definition of a function. A Rodrigues formula may be converted into a SCHLÄFLI INTEGRAL.

see also SCHLÄFLI INTEGRAL

Rogers-Ramanujan Continued Fraction

see RAMANUJAN CONTINUED FRACTION

Rogers-Ramanujan Identities

For $|q| < 1$ and using the NOTATION of the RAMANUJAN THETA FUNCTION, the Rogers-Ramanujan identities are

$$\frac{f(-q^5)}{f(-q,-q^4)} = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q)_k} \tag{1}$$

$$\frac{f(-q^5)}{f(-q^2,-q^3)} = \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q)_k}, \tag{2}$$

where $(q)_k$ are q-SERIES. Written out explicitly (Hardy 1959, p. 13),

$$1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\frac{q^9}{(1-q)(1-q^2)(1-q^3)}+\cdots = \frac{1}{(1-q)(1-q^6)\cdots(1-q^4)(1-q^9)\cdots} \tag{3}$$

$$1+\frac{q^2}{1-q}+\frac{q^6}{(1-q)(1-q^2)}+\frac{q^{12}}{(1-q)(1-q^2)(1-q^3)}+\cdots = \frac{1}{(1-q^2)(1-q^7)\cdots(1-q^3)(1-q^8)\cdots}. \tag{4}$$

The identities can also be written succinctly as

$$1+\sum_{k=1}^{\infty} \frac{q^{k^2+ak}}{(1-q)(1-q^2)\cdots(1-q^k)} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+a+1})(1-q^{5j-a+4})}, \tag{5}$$

where $a = 0, 1$.

Other forms of the Rogers-Ramanujan identities include

$$\sum_k \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}} = \sum_k \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}} \tag{6}$$

and

$$\sum_k \frac{2q^{k^2}}{(q;q)_k (q;q)_{n-k}} = \sum_k \frac{(-1)^k (1+q^k) q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}} \tag{7}$$

(Petkovšek *et al.* 1996).

see also ANDREWS-SCHUR IDENTITY

References

Andrews, G. E. *The Theory of Partitions.* Cambridge, England: Cambridge University Press, 1985.

Andrews, G. E. *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra.* Providence, RI: Amer. Math. Soc., pp. 17–20, 1986.

Andrews, G. E. and Baxter, R. J. "A Motivated Proof of the Rogers-Ramanujan Identities." *Amer. Math. Monthly* **96**, 401–409, 1989.

Bressoud, D. M. *Analytic and Combinatorial Generalizations of the Rogers-Ramanujan Identities.* Providence, RI: Amer. Math. Soc., 1980.

Hardy, G. H. *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work, 3rd ed.* New York: Chelsea, p. 13, 1959.

Paule, P "Short and Easy Computer Proofs of the Rogers-Ramanujan Identities and of Identities of Similar Type." *Electronic J. Combinatorics* **1**, R10, 1–9, 1994. `http://www.combinatorics.org/Volume_1/volume1.html#R10`.

Petkovšek, M.; Wilf, H. S.; and Zeilberger, D. *A=B.* Wellesley, MA: A. K. Peters, p. 117, 1996.

Robinson, R. M. "Comment to: 'A Motivated Proof of the Rogers-Ramanujan Identities.'" *Amer. Math. Monthly* **97**, 214–215, 1990.

Rogers, L. J. "Second Memoir on the Expansion of Certain Infinite Products." *Proc. London Math. Soc.* **25**, 318–343, 1894.

Sloane, N. J. A. Sequence A006141/M0260 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Rolle's Theorem

Let f be differentiable on (a,b) and continuous on $[a,b]$. If $f(a) = f(b) = 0$, then there is at least one point $c \in (a,b)$ where $f'(c) = 0$.

see also FIXED POINT THEOREM, MEAN-VALUE THEOREM

Roman Coefficient

A generalization of the BINOMIAL COEFFICIENT whose NOTATION was suggested by Knuth,

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \frac{\lfloor n\rceil!}{\lfloor k\rceil!\,\lfloor n-k\rceil!}. \tag{1}$$

The above expression is read "Roman n choose k." Whenever the BINOMIAL COEFFICIENT is defined (i.e., $n \geq k \geq 0$ or $k \geq 0 > n$), the Roman coefficient agrees with it. However, the Roman coefficients are defined for values for which the BINOMIAL COEFFICIENTS are not, e.g.,

$$\left\lfloor \begin{matrix} n \\ -1 \end{matrix} \right\rceil = \frac{1}{\lfloor n+1\rceil} \tag{2}$$

$$\left\lfloor \begin{matrix} 0 \\ k \end{matrix} \right\rceil = \frac{(-1)^{k+(k>0)}}{\lfloor k\rceil}, \tag{3}$$

where

$$n < 0 \equiv \begin{cases} 1 & \text{for } n < 0 \\ 0 & \text{for } n \geq 0. \end{cases} \tag{4}$$

The Roman coefficients also satisfy properties like those of the BINOMIAL COEFFICIENT,

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \left\lfloor \begin{matrix} n \\ n-k \end{matrix} \right\rceil \tag{5}$$

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \left\lfloor \begin{matrix} k \\ r \end{matrix} \right\rceil = \left\lfloor \begin{matrix} n \\ r \end{matrix} \right\rceil \left\lfloor \begin{matrix} n-r \\ k-r \end{matrix} \right\rceil, \tag{6}$$

an analog of PASCAL'S FORMULA

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rceil + \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rceil, \tag{7}$$

and a curious rotation/reflection law due to Knuth

$$(-1)^{k+(k>0)}\left\lfloor \begin{matrix}-n\\ k-1\end{matrix}\right\rceil = (-1)^{n+(n>0)}\left\lfloor \begin{matrix}-k\\ n-1\end{matrix}\right\rceil \qquad (8)$$

(Roman 1992).

see also BINOMIAL COEFFICIENT, ROMAN FACTORIAL

References

Roman, S. "The Logarithmic Binomial Formula." *Amer. Math. Monthly* **99**, 641–648, 1992.

Roman Factorial

$$\lfloor n\rceil! \equiv \begin{cases} n! & \text{for } n \geq 0 \\ \frac{(-1)^{-n-1}}{(-n-1)!} & \text{for } n < 0. \end{cases} \qquad (1)$$

The Roman factorial arises in the definition of the HARMONIC LOGARITHM and ROMAN COEFFICIENT. It obeys the identities

$$\lfloor n\rceil! = \lfloor n\rceil\, \lfloor n-1\rceil! \qquad (2)$$

$$\frac{\lfloor n\rceil!}{\lfloor n-k\rceil!} = \lfloor n\rceil\, \lfloor n-1\rceil \cdots \lfloor n-k+1\rceil \qquad (3)$$

$$\lfloor n\rceil!\, \lfloor -n-1\rceil! = (-1)^{n+(n<0)}, \qquad (4)$$

where

$$\lfloor n\rceil \equiv \begin{cases} n & \text{for } n \neq 0 \\ 1 & \text{for } n = 0 \end{cases} \qquad (5)$$

and

$$n < 0 \equiv \begin{cases} 1 & \text{for } n < 0 \\ 0 & \text{for } n \geq 0. \end{cases} \qquad (6)$$

see also HARMONIC LOGARITHM, HARMONIC NUMBER, ROMAN COEFFICIENT

References

Loeb, D. and Rota, G.-C. "Formal Power Series of Logarithmic Type." *Advances Math.* **75**, 1–118, 1989.

Roman, S. "The Logarithmic Binomial Formula." *Amer. Math. Monthly* **99**, 641–648, 1992.

Roman Numeral

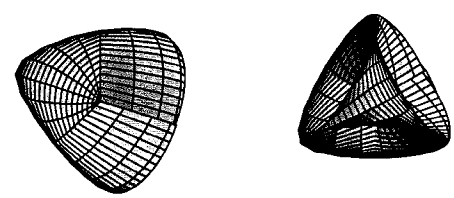
A system of numerical notations used by the Romans. It is an additive (and subtractive) system in which letters are used to denote certain "base" numbers, and arbitrary numbers are then denoted using combinations of symbols.

Character	Numerical Value
I	1
V	5
X	10
L	50
C	100
D	500
M	1000

For example, the number 1732 would be denoted MDCCXXXII. One additional rule states that, instead of using four symbols to represent a 4, 40, 9, 90, etc., such numbers are instead denoted by preceding the symbol for 5, 50, 10, 100, etc., with a symbol indicating *subtraction*. For example, 4 is denoted IV, 9 as IX, 40 as XL, etc. However, this rule is generally *not* followed on the faces of clocks, where IIII is usually encountered instead of IV.

Roman numerals are encountered in the release year for movies and occasionally on the numerals on the faces of watches and clocks, but in few other modern instances. They do have the advantage that ADDITION can be done "symbolically" (and without worrying about the "place" of a given DIGIT) by simply combining all the symbols together, grouping, writing groups of 5 Is as V, groups of 2 Vs as X, etc.

Roman Surface



A QUARTIC NONORIENTABLE SURFACE, also known as the STEINER SURFACE. The Roman surface is one of the three possible surfaces obtained by sewing a MÖBIUS STRIP to the edge of a DISK. The other two are the BOY SURFACE and CROSS-CAP, all of which are homeomorphic to the REAL PROJECTIVE PLANE (Pinkall 1986).

The center point of the Roman surface is an ordinary TRIPLE POINT with $(\pm 1, 0, 0) = (0, \pm 1, 0) = (0, 0, \pm 1)$, and the six endpoints of the three lines of self-intersection are singular PINCH POINTS, also known as WHITNEY SINGULARITIES. The Roman surface is essentially six CROSS-CAPS stuck together and contains a double INFINITY of CONICS.

The Roman surface can given by the equation

$$(x^2+y^2+z^2-k^2)^2 = [(z-k)^2-2x^2][(z+k)^2-2y^2]. \quad (1)$$

Solving for z gives the pair of equations

$$z = \frac{k(y^2-x^2) \pm (x^2-y^2)\sqrt{k^2-x^2-y^2}}{2(x^2+y^2)}. \qquad (2)$$

If the surface is rotated by 45° about the z-AXIS via the ROTATION MATRIX

$$\mathsf{R}_z(45°) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3)$$

to give

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \mathsf{R}_z(45°)\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad (4)$$

then the simple equation

$$x^2y^2 + x^2z^2 + y^2z^2 + 2kxyz = 0 \tag{5}$$

results. The Roman surface can also be generated using the general method for NONORIENTABLE SURFACES using the polynomial function

$$\mathbf{f}(x, y, z) = (xy, yz, zx) \tag{6}$$

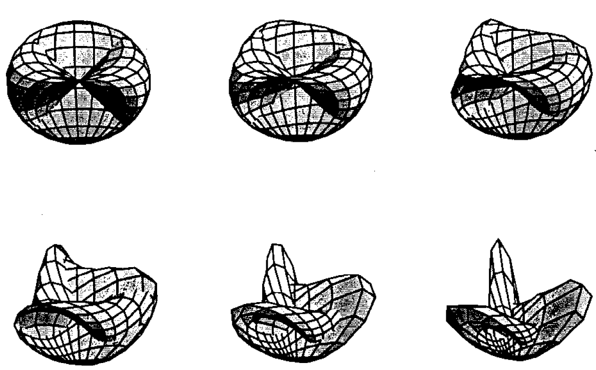
(Pinkall 1986). Setting

$$x = \cos u \sin v \tag{7}$$
$$y = \sin u \sin v \tag{8}$$
$$z = \cos v \tag{9}$$

in the former gives

$$x(u, v) = \tfrac{1}{2}\sin(2u)\sin^2 v \tag{10}$$
$$y(u, v) = \tfrac{1}{2}\sin u \cos(2v) \tag{11}$$
$$z(u, v) = \tfrac{1}{2}\cos u \sin(2v) \tag{12}$$

for $u \in [0, 2\pi)$ and $v \in [-\pi/2, \pi/2]$. Flipping $\sin v$ and $\cos v$ and multiplying by 2 gives the form shown by Wang.



A HOMOTOPY (smooth deformation) between the Roman surface and BOY SURFACE is given by the equations

$$x(u, v) = \frac{\sqrt{2}\cos(2u)\cos^2 v + \cos u \sin(2v)}{2 - \alpha\sqrt{2}\sin(3u)\sin(2v)} \tag{13}$$

$$y(u, v) = \frac{\sqrt{2}\sin(2u)\cos^2 v - \sin u \sin(2v)}{2 - \alpha\sqrt{2}\sin(3u)\sin(2v)} \tag{14}$$

$$z(u, v) = \frac{3\cos^2 v}{2 - \alpha\sqrt{2}\sin(3u)\sin(2v)} \tag{15}$$

for $u \in [-\pi/2, \pi/2]$ and $v \in [0, \pi]$ as α varies from 0 to 1. $\alpha = 0$ corresponds to the Roman surface and $\alpha = 1$ to the BOY SURFACE (Wang).

see also BOY SURFACE, CROSS-CAP, HEPTAHEDRON, MÖBIUS STRIP, NONORIENTABLE SURFACE, QUARTIC SURFACE, STEINER SURFACE

References

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Fischer, G. (Ed.). Plates 42–44 and 108–114 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, pp. 42–44 and 108–109, 1986.

Geometry Center. "The Roman Surface." `http://www.geom.umn.edu/zoo/toptype/pplane/roman/`.

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 242–243, 1993.

Nordstrand, T. "Steiner's Roman Surface." `http://www.uib.no/people/nfytn/steintxt.htm`.

Pinkall, U. *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, p. 64, 1986.

Wang, P. "Renderings." `http://www.ugcs.caltech.edu/~peterw/portfolio/renderings/`.

Roman Symbol

$$\lfloor n \rceil \equiv \begin{cases} n & \text{for } n \neq 0 \\ 1 & \text{for } n = 0. \end{cases}$$

see also ROMAN FACTORIAL, HARMONIC LOGARITHM

References

Roman, S. "The Logarithmic Binomial Formula." *Amer. Math. Monthly* **99**, 641–648, 1992.

Romberg Integration

A powerful NUMERICAL INTEGRATION technique which uses k refinements of the extended TRAPEZOIDAL RULE to remove error terms less than order $\mathcal{O}(N^{-2k})$. The routine advocated by Press *et al.* (1992) makes use of NEVILLE'S ALGORITHM.

References

Acton, F. S. *Numerical Methods That Work, 2nd printing.* Washington, DC: Math. Assoc. Amer., pp. 106–107, 1990.

Dahlquist, G. and Bjorck, A. §7.4.1–7.4.2 in *Numerical Methods.* Englewood Cliffs, NJ: Prentice-Hall, 1974.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Romberg Integration." §4.3 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 134–135, 1992.

Ralston, A. and Rabinowitz, P. §4.10 in *A First Course in Numerical Analysis, 2nd ed.* New York: McGraw-Hill, 1978.

Stoer, J.; and Burlisch, R. §3.4–3.5 in *Introduction to Numerical Analysis.* New York: Springer-Verlag, 1980.

Rook Number

The rook numbers r_n^B of an $n \times n$ BOARD B are the number of subsets of size n such that no two elements have the same first or second coordinate. In other word, it is the number of ways of placing n rooks on B such that none attack each other. The rook numbers of a board determine the rook numbers of the complementary board $\overline{B}$, defined to be $\mathbf{d} \times \mathbf{d} \backslash B$. This is known as the ROOK RECIPROCITY THEOREM. The first few rook numbers are 1, 2, 7, 23, 115, 694, 5282, 46066, ... (Sloane's A000903). For an $n \times n$ board, each $n \times n$

PERMUTATION MATRIX corresponds to an allowed configuration of rooks.

see also ROOK RECIPROCITY THEOREM

References

Sloane, N. J. A. Sequence A000903/M1761 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Rook Reciprocity Theorem

$$\sum_{k=0}^{d} r_k^B (d-k)!x^k = \sum_{k=0}^{d} (-1)^k r_k^{\bar{B}} (d-k)!x^k (x+1)^{d-k}.$$

References

Chow, T. Y. "The Path-Cycle Symmetric Function of a Digraph." *Adv. Math.* **118**, 71–98, 1996.

Chow, T. "A Short Proof of the Rook Reciprocity Theorem." *Electronic J. Combinatorics* **3**, R10, 1–2, 1996. http://www.combinatorics.org/Volume_3/volume3.html#R10.

Goldman, J. R.; Joichi, J. T.; and White, D. E. "Rook Theory I. Rook Equivalence of Ferrers Boards." *Proc. Amer. Math. Soc.* **52**, 485–492, 1975.

Riordan, J. *An Introduction to Combinatorial Analysis.* New York: Wiley, 1958.

Rooks Problem

							R
						R	
					R		
				R			
			R				
		R					
	R						
R							

The rook is a CHESS piece which may move any number of spaces either horizontally or vertically per move. The maximum number of nonattacking rooks which may be placed on an $n \times n$ CHESSBOARD is n. This arrangement is achieved by placing the rooks along the diagonal (Madachy 1979). The total number of ways of placing n nonattacking rooks on an $n \times n$ board is $n!$ (Madachy 1979, p. 47). The number of rotationally and reflectively inequivalent ways of placing n nonattacking rooks on an $n \times n$ board are 1, 2, 7, 23, 115, 694, ... (Sloane's A000903; Dudeney 1970, p. 96; Madachy 1979, pp. 46–54).

							R
						R	
					R		
				R			
			R				
		R					
	R						
R							

The minimum number of rooks needed to occupy or attack all spaced on an 8×8 CHESSBOARD is 8, illustrated above (Madachy 1979).

Consider an $n \times n$ chessboard with the restriction that, for every subset of $\{1, \ldots, n\}$, a rook may not be put in column $s + j \pmod{n}$ when on row j, where the rows are numbered 0, 1, ..., $n - 1$. Vardi (1991) denotes the number of rook solutions so restricted as $\text{rook}(s, n)$. $\text{rook}(\{1\}, n)$ is simply the number of DERANGEMENTS on n symbols, known as a SUBFACTORIAL. The first few values are 1, 2, 9, 44, 265, 1854, ... (Sloane's A000166). $\text{rook}(\{1, 2\}, n)$ is a solution to the MARRIED COUPLES PROBLEM, sometimes known as MÉNAGE NUMBERS. The first few MÉNAGE NUMBERS are -1, 1, 0, 2, 13, 80, 579, ... (Sloane's A000179).

Although simple formulas are not known for general $\{1, \ldots, p\}$, RECURRENCE RELATIONS can be used to compute $\text{rook}(\{1, \ldots, p\}, n)$ in polynomial time for $p = 3$, ..., 6 (Metropolis *et al.* 1969, Minc 1978, Vardi 1991).

see also CHESS, MÉNAGE NUMBER, ROOK NUMBER, ROOK RECIPROCITY THEOREM

References

Dudeney, H. E. "The Eight Rooks." §295 in *Amusements in Mathematics.* New York: Dover, p. 88, 1970.

Kraitchik, M. "The Problem of the Rooks" and "Domination of the Chessboard." §10.2 and 10.4 in *Mathematical Recreations.* New York: W. W. Norton, pp. 240–247 and 255–256, 1942.

Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, pp. 36–37, 1979.

Metropolis, M.; Stein, M. L.; and Stein, P. R. "Permanents of Cyclic (0, 1) Matrices." *J. Combin. Th.* **7**, 291–321, 1969.

Minc, H. §3.1 in *Permanents.* Reading, MA: Addison-Wesley, 1978.

Riordan, J. Chs. 7–8 in *An Introduction to Combinatorial Analysis.* Princeton, NJ: Princeton University Press, 1978.

Sloane, N. J. A. Sequences A000903/M1761, A000166/M1937, and A000179/M2062 in "An On-Line Version of the Encyclopedia of Integer Sequences." http://www.research.att.com/~njas/sequences/eisonline.html.

Sloane, N. J. A. and Plouffe, S. Extended entry for M2062 in *The Encyclopedia of Integer Sequences.* San Diego: Academic Press, 1995.

Vardi, I. *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, pp. 123–124, 1991.

Room Square

A Room square (named after T. G. Room) of order n (for n ODD) is an arrangement in an $n \times n$ SQUARE MATRIX of $n + 1$ objects such that each cell is either empty or holds exactly two different objects. Furthermore, each object appears once in each row and column and each unordered pair occupies exactly one cell. The Room square of order 2 is shown below.

$$\boxed{1, 2}$$

The Room square of order 8 is

1, 8			5, 7		3, 4	2, 6
3, 7	2, 8			6, 1		4, 5
5, 6	4, 1	3, 8			7, 2	
	6, 7	5, 2	4, 8			1, 3
2, 4		7, 1	6, 3	5 ,8		
	3, 5		1, 2	7, 4	6, 8	
		4, 6		2, 3	1, 5	7, 8

References

Dinitz, J. H. and Stinson, D. R. In *Contemporary Design Theory: A Collection of Surveys* (Ed. J. H. Dinitz and D. R. Stinson). New York: Wiley, 1992.

Gardner, M. *Time Travel and Other Mathematical Bewilderments.* New York: W. H. Freeman, pp. 146–147 and 151–152, 1988.

Mullin, R. C. and Nemeth, E. "On Furnishing Room Squares." *J. Combin. Th.* **7**, 266–272, 1969.

Mullin, R. D. and Wallis, W. D. "The Existence of Room Squares." *Aequationes Math.* **13**, 1–7, 1975.

O'Shaughnessy, C. D. "On Room Squares of Order $6m+2$." *J. Combin. Th.* **13**, 306–314, 1972.

Room, T. G. "A New Type of Magic Square" (Note 2569). *Math. Gaz.* **39**, 307, 1955.

Wallis, W. D. "Solution of the Room Square Existence Problem." *J. Combin. Th.* **17**, 379–383, 1974.

Root

The roots of an equation

$$f(x) = 0 \tag{1}$$

are the values of x for which the equation is satisfied. The FUNDAMENTAL THEOREM OF ALGEBRA states that every POLYNOMIAL equation of degree n has exactly n roots, where some roots may have a multiplicity greater than 1 (in which case they are said to be degenerate).

To find the nth roots of a COMPLEX NUMBER, solve the equation $z^n = w$. Then

$$z^n = |z|^n[\cos(n\theta) + i\sin(n\theta)] = |w|\,(\cos\phi + i\sin\phi), \tag{2}$$

so

$$|z| = |w|^{1/n} \tag{3}$$

and

$$\arg(z) = \frac{\phi}{n}. \tag{4}$$

Rolle proved that any number has n nth roots (Boyer 1968, p. 476). Householder (1970) gives an algorithm for constructing root-finding algorithms with an arbitrary order of convergence. Special root-finding techniques can often be applied when the function in question is a POLYNOMIAL.

see also BAILEY'S METHOD, BISECTION PROCEDURE, BRENT'S METHOD, CROUT'S METHOD, DESCARTES' SIGN RULE, FALSE POSITION METHOD, FUNDAMENTAL THEOREM OF SYMMETRIC FUNCTIONS, GRAEFFE'S METHOD, HALLEY'S IRRATIONAL FORMULA, HALLEY'S METHOD, HALLEY'S RATIONAL FORMULA, HORNER'S METHOD, HOUSEHOLDER'S METHOD, HUTTON'S METHOD, ISOGRAPH, JENKINS-TRAUB METHOD, LAGUERRE'S METHOD, LAMBERT'S METHOD, LEHMER-SCHUR METHOD, LIN'S METHOD, MAEHLY'S PROCEDURE, MULLER'S METHOD, NEWTON'S METHOD, RIDDERS' METHOD, ROOT DRAGGING THEOREM, SCHRÖDER'S METHOD, POLYNOMIAL, SECANT METHOD, STURM FUNCTION, STURM THEOREM, TANGENT HYPERBOLAS METHOD, WEIERSTRAß APPROXIMATION THEOREM

References

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Householder, A. S. *The Numerical Treatment of a Single Nonlinear Equation.* New York: McGraw-Hill, 1970.

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Root Dragging Theorem

If any of the ROOTS of a POLYNOMIAL are increased, then *all* of the critical points increase.

References

Anderson, B. "Polynomial Root Dragging." *Amer. Math. Monthly* **100**, 864–866, 1993.

Root Linear Coefficient Theorem

The sum of the reciprocals of ROOTS of an equation equals the NEGATIVE COEFFICIENT of the linear term in the MACLAURIN SERIES.

see also NEWTON'S RELATIONS

Root-Mean-Square

The root-mean-square (RMS) of a variate x, sometimes called the QUADRATIC MEAN, is the SQUARE ROOT of the mean squared value of x:

$$R(x) \equiv \sqrt{\langle x^2\rangle} \tag{1}$$

$$= \begin{cases} \sqrt{\dfrac{\sum_{i=1}^n x_i^2}{n}} & \text{for a discrete distribution} \\ \sqrt{\dfrac{\int P(x)x^2\,dx}{\int P(x)\,dx}} & \text{for a continuous distribution.} \end{cases} \tag{2}$$

Hoehn and Niven (1985) show that

$$R(a_1+c, a_2+c, \ldots, a_n+c) < c + R(a_1, a_2, \ldots, a_n)$$

for any POSITIVE constant c.

Physical scientists often use the term root-mean-square as a synonym for STANDARD DEVIATION when they refer to the SQUARE ROOT of the mean squared deviation of a signal from a given baseline or fit.

see also ARITHMETIC-GEOMETRIC MEAN, ARITHMETIC-HARMONIC MEAN, GENERALIZED MEAN, GEOMETRIC MEAN, HARMONIC MEAN, HARMONIC-GEOMETRIC MEAN, MEAN, MEDIAN (STATISTICS), STANDARD DEVIATION, VARIANCE

References

Hoehn, L. and Niven, I. "Averages on the Move." *Math. Mag.* **58**, 151–156, 1985.

Root (Radical)

The nth root (or "RADICAL") of a quantity z is a value r such that $z = r^n$, and therefore is the INVERSE FUNCTION to the taking of a POWER. The nth root is denoted $r = \sqrt[n]{z}$ or, using POWER notation, $r = z^{1/n}$. The special case of the SQUARE ROOT is denoted $\sqrt{z}$. The quantities for which a general FUNCTION equals 0 are also called ROOTS, or sometimes ZEROS.

see also CUBE ROOT, ROOT, SQUARE ROOT, VINCULUM

Root Test

Let u_k be a SERIES with POSITIVE terms, and let

$$\rho \equiv \lim_{k\to\infty} {u_k}^{1/k}.$$

1. If $\rho < 1$, the SERIES CONVERGES.
2. If $\rho > 1$ or $\rho = \infty$, the SERIES DIVERGES.
3. If $\rho = 1$, the SERIES may CONVERGE or DIVERGE.

This test is also called the CAUCHY ROOT TEST.

see also CONVERGENCE TESTS

References

Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 281–282, 1985.

Bromwich, T. J. I'a and MacRobert, T. M. *An Introduction to the Theory of Infinite Series, 3rd ed.* New York: Chelsea, pp. 31–39, 1991.

Root (Tree)

A special node which is designated to turn a TREE into a ROOTED TREE. The root is sometimes also called "EVE," and each of the nodes which is one EDGE further away from a given EDGE is called a CHILD. Nodes connected to the same node are then called SIBLINGS.

see also CHILD, ROOTED TREE, SIBLING, TREE

Root of Unity

The nth ROOTS of UNITY are ROOTS $\zeta_k = e^{2\pi ik/p}$ of the CYCLOTOMIC EQUATION

$$x^p = 1,$$

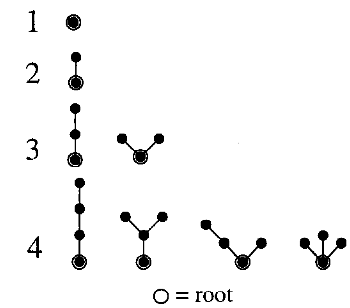
which are known as the DE MOIVRE NUMBERS.

see also CYCLOTOMIC EQUATION, DE MOIVRE'S IDENTITY, DE MOIVRE NUMBER, UNITY

References

Courant, R. and Robbins, H. "De Moivre's Formula and the Roots of Unity." §5.3 in *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, pp. 98–100, 1996.

Rooted Tree



A TREE with a special node called the "ROOT" or "EVE." Denote the number of rooted trees with n nodes by T_n, then the GENERATING FUNCTION is

$$T(x) \equiv \sum_{n=0}^{\infty} T_n x^n = x + x^2 + 2x^3 + 4x^4 + 9x^5 + 20x^6 + 48x^7 + 115x^8 + 286x^9 + 719x^{10} + \ldots \quad (1)$$

(Sloane's A000081). This POWER SERIES satisfies

$$T(x) = x \exp\left[\sum_{r=1}^{\infty} \frac{1}{r} T(x^r)\right] \quad (2)$$

$$t(x) = T(x) - \tfrac{1}{2}[T^2(x) - T(x^2)], \quad (3)$$

where $t(x)$ is the GENERATING FUNCTION for unrooted TREES. A GENERATING FUNCTION for T_n can be written using a product involving *the sequence itself* as

$$x \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{T_n}} = \sum_{n=1}^{\infty} T_n x^n. \quad (4)$$

The number of rooted trees can also be calculated from the RECURRENCE RELATION

$$T_{i+1} = \frac{1}{i} \sum_{j=1}^{i} \left(\sum_{d|j} dT_d\right) T_{i-j+1}, \quad (5)$$

with $T_0 = 0$ and $T_1 = 1$, where the second sum is over all d which DIVIDE j (Finch).

see also ORDERED TREE, RED-BLACK TREE, WEAKLY BINARY TREE

References

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/otter/otter.html`.

Ruskey, F. "Information on Rooted Trees." `http://sue.csc.uvic.ca/~cos/inf/tree/RootedTree.html`.

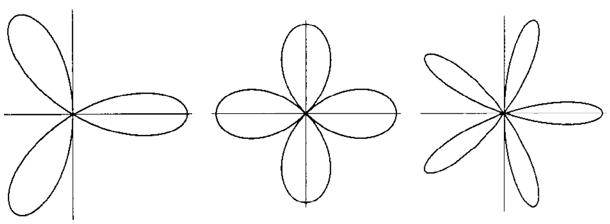
Sloane, N. J. A. Sequence A000081/M1180 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Rosatti's Theorem

There is a one-to-one correspondence between the sets of equivalent correspondences (not of value 0) on an irreducible curve of GENUS (CURVE) p, and the rational COLLINEATIONS of a projective space of $2p - 1$ dimensions which leave invariant a space of $p - 1$ dimensions. The number of linearly independent correspondences will be that of linearly independent COLLINEATIONS.

References
Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 339, 1959.

Rose



A curve which has the shape of a petalled flower. This curve was named RHODONEA by the Italian mathematician Guido Grandi between 1723 and 1728 because it resembles a rose (MacTutor Archive). The polar equation of the rose is

$$r = a\sin(n\theta),$$

or

$$r = a\cos(n\theta).$$

If n is ODD, the rose is n-petalled. If n is EVEN, the rose is $2n$-petalled. If n is IRRATIONAL, then there are an infinite number of petals.

The QUADRIFOLIUM is the rose with $n = 2$. The rose is the RADIAL CURVE of the EPICYCLOID.

see also DAISY, MAURER ROSE, STARR ROSE

References
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 175–177, 1972.
Lee, X. "Rose." `http://www.best.com/~xah/SpecialPlaneCurves_dir/Rose_dir/rose.html`.
MacTutor History of Mathematics Archive. "Rhodonea Curves." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Rhodonea.html`.
Wagon, S. "Roses." §4.1 in *Mathematica in Action.* New York: W. H. Freeman, pp. 96–102, 1991.

Rosenbrock Methods

A generalization of the RUNGE-KUTTA METHOD for solution of ORDINARY DIFFERENTIAL EQUATIONS, also called KAPS-RENTROP METHODS.

see also RUNGE-KUTTA METHOD

References
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 730–735, 1992.

Rössler Model

The nonlinear 3-D MAP

$$\begin{aligned}\dot{X} &= -(Y + Z)\\ \dot{Y} &= X + 0.2Y\\ \dot{Z} &= 0.2 + XZ - cZ.\end{aligned}$$

see also LORENZ SYSTEM

References
Dickau, R. M. "Rössler Attractor." `http://www.prairienet.org/~pops/rossler.html`.
Peitgen, H.-O.; Jürgens, H.; and Saupe, D. §12.3 in *Chaos and Fractals: New Frontiers of Science.* New York: Springer-Verlag, pp. 686–696, 1992.

Rotation

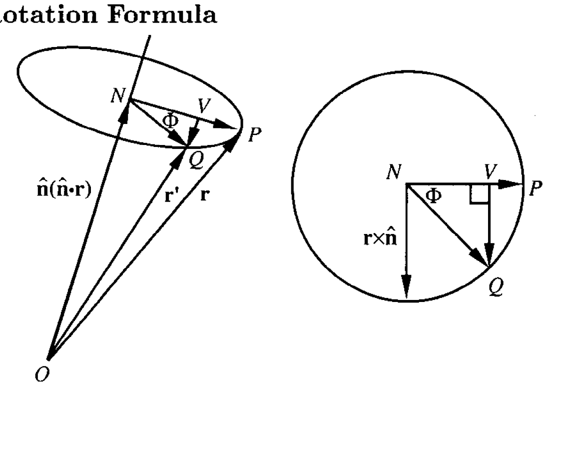
The turning of an object or coordinate system by an ANGLE about a fixed point. A rotation is an ORIENTATION-PRESERVING ORTHOGONAL TRANSFORMATION. EULER'S ROTATION THEOREM states that an arbitrary rotation can be parameterized using three parameters. These parameters are commonly taken as the EULER ANGLES. Rotations can be implemented using ROTATION MATRICES.

The rotation SYMMETRY OPERATION for rotation by $360°/n$ is denoted "n." For periodic arrangements of points ("crystals"), the CRYSTALLOGRAPHY RESTRICTION gives the only allowable rotations as 1, 2, 3, 4, and 6.

see also DILATION, EUCLIDEAN GROUP, EULER'S ROTATION THEOREM, EXPANSION, IMPROPER ROTATION, INFINITESIMAL ROTATION, INVERSION OPERATION, MIRROR PLANE, ORIENTATION-PRESERVING, ORTHOGONAL TRANSFORMATION, REFLECTION, ROTATION FORMULA, ROTATION GROUP, ROTATION MATRIX, ROTATION OPERATOR, ROTOINVERSION, SHIFT, TRANSLATION

References
Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 211, 1987.
Yates, R. C. "Instantaneous Center of Rotation and the Construction of Some Tangents." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 119–122, 1952.

Rotation Formula



A formula which relates the VECTOR $\mathbf{r}'$ to the ANGLE Φ in the above figure (Goldstein 1980). Referring to the figure,

$$\begin{aligned}\mathbf{r}' &= \overrightarrow{ON} + \overrightarrow{NV} + \overrightarrow{VQ} \\ &= \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r}) + [\mathbf{r} - \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r})]\cos\Phi + (\mathbf{r}\times\hat{\mathbf{n}})\sin\Phi \\ &= \mathbf{r}\cos\Phi + \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{r})(1-\cos\Phi) + (\mathbf{r}\times\hat{\mathbf{n}})\sin\Phi.\end{aligned}$$

The ANGLE Φ and unit normal $\hat{\mathbf{n}}$ may also be expressed as EULER ANGLES. In terms of the EULER PARAMETERS,

$$\mathbf{r}' = \mathbf{r}({e_0}^2 - {e_1}^2 - {e_2}^2 - {e_3}^2) + 2\mathbf{e}(\mathbf{e}\cdot\mathbf{r}) + 2(\mathbf{r}\times\hat{\mathbf{n}})\sin\Phi.$$

see also EULER ANGLES, EULER PARAMETERS

References

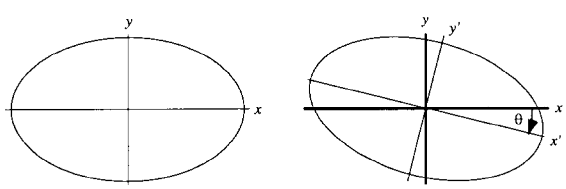
Goldstein, H. *Classical Mechanics, 2nd ed.* Reading, MA: Addison-Wesley, 1980.

Rotation Group

There are three representations of the rotation groups, corresponding to EXPANSION/DILATION, ROTATION, and SHEAR.

Rotation Matrix

When discussing a ROTATION, there are two possible conventions: rotation of the *axes* and rotation of the *object* relative to fixed axes.



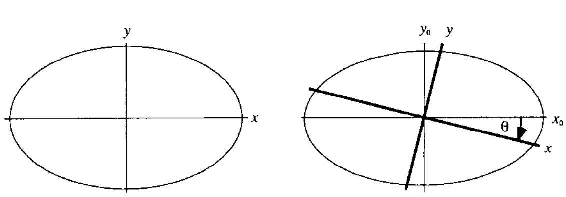
curve rotated by angle θ

In $\mathbb{R}^2$, let a curve be rotated by a clockwise ANGLE θ, so that the original axes of the curve are $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, and the new axes of the curve are $\hat{\mathbf{x}}'$ and $\hat{\mathbf{y}}'$. The MATRIX transforming the original curve to the rotated curve, referred to the original $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ axes, is

$$\mathsf{R}_\theta = \begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}, \tag{1}$$

i.e.,

$$\mathbf{x} = \mathsf{R}_\theta \mathbf{x}'. \tag{2}$$



axes rotated by angle θ

On the other hand, let the *axes* with respect to which a curve is measured be rotated by a clockwise ANGLE θ, so that the original axes are $\hat{\mathbf{x}}_0$ and $\hat{\mathbf{y}}_0$, and the new axes are $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$. Then the MATRIX transforming the coordinates of the curve with respect to $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ is given by the MATRIX TRANSPOSE of the above matrix:

$$\mathsf{R}'_\theta = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}, \tag{3}$$

i.e.,

$$\mathbf{x} = \mathsf{R}'_\theta \mathbf{x}_0. \tag{4}$$

In $\mathbb{R}^3$, rotations of the x-, y-, and z-axes give the matrices

$$\mathsf{R}_x(\alpha) = \begin{bmatrix}1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha\end{bmatrix} \tag{5}$$

$$\mathsf{R}_y(\beta) = \begin{bmatrix}\cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta\end{bmatrix} \tag{6}$$

$$\mathsf{R}_z(\gamma) = \begin{bmatrix}\cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1\end{bmatrix}. \tag{7}$$

see also EULER ANGLES, EULER'S ROTATION THEOREM, ROTATION

Rotation Number

The period for a QUASIPERIODIC trajectory to pass through the same point in a SURFACE OF SECTION. If the rotation number is IRRATIONAL, the trajectory will densely fill out a curve in the SURFACE OF SECTION. If the rotation number is RATIONAL, it is called the WINDING NUMBER, and only a finite number of points in the SURFACE OF SECTION will be visited by the trajectory.

see also QUASIPERIODIC FUNCTION, SURFACE OF SECTION, WINDING NUMBER (MAP)

Rotation Operator

The rotation operator can be derived from examining an INFINITESIMAL ROTATION

$$\left(\frac{d}{dt}\right)_{\text{space}} = \left(\frac{d}{dt}\right)_{\text{body}} + \boldsymbol{\omega}\times,$$

where d/dt is the time derivative, $\boldsymbol{\omega}$ is the ANGULAR VELOCITY, and $\times$ is the CROSS PRODUCT operator.

see also ACCELERATION, ANGULAR ACCELERATION, INFINITESIMAL ROTATION

Roth's Removal Rule

If the matrices A, X, B, and C satisfy

$$\mathsf{AX} - \mathsf{XB} = \mathsf{C},$$

then

$$\begin{bmatrix}\mathsf{I} & \mathsf{X} \\ 0 & \mathsf{I}\end{bmatrix}\begin{bmatrix}\mathsf{A} & \mathsf{C} \\ 0 & \mathsf{B}\end{bmatrix}\begin{bmatrix}\mathsf{I} & -\mathsf{X} \\ 0 & \mathsf{I}\end{bmatrix} = \begin{bmatrix}\mathsf{A} & 0 \\ 0 & \mathsf{B}\end{bmatrix},$$

where I is the IDENTITY MATRIX.

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Roth's Theorem

For ALGEBRAIC α

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}},$$

with $\epsilon > 0$, has finitely many solutions. Klaus Roth received a FIELDS MEDAL for this result.

see also HURWITZ EQUATION, HURWITZ'S IRRATIONAL NUMBER THEOREM, LAGRANGE NUMBER (RATIONAL APPROXIMATION), LIOUVILLE'S RATIONAL APPROXIMATION THEOREM, LIOUVILLE-ROTH CONSTANT, MARKOV NUMBER, SEGRE'S THEOREM, THUE-SIEGEL-ROTH THEOREM

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Rotkiewicz Theorem

If $n > 19$, there exists a base-2 PSEUDOPRIME between n and n^2. The theorem was proved in 1965.

see also PSEUDOPRIME

References
Rotkiewicz, A. "Les intervalles contenants les nombres pseudoprimiers." *Rend. Circ. Mat. Palermo Ser. 2* **14**, 278–280, 1965.
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Rotkiewicz, A. "Sur les nombres pseudoprimiers carrés." *Elem. Math.* **20**, 39–40, 1965.

Rotoinversion

see IMPROPER ROTATION

Rotor

A convex figure that can be rotated inside a POLYGON (or POLYHEDRON) while always touching every side (or face). The least AREA rotor in a SQUARE is the REULEAUX TRIANGLE. The least AREA rotor in an EQUILATERAL TRIANGLE is a LENS with two 60° ARCS of CIRCLES and RADIUS equal to the TRIANGLE ALTITUDE.

There exist nonspherical rotors for the TETRAHEDRON, OCTAHEDRON, and CUBE, but not for the DODECAHEDRON and ICOSAHEDRON.

see also LENS, REULEAUX TRIANGLE

References
Gardner, M. *The Unexpected Hanging and Other Mathematical Diversions.* Chicago, IL: Chicago University Press, p. 219, 1991.

Rotunda

A class of solids whose only true member is the PENTAGONAL ROTUNDA.

see also ELONGATED ROTUNDA, GYROELONGATED ROTUNDA, PENTAGONAL ROTUNDA, TRIANGULAR HEBESPHENOROTUNDA

References
Johnson, N. W. "Convex Polyhedra with Regular Faces." *Canad. J. Math.* **18**, 169–200, 1966.

Rouché's Theorem

Given two functions f and g ANALYTIC in A with γ a simple loop HOMOTOPIC to a point in A, if $|g(z)| < |f(z)|$ for all z on γ, then f and $f + g$ have the same number of ROOTS inside γ.

References
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Roulette

The curve traced by a fixed point on a closed convex curve as that curve rolls without slipping along a second curve. The roulettes described by the FOCI of CONICS when rolled upon a line are sections of MINIMAL SURFACES (i.e., they yield MINIMAL SURFACES when revolved about the line) known as UNDULOIDS.

Curve 1	Curve 2	Pole	Roulette
circle	exterior circle	on c.	epicycloid
circle	interior circle	on c.	hypocycloid
circle	line	on c.	cycloid
circle	same circle	any point	rose
circle involute	line	center	parabola
cycloid	line	center	ellipse
ellipse	line	focus	elliptic catenary
hyperbola	line	focus	hyperbolic catenary
hyperbolic spiral	line	origin	tractrix
line	any curve	on line	involute of curve
logarithmic spiral	line	any point	line
parabola	equal parabola	vertex	cissoid of Diocles
parabola	line	focus	catenary

see also GLISSETTE, UNDULOID

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Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 56–58 and 206, 1972.
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Round

see NINT

Rounding

The process of approximating a quantity, usually done for convenience or, in the case of numerical computations, of necessity. If rounding is performed on each of a series of numbers in a long computation, round-off errors can become important, especially if division by a small number ever occurs.

see also SHADOWING THEOREM

References

Wilkinson, J. H. *Rounding Errors in Algebraic Processes.* New York: Dover, 1994.

Routh-Hurwitz Theorem

Consider the CHARACTERISTIC EQUATION

$$|\lambda \mathsf{I} - \mathsf{A}| = \lambda^n + b_1\lambda^{n-1} + \ldots + b_{n-1}\lambda + b_n = 0$$

determining the n EIGENVALUES λ of a REAL $n \times n$ MATRIX A, where I is the IDENTITY MATRIX. Then the EIGENVALUES λ all have NEGATIVE REAL PARTS if

$$\Delta_1 > 0, \Delta_2 > 0, \ldots, \Delta_n > 0,$$

where

$$\Delta_k = \begin{vmatrix} b_1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ b_3 & b_2 & b_1 & 1 & 0 & 0 & \cdots & 0 \\ b_5 & b_4 & b_3 & b_2 & b_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{2k-1} & b_{2k-2} & b_{2k-3} & b_{2k-4} & b_{2k-5} & b_{2k-6} & \cdots & b_k \end{vmatrix}.$$

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Routh's Theorem

If the sides of a TRIANGLE are divided in the ratios $\lambda : 1$, $\mu : 1$, and $\nu : 1$, the CEVIANS form a central TRIANGLE whose AREA is

$$A = \frac{(\lambda\mu\nu - 1)^2}{(\lambda\mu + \lambda + 1)(\mu\nu + \mu + 1)(\nu\lambda + \nu + 1)}\Delta, \quad (1)$$

where Δ is the AREA of the original TRIANGLE. For $\lambda = \mu = \nu \equiv n$,

$$A = \frac{(n-1)^2}{n^2 + n + 1}\Delta. \quad (2)$$

For $n = 2, 3, 4, 5$, the areas are $\frac{1}{7}$, $\frac{3}{7}$, and $\frac{16}{31}$. The AREA of the TRIANGLE formed by connecting the division points on each side is

$$A' = \frac{\lambda\mu\nu}{(\lambda + 1)(\mu + 1)(\nu + 1)}\Delta. \quad (3)$$

Routh's theorem gives CEVA'S THEOREM and MENELAUS' THEOREM as special cases.

see also CEVA'S THEOREM, CEVIAN, MENELAUS' THEOREM

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RSA Encryption

A PUBLIC-KEY CRYPTOGRAPHY ALGORITHM which uses PRIME FACTORIZATION as the TRAPDOOR FUNCTION. Define

$$n \equiv pq \quad (1)$$

for p and q PRIMES. Also define a private key d and a public key e such that

$$de \equiv 1 \pmod{\phi(n)} \quad (2)$$

$$(e, \phi(n)) = 1, \quad (3)$$

where $\phi(n)$ is the TOTIENT FUNCTION.

Let the message be converted to a number M. The sender then makes n and e public and sends

$$E = M^e \pmod{n}. \quad (4)$$

To decode, the receiver (who knows d) computes

$$E^d \equiv (M^e)^d \equiv M^{ed} \equiv M^{N\phi(n)+1} \equiv M \pmod{n}, \quad (5)$$

since N is an INTEGER. In order to crack the code, d must be found. But this requires factorization of n since

$$\phi(n) = (p-1)(q-1). \quad (6)$$

Both p and q should be picked so that $p \pm 1$ and $q \pm 1$ are divisible by large PRIMES, since otherwise the POLLARD $p-1$ FACTORIZATION METHOD or WILLIAMS $p+1$ FACTORIZATION METHOD potentially factor n easily. It is also desirable to have $\phi(\phi(pq))$ large and divisible by large PRIMES.

It is possible to break the cryptosystem by repeated encryption if a unit of $\mathbb{Z}/\phi(n)\mathbb{Z}$ has small ORDER (Simmons and Norris 1977, Meijer 1996), where $\mathbb{Z}/s\mathbb{Z}$ is the RING of INTEGERS between 0 and $s-1$ under addition and multiplication (mod s). Meijer (1996) shows that "almost" every encryption exponent e is safe from breaking using repeated encryption for factors of the form

$$p = 2p_1 + 1 \qquad (7)$$

$$q = 2q_1 + 1, \qquad (8)$$

where

$$p_1 = 2p_2 + 1 \qquad (9)$$

$$q_1 = 2q_2 + 1, \qquad (10)$$

and p, p_1, p_2, q, q_1, and q_2 are all PRIMES. In this case,

$$\phi(n) = 4p_1q_1 \qquad (11)$$

$$\phi(\phi(n)) = 8p_2q_2. \qquad (12)$$

Meijer (1996) also suggests that p_2 and q_2 should be of order 10^{75}.

Using the RSA system, the identity of the sender can be identified as genuine without revealing his private code.

see also PUBLIC-KEY CRYPTOGRAPHY

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Simmons, G. J. and Norris, M. J. "Preliminary Comments on the MIT Public-Key Cryptosystem." *Cryptologia* **1**, 406–414, 1977.

RSA Number

Numbers contained in the "factoring challenge" of RSA Data Security, Inc. An additional number which is not part of the actual challenge is the RSA-129 number. The RSA numbers which have been factored are RSA-100, RSA-110, RSA-120, RSA-129, and RSA-130 (Cowie *et al.* 1996).

RSA-129 is a 129-digit number used to encrypt one of the first public-key messages. This message was published by R. Rivest, A. Shamir, and L. Adleman (Gardner 1977), along with the number and a $100 reward for its decryption. Despite belief that the message encoded by RSA-129 "would take millions of years of break," RSA-129 was factored in 1994 using a distributed computation which harnessed networked computers spread around the globe performing a multiple polynomial QUADRATIC SIEVE FACTORIZATION METHOD. The effort was coordinated by P. Leylad, D. Atkins, and M. Graff. They received 112,011 full factorizations, 1,431,337 single partial factorizations, and 8,881,138 double partial factorizations out of a factor base of 524,339 PRIMES. The final MATRIX obtained was $188,346 \times 188,346$ square.

The text of the message was "The magic words are squeamish ossifrage" (an ossifrage is a rare, predatory vulture found in the mountains of Europe), and the FACTORIZATION (into a 64-DIGIT number and a 65-DIGIT number) is

$$\begin{aligned}&114381625757888867669235779976146612010218296\cdots\\&\cdots 7212423625625618429357069352457338978305971\cdots\\&\cdots 235639587050589890751475992900268795435 41\\&= 3490529510847650949147849619903898133417764\cdots\\&\cdots 638493387843990820577 \cdot 3276913299326\cdots\\&\cdots 670954996198819083446141317764296799 2\cdots\\&\cdots 942539798288533\end{aligned}$$

(Leutwyler 1994, Cipra 1995).

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Rubber-Sheet Geometry

see ALGEBRAIC TOPOLOGY

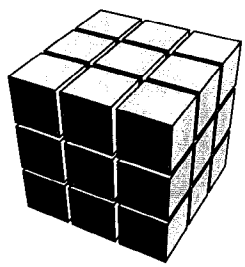
Rubik's Clock

A puzzle consisting of 18 small clocks. There are 12^{18} possible configurations, although not all are realizable.

see also RUBIK'S CUBE

References
Dénes, J. and Mullen, G. L. "Rubik's Clock and Its Solution." *Math. Mag.* **68**, 378–381, 1995.
Zeilberger, D. "Doron Zeilberger's Maple Packages and Programs: RubikClock." http://www.math.temple.edu/~zeilberg/programs.html.

Rubik's Cube



A $3\times3\times3$ CUBE in which the 26 subcubes on the outside are internally hinged in such a way that rotation (by a quarter turn in either direction or a half turn) is possible in any plane of cubes. Each of the six sides is painted a distinct color, and the goal of the puzzle is to return the cube to a state in which each side has a single color after it has been randomized by repeated rotations. The PUZZLE was invented in the 1970s by the Hungarian Erno Rubik and sold millions of copies worldwide over the next decade.

The number of possible positions of Rubik's cube is

$$\frac{8!12!3^8 2^{12}}{2\cdot 3\cdot 2} = 43{,}252{,}003{,}274{,}489{,}856{,}000$$

(Turner and Gold 1985). Hoey showed using the PÓLYA-BURNSIDE LEMMA that there are 901,083,404,981,813,-616 positions up to conjugacy by whole-cube symmetries.

Algorithms exist for solving a cube from an arbitrary initial position, but they are not necessarily optimal (i.e., requiring a minimum number of turns). The maximum number of turns required for an arbitrary starting position is still not known, although it is bounded from above. Michael Reid (1995) produced the best proven bound of 29 turns (or 42 "quarter-turns"). The proof involves large tables of "subroutines" generated by computer.

However, Dik Winter has produced a program based on work by Kociemba which has solved each of millions of cubes in at most 21 turns. Recently, Richard Korf (1997) has produced a different algorithm which is practical for cubes up to 18 moves away from solved. Out of 10 randomly generated cubes, one was solved in 16 moves, three required 17 moves, and six required 18 moves.

see also RUBIK'S CLOCK

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Turner, E. C. and Gold, K. F. "Rubik's Groups." *Amer. Math. Monthly* **92**, 617–629, 1985.

Rudin-Shapiro Sequence

The sequence of numbers given by

$$a_n = (-1)^{\sum_{i=1}^{k-1} \epsilon_i \epsilon_{i+1}}, \tag{1}$$

where n is written in binary

$$n = \epsilon_1 \epsilon_2 \ldots \epsilon_k. \tag{2}$$

It is therefore the parity of the number of pairs of consecutive 1s in the BINARY expansion of n. The SUMMATORY sequence is

$$s_n \equiv \sum_{j=0}^{n} a_j, \tag{3}$$

which gives

$$s_n = \begin{cases} 2^{k/2}+1 & \text{if } k \text{ is even} \\ 2^{(k-1)/2}+1 & \text{if } k \text{ is odd} \end{cases} \tag{4}$$

(Blecksmith and Laud 1995).

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Rudvalis Group

The SPORADIC GROUP *Ru*.

see also SPORADIC GROUP

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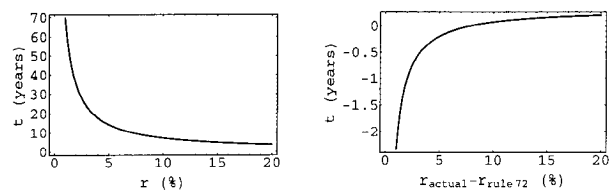
Wilson, R. A. "ATLAS of Finite Group Representation." http://for.mat.bham.ac.uk/atlas/Ru.html.

Rule

A usually simple ALGORITHM or IDENTITY. The term is frequently applied to specific orders of NEWTON-COTES FORMULAS.

see also ALGORITHM, BAC-CAB RULE, BODE'S RULE, CHAIN RULE, CRAMER'S RULE, DESCARTES' SIGN RULE, DURAND'S RULE, ESTIMATOR, EULER'S RULE, EULER'S TOTIENT RULE, GOLDEN RULE, HARDY'S RULE, HORNER'S RULE, IDENTITY, L'HOSPITAL'S RULE, LEIBNIZ INTEGRAL RULE, METHOD, OSBORNE'S RULE, PASCAL'S RULE, POWER RULE, PRODUCT RULE, QUARTER SQUARES RULE, QUOTA RULE, QUOTIENT RULE, ROTH'S REMOVAL RULE, RULE OF 72, SIMPSON'S RULE, SLIDE RULE, SUM RULE, TRAPEZOIDAL RULE, WEDDLE'S RULE, ZEUTHEN'S RULE

Rule of 72



The time required for a given PRINCIPAL to double (assuming $n = 1$ CONVERSION PERIOD) for COMPOUND INTEREST is given by solving

$$2P = P(1+r)^t, \quad (1)$$

or

$$t = \frac{\ln 2}{\ln(1+r)}, \quad (2)$$

where LN is the NATURAL LOGARITHM. This function can be approximated by the so-called "rule of 72":

$$t \approx \frac{0.72}{r}. \quad (3)$$

The above plots show the actual doubling time t (left plot) and difference between actual and time calculated using the rule of 72 (right plot) as a function of the interest rate r.

see also COMPOUND INTEREST, INTEREST

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Ruled Surface

A SURFACE which can be swept out by a moving LINE in space and therefore has a parameterization of the form

$$\mathbf{x}(u,v) = \mathbf{b}(u) + v\boldsymbol{\delta}(u), \quad (1)$$

where $\mathbf{b}$ is called the DIRECTRIX (also called the BASE CURVE) and $\boldsymbol{\delta}$ is the DIRECTOR CURVE. The straight lines themselves are called RULINGS. The rulings of a ruled surface are ASYMPTOTIC CURVES. Furthermore, the GAUSSIAN CURVATURE on a ruled REGULAR SURFACE is everywhere NONPOSITIVE.

Examples of ruled surfaces include the elliptic HYPERBOLOID of one sheet (a doubly ruled surface)

$$\begin{bmatrix} a(cosu \mp v\sin u) \\ b(\sin u \pm \cos u) \\ \pm cv \end{bmatrix} = \begin{bmatrix} a\cos u \\ b\sin u \\ 0 \end{bmatrix} \pm v \begin{bmatrix} -a\cos u \\ b\sin u \\ c \end{bmatrix}, \quad (2)$$

the HYPERBOLIC PARABOLOID (a doubly ruled surface)

$$\begin{bmatrix} a(u+v) \\ \pm bv \\ u^2+2uv \end{bmatrix} = \begin{bmatrix} au \\ 0 \\ u^2 \end{bmatrix} + v \begin{bmatrix} a \\ \pm b \\ 2u \end{bmatrix}, \quad (3)$$

PLÜCKER'S CONOID

$$\begin{bmatrix} r\cos\theta \\ r\sin\theta \\ 2\cos\theta\sin\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2\cos\theta\sin\theta \end{bmatrix} + r \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \quad (4)$$

and the MÖBIUS STRIP

$$a \begin{bmatrix} \cos u + v\cos(\tfrac{1}{2}u)\cos u \\ \sin u + v\cos(\tfrac{1}{2}u)\sin u \\ v\sin(\tfrac{1}{2}u) \end{bmatrix} = a \begin{bmatrix} \cos u \\ \sin u \\ 0 \end{bmatrix} + av \begin{bmatrix} \cos(\tfrac{1}{2}u)\cos u \\ \cos(\tfrac{1}{2}u)\sin u \\ \sin(\tfrac{1}{2}u) \end{bmatrix} \quad (5)$$

(Gray 1993).

The only ruled MINIMAL SURFACES are the PLANE and HELICOID (Catalan 1842, do Carmo 1986).

see also ASYMPTOTIC CURVE, CAYLEY'S RULED SURFACE, DEVELOPABLE SURFACE, DIRECTOR CURVE, DIRECTRIX (RULED SURFACE), GENERALIZED CONE, GENERALIZED CYLINDER, HELICOID, NONCYLINDRICAL RULED SURFACE, PLANE, RIGHT CONOID, RULING

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Ruler

A STRAIGHTEDGE with markings to indicate distances. Although GEOMETRIC CONSTRUCTIONS are sometimes said to be performed with a ruler and COMPASS, the term STRAIGHTEDGE is preferable to ruler since markings are not allowed by the classical Greek rules.

see also COASTLINE PARADOX, COMPASS, GEOMETRIC CONSTRUCTION, GEOMETROGRAPHY, GOLOMB RULER, PERFECT RULER, SIMPLICITY, SLIDE RULE, STRAIGHTEDGE

Ruler Function

The exponent of the largest POWER of 2 which DIVIDES a given number k. The values of the ruler function are 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, ... (Sloane's A001511).

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Ruling

One of the straight lines sweeping out a RULED SURFACE. The rulings on a ruled surface are ASYMPTOTIC CURVES.

see also ASYMPTOTIC CURVE, DIRECTOR CURVE, DIRECTRIX (RULED SURFACE), RULED SURFACE

Run

A run is a sequence of more than one consecutive identical outcomes, also known as a CLUMP. Given n BERNOULLI TRIALS (say, in the form of COIN TOSSINGS), the probability $P_t(n)$ of a run of t consecutive heads or tails is given by the RECURRENCE RELATION

$$P_t(n) = P_t(n-1) + 2^{-t}[1 - P_t(n-t)], \tag{1}$$

where $P_t(n) = 0$ for $n < t$ and $P_t(t) = 2^{1-t}$ (Bloom 1996).

Let $C_t(m,k)$ denote the number of sequences of m indistinguishable objects of type A and k indistinguishable objects of type B in which *no* t-run occurs. The probability that a t-run *does* occur is then given by

$$P_t(m,k) = 1 - \frac{C_t(m,k)}{\binom{m+k}{k}}, \tag{2}$$

where $\binom{a}{b}$ is a BINOMIAL COEFFICIENT. Bloom (1996) gives the following recurrence sequence for $C_t(m,k)$,

$$C_t(m,k) = \sum_{i=0}^{t-1} C_t(m-1, k-i) - \sum_{i=1}^{t-1} C_t(m-t, k-i) + e_t(m,k), \tag{3}$$

where

$$e_t(m,k) \equiv \begin{cases} 1 & \text{if } m = 0 \text{ and } 0 \le k < t \\ -1 & \text{if } m = t \text{ and } 0 \le k < t \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Another recurrence which has only a fixed number of terms is given by

$$C_t(m,k) = C_t(m-1,k) + C_t(m,k-1) - C_t(m-t,k-1) - C_t(m-1,k-t) + C_t(m-t,k-t) + e_t^*(m,k), \tag{5}$$

where

$$e_t^*(m,k) \equiv \begin{cases} 1 & \text{if } (m,k) = (0,0) \text{ or } (t,t) \\ -1 & \text{if } (m,k) = (0,t) \text{ or } (t,0) \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

(Goulden and Jackson 1983, Bloom 1996). These formulas disprove the assertion of Gardner (1982) that "there will almost always be a clump of six or seven CARDS of the same color" in a normal deck of cards by giving $P_6(26,26) = 0.46424$.

Given n BERNOULLI TRIALS with a probability of success (heads) p, the expected number of tails is $n(1-p)$, so the expected number of tail runs ≥ 1 is $\approx n(1-p)p$. Continuing,

$$N_R = n(1-p)p^R \tag{7}$$

is the expected number of runs $\ge R$. The longest expected run is therefore given by

$$R = \log_{1/p}[n(1-p)] \tag{8}$$

(Gordon *et al.* 1986, Schilling 1990). Given m 0s and n 1s, the number of possible arrangements with u runs is

$$f_u = \begin{cases} 2\binom{m-1}{k-1}\binom{n-1}{k-1} & u \equiv 2k \\ \binom{m-1}{k-1}\binom{n-1}{k-2} + \binom{m-1}{k-2}\binom{n-1}{k-1} & u \equiv 2k+1 \end{cases} \tag{9}$$

for k an INTEGER, where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT. Then

$$P(u \le u') = \sum_{u=2}^{u'} \frac{f_u}{\binom{m+n}{m}}. \tag{10}$$

Bloom (1996) gives the expected number of noncontiguous t-runs in a sequence of m 0s and n 1s as

$$E(n,m,t) = \frac{(m+1)(n)_t + (n+1)(m)_t}{(m+n)_t}, \tag{11}$$

where $(a)_n$ is the POCHHAMMER SYMBOL. For $m > 10$, u has an approximately NORMAL DISTRIBUTION with MEAN and VARIANCE

$$\mu_u = 1 + \frac{2mn}{m+n} \tag{12}$$

$$\sigma_u{}^2 = \frac{2mn(2mn-m-n)}{(m+n)^2(m+n-1)}. \tag{13}$$

see also COIN TOSSING, EULERIAN NUMBER, PERMUTATION, s-RUN

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Runge-Kutta Method

A method of integrating ORDINARY DIFFERENTIAL EQUATIONS by using a trial step at the midpoint of an interval to cancel out lower-order error terms. The second-order formula is

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1) \\ y_{n+1} &= y_n + k_2 + \mathcal{O}(h^3), \end{aligned}$$

and the fourth-order formula is

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1) \\ k_3 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2) \\ k_4 &= hf(x_n + h, y_n + k_3) \\ y_{n+1} &= y_n + \tfrac{1}{6}k_1 + \tfrac{1}{3}k_2 + \tfrac{1}{3}k_3 + \tfrac{1}{6}k_4 + \mathcal{O}(h^5). \end{aligned}$$

(Press *et al.* 1992). This method is reasonably simple and robust and is a good general candidate for numerical solution of differential equations when combined with an intelligent adaptive step-size routine.

see also ADAMS' METHOD, GILL'S METHOD, MILNE'S METHOD, ORDINARY DIFFERENTIAL EQUATION, ROSENBROCK METHODS

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Runge-Walsh Theorem

Let $f(x)$ be an ANALYTIC FUNCTION which is REGULAR in the interior of a JORDAN CURVE C and continuous in the closed DOMAIN bounded by C. Then $f(x)$ can be approximated with an arbitrary accuracy by POLYNOMIALS.

see also ANALYTIC FUNCTION, JORDAN CURVE

References

Szegő, G. *Orthogonal Polynomials, 4th ed.* Providence, RI: Amer. Math. Soc., p. 7, 1975.

Running Knot

A KNOT which tightens around an object when strained but slackens when the strain is removed. Running knots are sometimes also known as slip knots or nooses.

References

Owen, P. *Knots.* Philadelphia, PA: Courage, p. 60, 1993.

Russell's Antinomy

Let R be the set of all sets which are not members of themselves. Then R is neither a member of itself nor not a member of itself. Symbolically, let $R = \{x : x \notin x\}$. Then $R \in R$ IFF $R \notin R$.

Bertrand Russell discovered this PARADOX and sent it in a letter to G. Frege just as Frege was completing *Grundlagen der Arithmetik.* This invalidated much of the rigor of the work, and Frege was forced to add a note at the end stating, "A scientist can hardly meet with anything more undesirable than to have the foundation give way just as the work is finished. I was put in this position by a letter from Mr. Bertrand Russell when the work was nearly through the press."

see also GRELLING'S PARADOX

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Russell's Paradox

see RUSSELL'S ANTINOMY

Russian Multiplication

Also called ETHIOPIAN MULTIPLICATION. To multiply two numbers a and b, write $a_0 \equiv a$ and $b_0 \equiv b$ in two columns. Under a_0, write $\lfloor a_0/2 \rfloor$, where $\lfloor x \rfloor$ is the FLOOR FUNCTION, and under b_0, write $2b_0$. Continue until $a_i = 1$. Then cross out any entries in the b column which are opposite an EVEN NUMBER in the a column and add the b column. The result is the desired product. For example, for $a = 27, b = 35$

$$\begin{array}{rr} 27 & 35 \\ 13 & 70 \\ 6 & \cancel{140} \\ 3 & 280 \\ 1 & \underline{560} \\ & 945 \end{array}$$

Russian Roulette

Russian roulette is a GAME of chance in which one or more of the six chambers of a gun are filled with bullets, the magazine is rotated at random, and the gun is shot. The shooter bets on whether the chamber which rotates into place will be loaded. If it is, he loses not only his bet but his life.

A modified version is considered by Blom *et al.* (1996) and Blom (1989). In this variant, the revolver is loaded with a single bullet, and two duelists alternately spin the chamber and fire at themselves until one is killed. The probability that the first duelist is killed is then 6/11.

References

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Blom, G.; Englund, J,.-E.; and Sandell, D. "General Russian Roulette." *Math. Mag.* **69**, 293–297, 1996.

Ryser Formula

A formula for the PERMANENT of a MATRIX

$$\operatorname{perm}(a_{ij}) = (-1)^n \sum_{s\subseteq\{1,\ldots,n\}} (-1)^{|s|} \prod_{i=1}^n \sum_{j\in s} a_{ij},$$

where the SUM is over all SUBSETS of $\{1, \ldots, n\}$, and $|s|$ is the number of elements in s. The formula can be optimized by picking the SUBSETS so that only a single element is changed at a time (which is precisely a GRAY CODE), reducing the number of additions from n^2 to n.

It turns out that the number of disks moved after the kth step in the TOWERS OF HANOI is the same as the element which needs to be added or deleted in the kth ADDEND of the RYSER FORMULA (Gardner 1988, Vardi 1991, p. 111)

see also DETERMINANT, GRAY CODE, PERMANENT, TOWERS OF HANOI

References

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S

s-Additive Sequence

A generalization of an ULAM SEQUENCE in which each term is the SUM of two earlier terms in exactly s ways. (s,t)-additive sequences are a further generalization in which each term has exactly s representations as the SUM of t distinct earlier numbers. It is conjectured that 0-additive sequences ultimately have periodic differences of consecutive terms (Guy 1994, p. 233).

see also GREEDY ALGORITHM, STÖHR SEQUENCE, ULAM SEQUENCE

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s-Cluster

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let an $n \times n$ MATRIX have entries which are either 1 (with probability p) or 0 (with probability $q = 1 - p$). An s-cluster is an isolated group of s adjacent (i.e., horizontally or vertically connected) 1s. Let C_n be the total number of "SITE" clusters. Then the value

$$K_S(p) = \lim_{n\to\infty} \frac{\langle C_n \rangle}{n^2}, \qquad (1)$$

called the MEAN CLUSTER COUNT PER SITE or MEAN CLUSTER DENSITY, exists. Numerically, it is found that $K_S(1/2) \approx 0.065770\ldots$ (Ziff *et al.* 1997).

Considering instead "BOND" clusters (where numbers are assigned to the edges of a grid) and letting C_n be the total number of bond clusters, then

$$K_B(p) \equiv \lim_{n\to\infty} \frac{\langle C_n \rangle}{n^2} \qquad (2)$$

exists. The analytic value is known for $p = 1/2$,

$$K_B(\tfrac{1}{2}) = \tfrac{3}{2}\sqrt{3} - \tfrac{41}{16} \qquad (3)$$

(Ziff *et al.* 1997).

see also BOND PERCOLATION, PERCOLATION THEORY, s-RUN, SITE PERCOLATION

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Ziff, R.; Finch, S.; and Adamchik, V. "Universality of Finite-Sized Corrections to the Number of Percolation Clusters." *Phys. Rev. Let.* To appear, 1998.

s-Run

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let $\mathbf{v}$ be a n-VECTOR whose entries are each 1 (with probability p) or 0 (with probability $q = 1 - p$). An s-run is an isolated group of s consecutive 1s. Ignoring the boundaries, the total number of runs R_n satisfies

$$K_n = \frac{\langle R_n \rangle}{n} = (1-p)^2 \sum_{s=1}^n p^s = p(1-p)(1-p^n),$$

so

$$K(p) \equiv \lim_{n\to\infty} K_n = p(1-p),$$

which is called the MEAN RUN COUNT PER SITE or MEAN RUN DENSITY in PERCOLATION THEORY.

see also PERCOLATION THEORY, s-CLUSTER

References

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S-Signature

see SIGNATURE (RECURRENCE RELATION)

Saalschützian

For a GENERALIZED HYPERGEOMETRIC FUNCTION

$${}_{p+1}F_p \left[\begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_{p+1} \\ \beta_1, \beta_2, \ldots, \beta_p \end{matrix} ; z \right],$$

the Saalschützian S is defined if

$$\sum \beta = \sum \alpha + 1.$$

see also GENERALIZED HYPERGEOMETRIC FUNCTION

Saalschütz's Theorem

$${}_3F_2 \left[\begin{matrix} -x, -y, -z \\ n+1, -x-y-z \end{matrix} \right] = \frac{\Gamma(n+1)\Gamma(x+y+n+1)}{\Gamma(x+n+1)\Gamma(y+n+1)} \times \frac{\Gamma(y+z+n+1)\Gamma(z+x+n+1)}{\Gamma(z+n+1)\Gamma(x+y+z+n+1)},$$

where ${}_3F_2(a,b,c;d,e;z)$ is a GENERALIZED HYPERGEOMETRIC FUNCTION and $\Gamma(z)$ is the GAMMA FUNCTION.

It can be derived from the DOUGALL-RAMANUJAN IDENTITY and written in the symmetric form

$$ {}_3F_2(a,b,c;d,e;1) = \frac{(d-a)_{|c|}(d-b)_{|c|}}{d_{|c|}(d-a-b)_{|c|}} $$

for $d+e=a+b+c+1$ with c a negative integer and $(a)_n$ the POCHHAMMER SYMBOL (Petkovšek *et al.* 1996).

see also DOUGALL-RAMANUJAN IDENTITY, GENERALIZED HYPERGEOMETRIC FUNCTION

References

Petkovšek, M.; Wilf, H. S.; and Zeilberger, D. *A=B*. Wellesley, MA: A. K. Peters, pp. 43 and 126, 1996.

Saddle

A SURFACE possessing a SADDLE POINT.

see also HYPERBOLIC PARABOLOID, MONKEY SADDLE, SADDLE POINT (FUNCTION)

Saddle-Node Bifurcation

see FOLD BIFURCATION

Saddle Point (Fixed Point)

see HYPERBOLIC FIXED POINT (DIFFERENTIAL EQUATIONS), HYPERBOLIC FIXED POINT (MAP)

Saddle Point (Function)

A POINT of a FUNCTION or SURFACE which is a STATIONARY POINT but not an EXTREMUM. An example of a 1-D FUNCTION with a saddle point is $f(x)=x^3$, which has

$$ \begin{aligned} f'(x) &= 3x^2 \\ f''(x) &= 6x \\ f'''(x) &= 6. \end{aligned} $$

This function has a saddle point at $x_0=0$ by the EXTREMUM TEST since $f''(x_0)=0$ and $f'''(x_0)=6\neq 0$. An example of a SURFACE with a saddle point is the MONKEY SADDLE.

Saddle Point (Game)

For a general two-player ZERO-SUM GAME,

$$ \min_{i\leq m}\min_{j\leq n} a_{ij} \leq \min_{j\leq n}\max_{i\leq m} a_{ij}. $$

If the two are equal, then write

$$ \min_{i\leq m}\min_{j\leq n} a_{ij} = \min_{j\leq n}\max_{i\leq m} a_{ij} \equiv v, $$

where v is called the VALUE of the GAME. In this case, there exist optimal strategies for the first and second players.

A NECESSARY and SUFFICIENT condition for a saddle point to exist is the presence of a PAYOFF MATRIX element which is both a minimum of its row and a maximum of its column. A GAME may have more than one saddle point, but all must have the same VALUE.

see also GAME, PAYOFF MATRIX, VALUE

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Dresher, M. "Saddle Points." §1.5 in *The Mathematics of Games of Strategy: Theory and Applications*. New York: Dover, pp. 12–14, 1981.
Llewellyn, D. C.; Tovey, C.; and Trick, M. "Finding Saddlepoints of Two-Person, Zero Sum Games." *Amer. Math. Monthly* **95**, 912–918, 1988.

Safarevich Conjecture

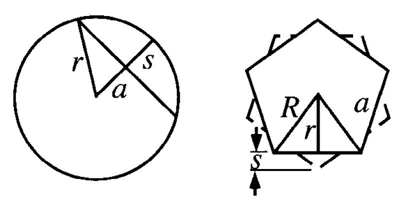
see SHAFAREVICH CONJECTURE

Safe

A position in a GAME is safe if the person who plays next will lose.

see also GAME, UNSAFE

Sagitta



The PERPENDICULAR distance s from an ARC's MIDPOINT to the CHORD across it, equal to the RADIUS r minus the APOTHEM a,

$$ s = r - a. \tag{1} $$

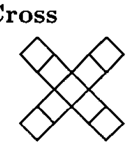
For a regular POLYGON of side length a,

$$ \begin{aligned} s \equiv R - r &= \tfrac{1}{2}a\left[\csc\left(\frac{\pi}{n}\right) - \cot\left(\frac{\pi}{n}\right)\right] \\ &= \tfrac{1}{2}a\tan\left(\frac{\pi}{2n}\right) && (2) \\ &= r\tan\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{2n}\right) && (3) \\ &= 2R\sin^2\left(\frac{\pi}{2n}\right), && (4) \end{aligned} $$

where R is the CIRCUMRADIUS, r the INRADIUS, a is the side length, and n is the number of sides.

see also APOTHEM, CHORD, SECTOR, SEGMENT

Saint Andrew's Cross



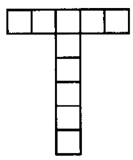
A GREEK CROSS rotated by 45°, also called the crux decussata. The MULTIPLICATION SIGN × is based on Saint Andrew's cross (Bergamini 1969).

see also CROSS, GREEK CROSS, MULTIPLICATION SIGN

References

Bergamini, D. *Mathematics.* New York: Time-Life Books, p. 11, 1969.

Saint Anthony's Cross



A CROSS also called the tau cross or crux commissa.

see also CROSS

Saint Petersburg Paradox

Consider a game in which a player bets on whether a given TOSS of a COIN will turn up heads or tails. If he bets $1 that heads will turn up on the first throw, $2 that heads will turn up on the second throw (if it did not turn up on the first), $4 that heads will turn up on the third throw, etc., his expected *payoff* is

$$\tfrac{1}{2}(1)+\tfrac{1}{4}(2)+\tfrac{1}{8}(4)+\ldots=\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\ldots=\infty.$$

Apparently, the first player can be in the hole by any amount of money and still come out ahead in the end. This PARADOX was first proposed by Daniel Bernoulli.

The paradox arises as a result of muddling the distinction between the amount of the final payoff and the net amount won in the game. It is misleading to consider the payoff without taking into account the amount lost on previous bets, as can be shown as follows. At the time the player first wins (say, on the nth toss), he will have lost

$$\sum_{k=1}^{n-1} 2^{k-1}=2^{n-1}-1$$

dollars. In this toss, however, he wins 2^{n-1} dollars. This means that the net gain for the player is a whopping $1, no matter how many tosses it takes to finally win. As expected, the large payoff after a long run of tails is exactly balanced by the large amount that the player has to invest.

In fact, by noting that the probability of winning on the nth toss is $1/2^n$, it can be seen that the probability distribution for the number of tosses needed to win is simply a GEOMETRIC DISTRIBUTION with $p=1/2$.

see also COIN TOSSING, GAMBLER'S RUIN, GEOMETRIC DISTRIBUTION, MARTINGALE

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Sal

see WALSH FUNCTION

Salamin Formula

see BRENT-SALAMIN FORMULA

Salem Constants

Each point of the PISOT-VIJAYARAGHAVAN CONSTANTS S is a LIMIT POINT from both sides of a set T known as the Salem constants (Salem 1945). The Salem constants are algebraic INTEGERS > 1 in which one or more of the conjugates is on the UNIT CIRCLE with the others inside (Le Lionnais 1983, p. 150). The smallest known Salem number was found by Lehmer (1933) as the largest REAL ROOT of

$$x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1=0,$$

which is

$$\sigma_1=1.176280818\ldots$$

(Le Lionnais 1983, p. 35). Boyd (1977) found the following table of small Salem numbers, and suggested that σ_1, σ_2, σ_3, and σ_4 are the smallest Salem numbers. The NOTATION 1 1 0 −1 −1 −1 is short for 1 1 0 −1 −1 −1 −1 −1 0 1 1, the coefficients of the above polynomial.

k	σ_k	°	polynomial
1	1.1762808183	10	1 1 0 −1 −1 −1
2	1.1883681475	18	1 −1 1 −1 0 0 −1 1 −1 1
3	1.2000265240	14	1 0 0 −1 −1 0 0 1
4	1.2026167437	14	1 0 −1 0 0 0 0 −1
5	1.2163916611	10	1 0 0 0 −1 −1
6	1.2197208590	18	1 −1 0 0 0 0 0 0 −1 1
7	1.2303914344	10	1 0 0 −1 0 −1
8	1.2326135486	20	1 −1 0 0 0 −1 1 0 0 −1 1
9	1.2356645804	22	1 0 −1 −1 0 0 0 1 1 0 −1 −1
10	1.2363179318	16	1 −1 0 0 0 0 0 0 −1
11	1.2375048212	26	1 0 −1 0 0 −1 0 0 −1 0 1 0 0 1
12	1.2407264237	12	1 −1 1 −1 0 0 −1
13	1.2527759374	18	1 0 0 0 0 0 −1 −1 −1 −1
14	1.2533306502	20	1 0 −1 0 0 −1 0 0 0 0 0
15	1.2550935168	14	1 0 −1 −1 0 1 0 −1
16	1.2562211544	18	1 −1 0 0 −1 1 0 0 0 −1
17	1.2601035404	24	1 −1 0 0 −1 1 0 −1 1 −1 0 1 −1
18	1.2602842369	22	1 −1 0 −1 1 0 0 0 −1 1 −1 1
19	1.2612309611	10	1 0 −1 0 0 −1
20	1.2630381399	26	1 −1 0 0 0 0 −1 0 0 0 0 0 0 1
21	1.2672964425	14	1 −1 0 0 0 0 −1 1
22	1.2806381563	8	1 0 0 −1 −1
23	1.2816913715	26	1 0 0 0 0 0 −1 −1 −1 −1 −1 −1 −1 −1
24	1.2824955606	20	1 −2 2 −2 2 −2 1 0 −1 1 −1
25	1.2846165509	18	1 0 0 0 −1 0 −1 −1 0 −1
26	1.2847468215	26	1 −2 1 1 −2 1 0 0 −1 1 0 −1 1 −1
27	1.2850993637	30	1 0 0 0 0 −1 −1 −1 −1 −1 −1 0 0 0 0 1
28	1.2851215202	30	1 −2 2 −2 1 0 −1 2 −2 1 0 −1 1 −1 1 −1
29	1.2851856708	30	1 −1 0 0 0 0 0 0 −1 0 0 0 −1 0 0 −1
30	1.2851967268	26	1 0 −1 −1 0 0 0 1 0 −1 −1 0 1 1
31	1.2851991792	44	1 −1 0 0 0 0 0 −1 0 0 0 −1 0 0 0 0 0 0 0 1 0 0 1
32	1.2852354362	30	1 0 −1 0 0 −1 −1 0 0 0 1 0 0 1 0 −1
33	1.2854090648	34	1 −1 0 0 −1 1 −1 0 1 −1 1 0 −1 1 −1 0 1 −1
34	1.2863959668	18	1 −2 2 −2 2 −2 2 −3 3 −3
35	1.2867301820	26	1 −1 0 0 −1 1 −1 0 1 −1 1 0 −1 1
36	1.2917414257	24	1 −1 0 0 0 0 −1 0 0 0 0 0 0
37	1.2920391602	20	1 0 −1 0 0 −1 0 0 −1 0 1
38	1.2934859531	10	1 0 −1 −1 0 1
39	1.2956753719	18	1 −1 0 0 −1 1 −1 0 1 −1

see also PISOT-VIJAYARAGHAVAN CONSTANTS

References

Boyd, D. W. "Small Salem Numbers." *Duke Math. J.* **44**, 315–328, 1977.

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Lehmer, D. H. "Factorization of Certain Cyclotomic Functions." *Ann. Math., Ser. 2* **34**, 461–479, 1933.

Salem, R. "Power Series with Integral Coefficients." *Duke Math. J.* **12**, 153–172, 1945.

Stewart, C. L. "Algebraic Integers whose Conjugates Lie Near the Unit Circle." *Bull. Soc. Math. France* **106**, 169–176, 1978.

Salesman Problem

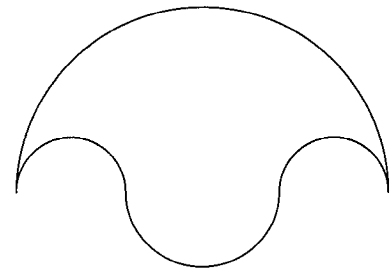
see TRAVELING SALESMAN PROBLEM

Salient Point

A point at which two noncrossing branches of a curve meet with different tangents.

see also CUSP

Salinon



The above figure formed from four connected SEMICIRCLES. The word salinon is Greek for salt cellar, which the figure resembles.

see also ARBELOS, PIECEWISE CIRCULAR CURVE, SEMICIRCLE

Salmon's Theorem

Given a track bounded by two confocal ELLIPSES, if a ball is rolled so that its trajectory is tangent to the inner ELLIPSE, the ball's trajectory will be tangent to the inner ELLIPSE following all subsequent caroms as well.

References

Salmon, G. *A Treatise on Conic Sections.* New York: Chelsea, p. 182, 1954.

Saltus

The word saltus has two different meanings: either a jump or an oscillation of a function.

Sample Proportion

Let there be x successes out of n BERNOULLI TRIALS. The sample proportion is the fraction of samples which were successes, so

$$\hat{p} = \frac{x}{n}. \tag{1}$$

For large n, $\hat{p}$ has an approximately NORMAL DISTRIBUTION. Let RE be the RELATIVE ERROR and SE the STANDARD ERROR, then

$$\langle p \rangle = p \tag{2}$$

$$\mathrm{SE}(\hat{p}) \equiv \sigma(\hat{p}) = \sqrt{\frac{p(1-p)}{n}} \tag{3}$$

$$\mathrm{RE}(\hat{p}) = \sqrt{\frac{2\hat{p}(1-\hat{p})}{n}}\,\mathrm{erf}^{-1}(\mathrm{CI}), \tag{4}$$

where CI is the CONFIDENCE INTERVAL and $\operatorname{erf} x$ is the ERF function. The number of tries needed to determine p with RELATIVE ERROR RE and CONFIDENCE INTERVAL CI is

$$n = \frac{2[\mathrm{erf}^{-1}(\mathrm{CI})]^2 \hat{p}(1-\hat{p})}{(\mathrm{RE})^2}. \tag{5}$$

Sample Space

Informally, the sample space for a given set of events is the set of all possible values the events may assume. Formally, the set of possible events for a given variate forms a SIGMA ALGEBRA, and sample space is defined as the largest set in the SIGMA ALGEBRA.

see also PROBABILITY SPACE, RANDOM VARIABLE, SIGMA ALGEBRA, STATE SPACE

Sample Variance

To estimate the population VARIANCE from a sample of N elements with a priori *unknown* MEAN (i.e., the MEAN is estimated from the sample itself), we need an unbiased ESTIMATOR for σ. This is the k-STATISTIC k_2, where

$$k_2 = \frac{N}{N-1} m_2 \tag{1}$$

and $m_2 \equiv s^2$ is the sample variance

$$s^2 \equiv \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2. \tag{2}$$

Note that some authors prefer the definition

$$s'^2 \equiv \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2, \tag{3}$$

since this makes the sample variance an UNBIASED ESTIMATOR for the population variance.

see also k-STATISTIC, VARIANCE

Sampling

For infinite precision sampling of a band-limited signal at the NYQUIST FREQUENCY, the signal-to-noise ratio after N_q samples is

$$\mathrm{SNR} = \frac{\langle r_\infty \rangle}{\sigma_\infty} = \frac{\rho\sigma^2}{\sigma^2 {N_q}^{-1/2} \sqrt{1+\rho^2}} = \frac{\rho}{\sqrt{1+\rho^2}} \sqrt{N_q}, \tag{1}$$

where ρ is the normalized cross-correlation COEFFICIENT

$$\rho \equiv \frac{\langle x(t) \rangle \langle y(t) \rangle}{\sqrt{\langle x^2(t) \rangle \langle y^2(t) \rangle}}. \tag{2}$$

For $\rho \ll 1$,

$$\mathrm{SNR} \approx \rho \sqrt{N_q}. \tag{3}$$

The identical result is obtained for oversampling. For undersampling, the SNR decreases (Thompson *et al.* 1986).

see also NYQUIST SAMPLING, OVERSAMPLING, QUANTIZATION EFFICIENCY, SAMPLING FUNCTION, SHANNON SAMPLING THEOREM, SINC FUNCTION

References

Feuer, A. *Sampling in Digital Signal Processing and Control.* Boston, MA: Birkhäuser, 1996.

Thompson, A. R.; Moran, J. M.; and Swenson, G. W. Jr. *Interferometry and Synthesis in Radio Astronomy.* New York: Wiley, pp. 214–216, 1986.

Sampling Function

The 1-D sampling function is given by

$$S(x) = \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x),$$

where δ is the DIRAC DELTA FUNCTION. The 2-D version is

$$S(u,v) = \sum \delta(u - u_n, v - v_n),$$

which can be weighted to

$$S(u,v) = \sum R_n T_n D_n \delta(u - u_n, v - v_n),$$

where R_n is a reliability weight, D_n is a density weight (WEIGHTING FUNCTION), and T_n is a taper.

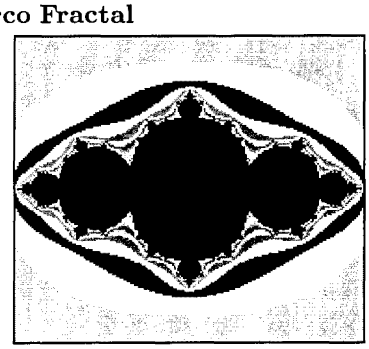
see also SHAH FUNCTION, SINC FUNCTION

Sampling Theorem

In order for a band-limited (i.e., one with a zero POWER SPECTRUM for frequencies $f > B$) baseband ($f > 0$) signal to be reconstructed fully, it must be sampled at a rate $f \geq 2B$. A signal sampled at $f = 2B$ is said to be NYQUIST SAMPLED, and $f = 2B$ is called the NYQUIST FREQUENCY. No information is lost if a signal is sampled at the NYQUIST FREQUENCY, and no additional information is gained by sampling faster than this rate.

see also ALIASING, NYQUIST FREQUENCY, NYQUIST SAMPLING, OVERSAMPLING

San Marco Fractal



The FRACTAL $J(-3/4, 0)$, where J is the JULIA SET. It slightly resembles the MANDELBROT SET.

see also DOUADY'S RABBIT FRACTAL, JULIA SET, MANDELBROT SET

References

Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, p. 173, 1991.

Sandwich Theorem

see HAM SANDWICH THEOREM, SQUEEZING THEOREM

Sard's Theorem

The set of "critical values" of a MAP $u: \mathbb{R}^n \to \mathbb{R}^n$ of CLASS C^1 has LEBESGUE MEASURE 0 in $\mathbb{R}^n$.

see also CLASS (MAP), LEBESGUE MEASURE

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 682, 1980.

Šarkovskii's Theorem

Order the NATURAL NUMBERS as follows:

$$\begin{aligned}3 \prec 5 \prec 7 \prec 9 \prec 11 \prec 13 \prec 15 \prec \ldots \prec 2\cdot 3 \prec 2\cdot 5 \prec 2\cdot 7\\ \prec 2\cdot 9 \prec \ldots \prec 2\cdot 2\cdot 3 \prec 2\cdot 2\cdot 5 \prec 2\cdot 2\cdot 7\\ \prec 2\cdot 2\cdot 9 \prec \ldots \prec 2\cdot 2\cdot 2\cdot 3 \prec \ldots\\ \prec 2^5 \prec 2^4 \prec 2^3 \prec 2^2 \prec 2 \prec 1.\end{aligned}$$

Now let F be a CONTINUOUS FUNCTION from the REALS to the REALS and suppose $p \prec q$ in the above ordering. Then if F has a point of LEAST PERIOD p, then F also has a point of LEAST PERIOD q.

A special case of this general result, also known as Šarkovskii's theorem, states that if a CONTINUOUS REAL function has a PERIODIC POINT with period 3, then there is a PERIODIC POINT of period n for every INTEGER n.

A converse to Šarkovskii's theorem says that if $p \prec q$ in the above ordering, then we can find a CONTINUOUS FUNCTION which has a point of LEAST PERIOD q, but does not have any points of LEAST PERIOD p (Elaydi 1996). For example, there is a CONTINUOUS FUNCTION with no points of LEAST PERIOD 3 but having points of all other LEAST PERIODS.

see also LEAST PERIOD

References

Conway, J. H. and Guy, R. K. "Periodic Points." In *The Book of Numbers.* New York: Springer-Verlag, pp. 207–208, 1996.
Devaney, R. L. *An Introduction to Chaotic Dynamical Systems, 2nd ed.* Reading, MA: Addison-Wesley, 1989.
Elaydi, S. "On a Converse of Sharkovsky's Theorem." *Amer. Math. Monthly* **103**, 386–392, 1996.
Ott, E. *Chaos in Dynamical Systems.* New York: Cambridge University Press, p. 49, 1993.
Sharkovsky, A. N. "Co-Existence of Cycles of a Continuous Mapping of a Line onto Itself." *Ukranian Math. Z.* **16**, 61–71, 1964.
Stefan, P. "A Theorem of Sharkovsky on the Existence of Periodic Orbits of Continuous Endomorphisms of the Real Line." *Comm. Math. Phys.* **54**, 237–248, 1977.

Sárközy's Theorem

A partial solution to the ERDŐS SQUAREFREE CONJECTURE which states that the BINOMIAL COEFFICIENT $\binom{2n}{n}$ is never SQUAREFREE for all sufficiently large $n \geq n_0$. Sárközy (1985) showed that if $s(n)$ is the square part of the BINOMIAL COEFFICIENT $\binom{2n}{n}$, then

$$\ln s(n) \sim (\sqrt{2} - 2)\zeta(\tfrac{1}{2})\sqrt{n},$$

where $\zeta(z)$ is the RIEMANN ZETA FUNCTION. An upper bound on n_0 of $2^{8,000}$ has been obtained.

see also BINOMIAL COEFFICIENT, ERDŐS SQUAREFREE CONJECTURE

References

Erdős, P. and Graham, R. L. *Old and New Problems and Results in Combinatorial Number Theory.* Geneva, Switzerland: L'Enseignement Mathématique Université de Genève, Vol. 28, 1980.
Sander, J. W. "A Story of Binomial Coefficients and Primes." *Amer. Math. Monthly* **102**, 802–807, 1995.
Sárközy, A. "On the Divisors of Binomial Coefficients, I." *J. Number Th.* **20**, 70–80, 1985.
Vardi, I. "Applications to Binomial Coefficients." *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, pp. 25–28, 1991.

Sarrus Linkage

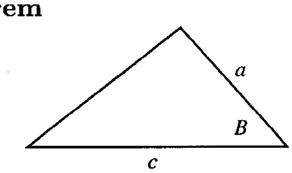
A LINKAGE which converts circular to linear motion using a hinged square.

see also HART'S INVERSOR, LINKAGE, PEAUCELLIER INVERSOR

Sarrus Number

see POULET NUMBER

SAS Theorem



Specifying two sides and the ANGLE between them uniquely determines a TRIANGLE. Let b be the base length and h be the height. Then the AREA is

$$K = \tfrac{1}{2}ch = \tfrac{1}{2}ac\sin B. \quad (1)$$

The length of the third side is given by the LAW OF COSINES,

$$b^2 = a^2 + c^2 - 2ac\cos B,$$

so

$$b = \sqrt{a^2 + c^2 - 2ac\cos B}. \quad (2)$$

Using the LAW OF SINES

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad (3)$$

then gives the two other ANGLES as

$$A = \sin^{-1}\left(\frac{a\sin B}{\sqrt{a^2 + c^2 - 2ac\cos B}}\right) \quad (4)$$

$$C = \sin^{-1}\left(\frac{c\sin B}{\sqrt{a^2 + c^2 - 2ac\cos B}}.\right) \quad (5)$$

see also AAA THEOREM, AAS THEOREM, ASA THEOREM, ASS THEOREM, SSS THEOREM, TRIANGLE

Satellite Knot

Let K_1 be a knot inside a TORUS. Now knot the TORUS in the shape of a second knot (called the COMPANION KNOT) K_2. Then the new knot resulting from K_1 is called the satellite knot K_3. COMPOSITE KNOTS are special cases of satellite knots. The only KNOTS which are not HYPERBOLIC KNOTS are TORUS KNOTS and satellite knots (including COMPOSITE KNOTS). No satellite knot is an ALMOST ALTERNATING KNOT.

see also ALMOST ALTERNATING KNOT, COMPANION KNOT, COMPOSITE KNOT, HYPERBOLIC KNOT, TORUS KNOT

References

Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, pp. 115–118, 1994.

Satisfiability Problem

Deciding whether a given Boolean formula in conjunctive normal form has an assignment that makes the formula "true." In 1971, Cook showed that the problem is NP-COMPLETE.

see also BOOLEAN ALGEBRA

References

Cook, S. A. and Mitchell, D. G. "Finding Hard Instances of the Satisfiability Problem: A Survey." In *Satisfiability problem: theory and applications (Piscataway, NJ, 1996). Theoret. Comput. Sci., Vol. 35.* Providence, RI: Amer. Math. Soc., pp. 1–17, 1997.

Sausage Conjecture

In n-D for $n \geq 5$ the arrangement of HYPERSPHERES whose CONVEX HULL has minimal CONTENT is always a "sausage" (a set of HYPERSPHERES arranged with centers along a line), independent of the number of n-spheres. The CONJECTURE was proposed by Fejes Tóth, and solved for dimensions ≥ 42 by Betke *et al.* (1994) and Betke and Henk (1998).

see also CONTENT, CONVEX HULL, HYPERSPHERE, HYPERSPHERE PACKING, SPHERE PACKING

References

Betke, U.; Henk, M.; and Wills, J. M. "Finite and Infinite Packings." *J. Reine Angew. Math.* **453**, 165–191, 1994.

Betke, U. and Henk, M. "Finite Packings of Spheres." *Discrete Comput. Geom.* **19**, 197–227, 1998.

Croft, H. T.; Falconer, K. J.; and Guy, R. K. Problem D9 in *Unsolved Problems in Geometry.* New York: Springer-Verlag, 1991.

Fejes Tóth, L. "Research Problems." *Periodica Methematica Hungarica* **6**, 197–199, 1975.

Savitzky-Golay Filter

A low-pass filter which is useful for smoothing data.

see also FILTER

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 183 and 644–645, 1992.

Savoy Knot

see FIGURE-OF-EIGHT KNOT

Scalar

A one-component quantity which is invariant under ROTATIONS of the coordinate system.

see also PSEUDOSCALAR, SCALAR FIELD, SCALAR FUNCTION, SCALAR POTENTIAL, SCALAR TRIPLE PRODUCT, TENSOR, VECTOR

Scalar Curvature

see CURVATURE SCALAR

Scalar Field

A MAP $f : \mathbb{R}^n \mapsto \mathbb{R}$ which assigns each $\mathbf{x}$ a SCALAR FUNCTION $f(\mathbf{x})$.

see also VECTOR FIELD

References

Morse, P. M. and Feshbach, H. "Scalar Fields." §1.1 in *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 4–8, 1953.

Scalar Function

A function $f(x_1, \ldots, x_n)$ of one or more variables whose RANGE is one-dimensional, as compared to a VECTOR FUNCTION, whose RANGE is three-dimensional (or, in general, n-dimensional).

see also COMPLEX FUNCTION, REAL FUNCTION, VECTOR FUNCTION

Scalar Potential

A conservative VECTOR FIELD (for which the CURL $\nabla \times \mathbf{F} = \mathbf{0}$) may be assigned a scalar potential

$$\begin{aligned}\phi(x,y,z) - \phi(0,0,0) &\equiv -\int_C \mathbf{F} \cdot d\mathbf{s} \\ &= -\int_{(0,0,0)}^{(x,0,0)} F_1(t,0,0)\,dt + \int_{(x,0,0)}^{(x,y,0)} F_2(x,t,0)\,dt \\ &\quad + \int_{(x,y,0)}^{x,y,z} F_3(x,y,t)\,dt,\end{aligned}$$

where $\int_C \mathbf{F} \cdot d\mathbf{s}$ is a LINE INTEGRAL.

see also POTENTIAL FUNCTION, VECTOR POTENTIAL

Scalar Triple Product

The VECTOR product

$$\begin{aligned}[\mathbf{A},\mathbf{B},\mathbf{C}] &\equiv \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) \\ &= \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix},\end{aligned}$$

which yields a SCALAR (actually, a PSEUDOSCALAR).

The VOLUME of a PARALLELEPIPED whose sides are given by the vectors **A**, **B**, and **C** is

$$V_{\text{parallelepiped}} = |\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|.$$

see also CROSS PRODUCT, DOT PRODUCT, PARALLELEPIPED, VECTOR TRIPLE PRODUCT

References

Arfken, G. "Triple Scalar Product, Triple Vector Product." §1.5 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 26–33, 1985.

Scale

see BASE (NUMBER)

Scale Factor

For a diagonal METRIC TENSOR $g_{ij} = g_{ii}\delta_{ij}$, where δ_{ij} is the KRONECKER DELTA, the scale factor is defined by

$$h_i \equiv \sqrt{g_{ii}}. \tag{1}$$

The LINE ELEMENT (first FUNDAMENTAL FORM) is then given by

$$\begin{aligned} ds^2 &= g_{11}\, dx_{11}{}^2 + g_{22}\, dx_{22}{}^2 + g_{33}\, dx_{33}{}^2 && (2)\\ &= h_1{}^2\, dx_{11}{}^2 + h_2{}^2\, dx_{22}{}^2 + h_3{}^2\, dx_{33}{}^2. && (3)\end{aligned}$$

The scale factor appears in vector derivatives of coordinates in CURVILINEAR COORDINATES.

see also CURVILINEAR COORDINATES, FUNDAMENTAL FORMS, LINE ELEMENT

Scalene Triangle

A TRIANGLE with three unequal sides.

see also ACUTE TRIANGLE, EQUILATERAL TRIANGLE, ISOSCELES TRIANGLE, OBTUSE TRIANGLE, TRIANGLE

Scaling

Increasing a plane figure's linear dimensions by a scale factor s increases the PERIMETER $p' \to sp$ and the AREA $A' \to s^2 A$.

see also DILATION, EXPANSION, FRACTAL, SELF-SIMILARITY

Scattering Operator

An OPERATOR relating the past asymptotic state of a DYNAMICAL SYSTEM governed by the Schrödinger equation

$$i\frac{d}{dt}\psi(t) = H\psi(t)$$

to its future asymptotic state.

see also WAVE OPERATOR

Scattering Theory

The mathematical study of the SCATTERING OPERATOR and Schrödinger equation.

see also SCATTERING OPERATOR

References

Yafaev, D. R. *Mathematical Scattering Theory: General Theory.* Providence, RI: Amer. Math. Soc., 1996.

Schaar's Identity

A generalization of the GAUSSIAN SUM. For p and q of opposite PARITY (i.e., one is EVEN and the other is ODD), Schaar's identity states

$$\frac{1}{\sqrt{q}}\sum_{r=0}^{q-1} e^{-\pi i r^2 p/q} = \frac{e^{-\pi i/4}}{\sqrt{p}}\sum_{r=0}^{p-1} e^{\pi i r^2 q/p}.$$

see also GAUSSIAN SUM

References

Evans, R. and Berndt, B. "The Determination of Gauss Sums." *Bull. Amer. Math. Soc.* **5**, 107–129, 1981.

Schanuel's Conjecture

Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be linearly independent over the RATIONALS $\mathbb{Q}$, then

$$\mathbb{Q}(\lambda_1, \ldots, \lambda_n, e^{\lambda_1}, \ldots, e^{\lambda_n})$$

has TRANSCENDENCE degree at least n over $\mathbb{Q}$. Schanuel's conjecture is a generalization of the LINDEMANN-WEIERSTRAß THEOREM. If the conjecture is true, then it follows that e and π are algebraically independent. Mcintyre (1991) proved that the truth of Schanuel's conjecture also guarantees that there are no unexpected exponential-algebraic relations on the INTEGERS $\mathbb{Z}$ (Marker 1996).

see also CONSTANT PROBLEM

References

Macintyre, A. "Schanuel's Conjecture and Free Exponential Rings." *Ann. Pure Appl. Logic* **51**, 241–246, 1991.
Marker, D. "Model Theory and Exponentiation." *Not. Amer. Math. Soc.* **43**, 753–759, 1996.

Schauder Fixed Point Theorem

Let A be a closed convex subset of a BANACH SPACE and assume there exists a continuous MAP T sending A to a countably compact subset $T(A)$ of A. Then T has fixed points.

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 543, 1980.
Schauder, J. "Der Fixpunktsatz in Funktionalräumen." *Studia Math.* **2**, 171–180, 1930.
Zeidler, E. *Applied Functional Analysis: Applications to Mathematical Physics.* New York: Springer-Verlag, 1995.

Scheme

A local-ringed SPACE which is locally isomorphic to an AFFINE SCHEME.

see also AFFINE SCHEME

References

Iyanaga, S. and Kawada, Y. (Eds.). "Schemes." §18E in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 69, 1980.

Schensted Correspondence

A correspondence between a PERMUTATION and a pair of YOUNG TABLEAUX.

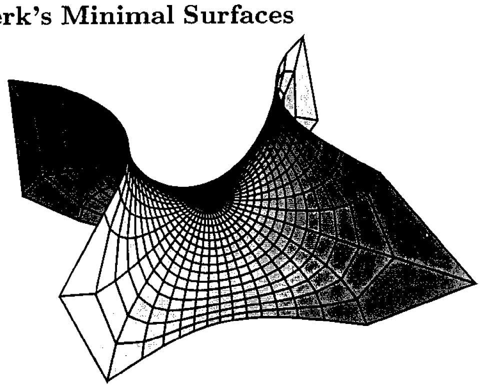
see also PERMUTATION, YOUNG TABLEAU

References

Knuth, D. E. *The Art of Computer Programming, Vol. 3: Sorting and Searching, 2nd ed.* Reading, MA: Addison-Wesley, 1973.

Stanton, D. W. and White, D. E. §3.6 in *Constructive Combinatorics.* New York: Springer-Verlag, pp. 85–87, 1986.

Scherk's Minimal Surfaces



A class of MINIMAL SURFACES discovered by Scherk (1834) which were the first new surfaces discovered since Meusnier in 1776. Scherk's first surface is doubly periodic. Scherk's second surface, illustrated above, can be written parametrically as

$$\begin{aligned} x &= 2\Re[\ln(1+re^{i\theta})-\ln(1-re^{i\theta})] \\ y &= \Re[4i\tan^{-1}(re^{i\theta})] \\ z &= \Re\left\{2i(-\ln[1-r^2e^{2i\theta}]+\ln[1+r^2e^{2i\theta}])\right\} \end{aligned}$$

for $\theta \in [0, 2\pi)$, and $r \in (0,1)$. Scherk's first surface has been observed to form in layers of block copolymers (Peterson 1988).

von Seggern (1993) calls

$$z = c\ln\left[\frac{\cos(2\pi y)}{\cos(2\pi x)}\right]$$

"Scherk's surface." Beautiful images of wood sculptures of Scherk surfaces are illustrated by Séquin.

References

Dickson, S. "Minimal Surfaces." *Mathematica J.* **1**, 38–40, 1990.

do Carmo, M. P. *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, p. 41, 1986.

Meusnier, J. B. "Mémoire sur la courbure des surfaces." *Mém. des savans étrangers* **10** (lu 1776), 477–510, 1785.

Peterson, I. "Geometry for Segregating Polymers." *Sci. News*, 151, Sep. 3, 1988.

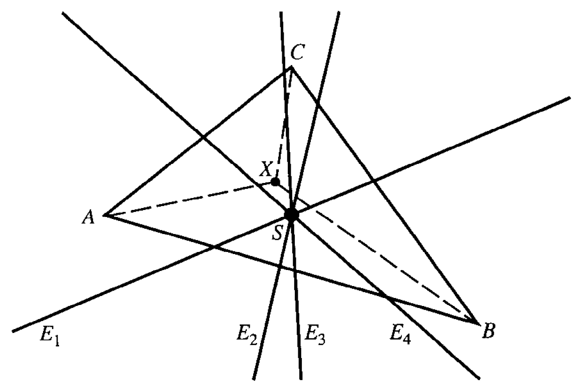
Scherk, H. F. "Bemerkung über der kleinste Fläche innerhalb gegebener Grenzen." *J. Reine. angew. Math.* **13**, 185–208, 1834.

Thomas, E. L.; Anderson, D. M.; Henkee, C. S.; and Hoffman, D. "Periodic Area-Minimizing Surfaces in Block Copolymers." *Nature* **334**, 598–601, 1988.

von Seggern, D. *CRC Standard Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 304, 1993.

Wolfram Research "Mathematica Version 2.0 Graphics Gallery." `http://www.mathsource.com/cgi-bin/MathSource/Applications/Graphics/3D/0207-155`.

Schiffler Point



The CONCURRENCE S of the EULER LINES E_n of the TRIANGLES ΔXBC, ΔXCA, ΔXAB, and ΔABC where X is the INCENTER. The TRIANGLE CENTER FUNCTION is

$$\alpha = \frac{1}{\cos B + \cos C} = \frac{b+c-a}{b+c}.$$

References

Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.

Kimberling, C. "Schiffler Point." `http://www.evansville.edu/~ck6/tcenters/recent/schiff.html`.

Schiffler, K.; Veldkamp, G. R.; and van der Spek, W. A. "Problem 1018 and Solution." *Crux Math.* **12**, 176–179, 1986.

Schinzel Circle

A CIRCLE having a given number of LATTICE POINTS on its CIRCUMFERENCE. The Schinzel circle halving n lattice points is given by the equation

$$\begin{cases} (x-\frac{1}{2})^2+y^2 = \frac{1}{4}5^{k-1} & \text{for } n=2k \text{ even} \\ (x-\frac{1}{3})^2+y^2 = \frac{1}{9}5^{2k} & \text{for } n=2k+1 \text{ odd.} \end{cases}$$

Note that these solutions do not necessarily have the smallest possible RADIUS. For example, while the Schinzel circle centered at (1/3, 0) and with radius 625/3 has nine lattice points on its circumference, so does the circle centered at (1/3, 0) with radius 65/3.

see also CIRCLE, CIRCLE LATTICE POINTS, KULIKOWSKI'S THEOREM, LATTICE POINT, SCHINZEL'S THEOREM, SPHERE

References
Honsberger, R. "Circles, Squares, and Lattice Points." Ch. 11 in *Mathematical Gems I.* Washington, DC: Math. Assoc. Amer., pp. 117–127, 1973.
Kulikowski, T. "Sur l'existence d'une sphère passant par un nombre donné aux coordonnées entières." *L'Enseignement Math. Ser. 2* **5**, 89–90, 1959.
Schinzel, A. "Sur l'existence d'un cercle passant par un nombre donné de points aux coordonnées entières." *L'Enseignement Math. Ser. 2* **4**, 71–72, 1958.
Sierpiński, W. "Sur quelques problèmes concernant les points aux coordonnées entières." *L'Enseignement Math. Ser. 2* **4**, 25–31, 1958.
Sierpiński, W. "Sur un problème de H. Steinhaus concernant les ensembles de points sur le plan." *Fund. Math.* **46**, 191–194, 1959.
Sierpiński, W. *A Selection of Problems in the Theory of Numbers.* New York: Pergamon Press, 1964.

Schinzel's Hypothesis

If $f_1(x)$, ..., $f_s(x)$ are irreducible POLYNOMIALS with INTEGER COEFFICIENTS such that no INTEGER $n > 1$ divides $f_1(x)$, ..., $f_s(x)$ for all INTEGERS x, then there should exist infinitely many x such that $f_1(x), \ldots, f_s(x)$ are simultaneous PRIME.

References
Schinzel, A. and Sierpiński, W. "Sur certaines hypothéses concernant les nombres premiers. Remarque." *Acta Arithm.* **4**, 185–208, 1958.

Schinzel's Theorem

For every POSITIVE INTEGER n, there exists a CIRCLE in the plane having exactly n LATTICE POINTS on its CIRCUMFERENCE. The theorem is based on the number $r(n)$ of integral solutions (x, y) to the equation

$$x^2 + y^2 = n, \tag{1}$$

given by

$$r(n) = 4(d_1 - d_3), \tag{2}$$

where d_1 is the number of divisors of n of the form $4k+1$ and d_3 is the number of divisors of the form $4k+3$. It explicitly identifies such circles (the SCHINZEL CIRCLES) as

$$\begin{cases} (x-\frac{1}{2})^2 + y^2 = \frac{1}{4}5^{k-1} & \text{for } n = 2k \\ (x-\frac{1}{3})^2 + y^2 = \frac{1}{9}5^{2k} & \text{for } n = 2k+1. \end{cases} \tag{3}$$

Note, however, that these solutions do not necessarily have the smallest possible radius.

see also BROWKIN'S THEOREM, KULIKOWSKI'S THEOREM, SCHINZEL CIRCLE

References
Honsberger, R. "Circles, Squares, and Lattice Points." Ch. 11 in *Mathematical Gems I.* Washington, DC: Math. Assoc. Amer., pp. 117–127, 1973.
Kulikowski, T. "Sur l'existence d'une sphère passant par un nombre donné aux coordonnées entières." *L'Enseignement Math. Ser. 2* **5**, 89–90, 1959.
Schinzel, A. "Sur l'existence d'un cercle passant par un nombre donné de points aux coordonnées entières." *L'Enseignement Math. Ser. 2* **4**, 71–72, 1958.
Sierpiński, W. "Sur quelques problèmes concernant les points aux coordonnées entières." *L'Enseignement Math. Ser. 2* **4**, 25–31, 1958.
Sierpiński, W. "Sur un problème de H. Steinhaus concernant les ensembles de points sur le plan." *Fund. Math.* **46**, 191–194, 1959.
Sierpiński, W. *A Selection of Problems in the Theory of Numbers.* New York: Pergamon Press, 1964.

Schisma

The musical interval by which eight fifths and a major third exceed five octaves,

$$\frac{(\frac{3}{2})^8(\frac{5}{4})}{2^5} = \frac{3^8 \cdot 5}{2^{15}} = \frac{32805}{32768} = 1.00112915\ldots.$$

see also COMMA OF DIDYMUS, COMMA OF PYTHAGORAS, DIESIS

Schläfli Double Six

see DOUBLE SIXES

Schläfli's Formula

For $\Re[z] > 0$,

$$J_\nu(z) = \frac{1}{\pi}\int_0^{\pi/2} \cos(z\sin t - \nu t)\,dt - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\sinh t}e^{-\nu t}\,dt,$$

where $J_\nu(z)$ is a BESSEL FUNCTION OF THE FIRST KIND.

References
Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1472, 1980.

Schläfli Function

The function giving the VOLUME of the spherical quadrectangular TETRAHEDRON:

$$V = \frac{\pi^2}{8} f\left(\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}\right),$$

where

$$\frac{\pi^2}{2} f\left(\frac{\pi}{2} - x, y, \frac{\pi}{2} - z\right) = \sum_{m=1}^\infty \left(\frac{D - \sin x \sin z}{D + \sin x \sin z}\right)^m \times \frac{\cos(2mx) - \cos(2my) + \cos(2mz) - 1}{m^2} - x^2 - y^2 - z^2,$$

and

$$D \equiv \sqrt{\cos^2 x \cos^2 z - \cos^2 y}.$$

see also TETRAHEDRON

Schläfli Integral

A definition of a function using a CONTOUR INTEGRAL. Schläfli integrals may be converted into RODRIGUES FORMULAS.

see also RODRIGUES FORMULA

Schläfli's Modular Form

The MODULAR EQUATION of degree 5 can be written

$$\left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3 = 2\left(u^2v^2 - \frac{1}{u^2v^2}\right).$$

see also MODULAR EQUATION

Schläfli Polynomial

A polynomial given in terms of the NEUMANN POLYNOMIALS $O_n(x)$ by

$$S_n(x) = \frac{2xO_n(x) - 2\cos^2(\frac{1}{2}n\pi)}{n}.$$

see also NEUMANN POLYNOMIAL

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1477, 1980.

von Seggern, D. *CRC Standard Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 196, 1993.

Schläfli Symbol

The symbol $\{p, q\}$ is used to denote a TESSELLATION of regular p-gons, with q of them surrounding each VERTEX. The Schläfli symbol can be used to describe PLATONIC SOLIDS, and a generalized version describes QUASIREGULAR POLYHEDRA and ARCHIMEDEAN SOLIDS.

see also ARCHIMEDEAN SOLID, PLATONIC SOLID, QUASIREGULAR POLYHEDRON, TESSELLATION

Schlegel Graph

A GRAPH corresponding to POLYHEDRA skeletons. The POLYHEDRAL GRAPHS are special cases.

References

Gardner, M. *Wheels, Life, and Other Mathematical Amusements.* New York: W. H. Freeman, p. 158, 1983.

Schlömilch's Function

$$S(\nu, z) \equiv \int_0^\infty (1+t)^{-\nu} e^{-zt}\, dt = z^{\nu-1} e^z \int_z^\infty u^{-\nu} e^{-u}\, du$$
$$= z^{\nu/2-1} e^{z/2} W_{-\nu/2,(1-\nu)/2}(z),$$

where $W_{k,m}(z)$ is the WHITTAKER FUNCTION.

Schlömilch's Series

A FOURIER SERIES-like expansion of a twice continuously differentiable function

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^\infty a_n J_0(nx)$$

for $0 < x < \pi$, where $J_0(x)$ is a zeroth order BESSEL FUNCTION OF THE FIRST KIND and

$$a_0 \equiv 2f(0) + \frac{2}{\pi}\int_0^\pi du \int_0^{\pi/2} f'(u\sin\phi)\, d\phi$$
$$a_n \equiv \frac{2}{\pi}\int_0^\pi du \int_0^{\pi/2} uf'(u\sin\phi)\cos(n\pi)\, d\phi.$$

A special case gives the amazing identity

$$1 = J_0(z) + 2\sum_{n=1}^\infty J_{2n}(z) = [J_0(z)]^2 + 2\sum_{n=1}^\infty [J_n(z)]^2.$$

see also BESSEL FUNCTION OF THE FIRST KIND, BESSEL FUNCTION FOURIER EXPANSION, FOURIER SERIES

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1473, 1980.

Schmitt-Conway Biprism

A CONVEX POLYHEDRON which is SPACE-FILLING, but only aperiodically, was found by Conway in 1993.

see also CONVEX POLYHEDRON, SPACE-FILLING POLYHEDRON

Schnirelmann Constant

The constant s_0 in SCHNIRELMANN'S THEOREM.

see also SCHNIRELMANN'S THEOREM

Schnirelmann Density

The Schnirelmann density of a sequence of natural numbers is the greatest lower bound of the fractions $A(n)/n$ where $A(n)$ is the number of terms in the sequence $\leq n$.

References

Khinchin, A. Y. "The Landau-Schnirelmann Hypothesis and Mann's Theorem." Ch. 2 in *Three Pearls of Number Theory.* New York: Dover, pp. 18–36, 1998.

Schnirelmann's Theorem

There exists a POSITIVE INTEGER s such that every sufficiently large INTEGER is the sum of at most s PRIMES. It follows that there exists a POSITIVE INTEGER $s_0 \geq s$ such that every INTEGER > 1 is a sum of at most s_0 PRIMES, where s_0 is the SCHNIRELMANN CONSTANT. The best current estimate is $s_0 = 19$.

see also PRIME NUMBER, SCHNIRELMANN DENSITY, WARING'S PROBLEM

References

Khinchin, A. Y. "The Landau-Schnirelmann Hypothesis and Mann's Theorem." Ch. 2 in *Three Pearls of Number Theory.* New York: Dover, pp. 18–36, 1998.

Schoenemann's Theorem

If the integral COEFFICIENTS $C_0, C_1, \ldots, C_{N-1}$ of the POLYNOMIAL

$$f(x) = C_0 + C_1 x + C_2 x^2 + \ldots + C_{N-1} x^{N-1} + x^N$$

are divisible by a PRIME NUMBER p, while the free term C_0 is not divisible by p^2, then $f(x)$ is irreducible in the natural rationality domain.

see also ABEL'S IRREDUCIBILITY THEOREM, ABEL'S LEMMA, GAUSS'S POLYNOMIAL THEOREM, KRONECKER'S POLYNOMIAL THEOREM

References
Dörrie, H. *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, p. 118, 1965.

Scholz Conjecture

Let the minimal length of an ADDITION CHAIN for a number n be denoted $l(n)$. Then the Scholz conjecture states that

$$l(2^n - 1) \leq n - 1 + l(n).$$

The conjecture has been proven for a variety of special cases but not in general.

see also ADDITION CHAIN

References
Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 111, 1994.

Schönflies Symbol

One of the set of symbols C_i, C_s, C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_{2h}, C_{3h}, C_{4h}, C_{5h}, C_{6h}, C_{2v}, C_{3v}, C_{4v}, C_{5v}, C_{6v}, $C_{\infty v}$, D_2, D_3, D_4, D_5, D_6, D_{2h}, D_{3h}, D_{4h}, D_{5h}, D_{6h}, D_{8h}, $D_{\infty h}$, D_{2d}, D_{3d}, D_{4d}, D_{5d}, D_{6d}, I, I_h, O, O_h, S_4, S_6, S_8, T, T_d, and T_h used to identify crystallographic symmetry GROUPS.

Cotton (1990), gives a table showing the translations between Schönflies symbols and HERMANN-MAUGUIN SYMBOLS. Some of the Schönflies symbols denote different sets of symmetry operations but correspond to the same abstract GROUP and so have the same CHARACTER TABLE.

see also CHARACTER TABLE, HERMANN-MAUGUIN SYMBOL, POINT GROUPS, SPACE GROUPS, SYMMETRY OPERATION

References
Cotton, F. A. *Chemical Applications of Group Theory, 3rd ed.* New York: Wiley, p. 379, 1990.

Schönflies Theorem

If J is a simple closed curve in $\mathbb{R}^2$, the closure of one of the components of $\mathbb{R}^2 - J$ is HOMEOMORPHIC with the unit 2-BALL. This theorem may be proved using the RIEMANN MAPPING THEOREM, but the easiest proof is via MORSE THEORY.

The generalization to n-D is called MAZUR'S THEOREM. It follows from the Schönflies theorem that any two KNOTS of $\mathbb{S}^1$ in $\mathbb{S}^2$ or $\mathbb{R}^2$ are equivalent.

see also JORDAN CURVE THEOREM, MAZUR'S THEOREM, RIEMANN MAPPING THEOREM

References
Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, p. 9, 1976.
Thomassen, C. "The Jordan-Schönflies Theorem and the Classification of Surfaces." *Amer. Math. Monthly* **99**, 116–130, 1992.

Schoolgirl Problem

see KIRKMAN'S SCHOOLGIRL PROBLEM

Schoute Coaxal System

The CIRCUMCIRCLE, BROCARD CIRCLE, LEMOINE LINE, and ISODYNAMIC POINTS belong to a COAXAL SYSTEM orthogonal to the the APOLLONIUS CIRCLES, called the Schoute coaxal system. In general, there are 12 points whose PEDAL TRIANGLES with regard to a given TRIANGLE have a given form. They lie six by six on two CIRCLES of the Schoute coaxal system.

References
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 297–299, 1929.

Schoute's Theorem

In any TRIANGLE, the LOCUS of a point whose PEDAL TRIANGLE has a constant BROCARD ANGLE and is described in a given direction is a CIRCLE of the SCHOUTE COAXAL SYSTEM.

References
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 297–299, 1929.
Schoute, P. H. *Proc. Amsterdam Acad.,* 39–62, 1887–1888.

Schrage's Algorithm

An algorithm for multiplying two 32-bit integers modulo a 32-bit constant without using any intermediates larger than 32 bits. It is also useful in certain types of RANDOM NUMBER generators.

References
Bratley, P.; Fox, B. L.; and Schrage, E. L. *A Guide to Simulation, 2nd ed.* New York: Springer-Verlag, 1996.
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Random Numbers." Ch. 7 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, p. 269, 1992.
Schrage, L. "A More Portable Fortran Random Number Generator." *ACM Trans. Math. Software* **5**, 132–138, 1979.

Schröder-Bernstein Theorem

The Schröder-Bernstein theorem for numbers states that if

$$n \leq m \leq n,$$

then $m = n$. For SETS, the theorem states that if there are INJECTIONS of the SET A into the SET B and of B into A, then there is a BIJECTIVE correspondence between A and B (i.e., they are EQUIPOLLENT).

see also BIJECTION, EQUIPOLLENT, INJECTION

Schröder's Equation

$$f(\lambda z) = R(z),$$

where $R(z) = \lambda x + a_2 x^2 + \ldots$, $\lambda \equiv R'(0)$, $|\lambda| = 1$, and $\lambda^n \neq 1$ for all $n \in \mathbb{N}$.

Schröder's Method

Two families of equations used to find roots of nonlinear functions of a single variable. The "B" family is more robust and can be used in the neighborhood of degenerate multiple roots while still providing a guaranteed convergence rate. Almost all other root-finding methods can be considered as special cases of Schröder's method. Householder humorously claimed that papers on root-finding could be evaluated quickly by looking for a citation of Schröder's paper; if the reference were missing, the paper probably consisted of a rediscovery of a result due to Schröder (Stewart 1993).

One version of the "A" method is obtained by applying NEWTON'S METHOD to f/f',

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{[f'(x_n)]^2 - f(x_n)f''(x_n)}$$

(Scavo and Thoo 1995).

see also NEWTON'S METHOD

References

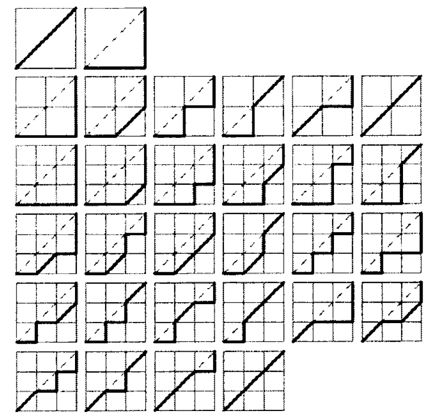
Householder, A. S. *The Numerical Treatment of a Single Nonlinear Equation.* New York: McGraw-Hill, 1970.

Scavo, T. R. and Thoo, J. B. "On the Geometry of Halley's Method." *Amer. Math. Monthly* **102**, 417–426, 1995.

Schröder, E. "Über unendlich viele Algorithmen zur Auflösung der Gleichungen." *Math. Ann.* **2**, 317–365, 1870.

Stewart, G. W. "On Infinitely Many Algorithms for Solving Equations." English translation of Schröder's original paper. College Park, MD: University of Maryland, Institute for Advanced Computer Studies, Department of Computer Science, 1993. `ftp://thales.cs.umd.edu/pub/reports/imase.ps`.

Schröder Number



The Schröder number S_n is the number of LATTICE PATHS in the Cartesian plane that start at (0, 0), end at (n, n), contain no points above the line $y = x$, and are composed only of steps (0, 1), (1, 0), and (1, 1), i.e., $\rightarrow$, $\uparrow$, and $\nearrow$. The diagrams illustrating the paths generating S_1, S_2, and S_3 are illustrated above. The numbers S_n are given by the RECURRENCE RELATION

$$S_n = S_{n-1} + \sum_{k=0}^{n-1} S_k S_{n-1-k},$$

where $S_0 = 1$, and the first few are 2, 6, 22, 90, ... (Sloane's A006318). The Schröder Numbers bear the same relation to the DELANNOY NUMBERS as the CATALAN NUMBERS do to the BINOMIAL COEFFICIENTS.

see also BINOMIAL COEFFICIENT, CATALAN NUMBER, DELANNOY NUMBER, LATTICE PATH, MOTZKIN NUMBER, p-GOOD PATH

References

Sloane, N. J. A. Sequence A006318/M1659 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Schroeder Stairs

see PENROSE STAIRWAY

Schröter's Formula

Let a general THETA FUNCTION be defined as

$$T(x, q) \equiv \sum_{n=-\infty}^{\infty} x^n q^{n^2},$$

then

$$T(x, q^a)T(x, q^b) = \sum_{k=0}^{a+b-1} y^k q^{bk^2} T(xyq^{2bk}, q^{a+b}) T(y^q x^{-b} q^{2abk}, q^{ab(1+b)}).$$

see also BLECKSMITH-BRILLHART-GERST THEOREM, JACOBI TRIPLE PRODUCT, RAMANUJAN THETA FUNCTIONS

References

Borwein, J. M. and Borwein, P. B. *Pi & the AGM: A Study in Analytic Number Theory and Computational Complexity.* New York: Wiley, p. 111, 1987.

Tannery, J. and Molk, J. *Elements de la Théorie des Fonctions Elliptiques, 4 vols.* Paris: Gauthier-Villars et fils, 1893–1902.

Schur Algebra

An Auslander algebra which connects the representation theories of the symmetric group of PERMUTATIONS and the GENERAL LINEAR GROUP $GL(n, \mathbb{C})$. Schur algebras are "quasihereditary."

References

Martin, S. *Schur Algebras and Representation Theory.* New York: Cambridge University Press, 1993.

Schur Functor

A FUNCTOR which defines an equivalence of module CATEGORIES.

References

Martin, S. *Schur Algebras and Representation Theory.* New York: Cambridge University Press, 1993.

Schur's Inequalities

Let $\mathsf{A} = a_{ij}$ be an $n \times n$ MATRIX with COMPLEX (or REAL) entries and EIGENVALUES $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$\sum_{i=1}^n |\lambda_i|^2 \leq \sum_{i,j=1}^n |a_{ij}|^2$$

$$\sum_{i=1}^n |\Re[\lambda_i]|^2 \leq \sum_{i,j=1}^n \left|\frac{a_{ij} + a_{ji}^*}{2}\right|^2$$

$$\sum_{i=1}^n |\Im[\lambda_i]|^2 \leq \sum_{i,j=1}^n \left|\frac{a_{ij} - a_{ji}^*}{2}\right|^2.$$

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1120, 1979.

Schur's Lemma

For each $k \in \mathbb{N}$ there exists a largest INTEGER $s(k)$ (known as the SCHUR NUMBER) such that no matter how the set of INTEGERS less than $\lfloor n!e \rfloor$ (where $\lfloor x \rfloor$ is the FLOOR FUNCTION) is partitioned into k classes, one class must contain INTEGERS x, y, z such that $x + y = z$, where x and y are not necessarily distinct. The upper bound has since been slightly improved to $\lfloor n!(e - 1/24) \rfloor$.

see also COMBINATORICS, SCHUR NUMBER, SCHUR'S THEOREM

References

Guy, R. K. "Schur's Problem. Partitioning Integers into Sum-Free Classes" and "The Modular Version of Schur's Problem." §E11 and E12 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 209–212, 1994.

Schur Matrix

The $p \times p$ SQUARE MATRIX formed by setting $s_{ij} = \xi^{ij}$, where ξ is an pth ROOT OF UNITY. The Schur matrix has a particularly simple DETERMINANT given by

$$\det \mathsf{S} = \epsilon_p p^{p/2},$$

where p is an ODD PRIME and

$$\epsilon_p \equiv \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4 \\ \mathrm{i} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

This determinant has been used to prove the QUADRATIC RECIPROCITY LAW (Landau 1958, Vardi 1991). The ABSOLUTE VALUES of the PERMANENTS of the Schur matrix of order $2p+1$ are given by 1, 3, 5, 105, 81, 6765, ... (Sloane's A003112, Vardi 1991).

Denote the Schur matrix S_p with the first row and first row column omitted by S'_p. Then

$$\operatorname{perm} \mathsf{S}_p = p \operatorname{perm} \mathsf{S}'_p,$$

where perm denoted the PERMANENT (Vardi 1991).

References

Graham, R. L. and Lehmer, D. H. "On the Permanent of Schur's Matrix." *J. Austral. Math. Soc.* **21**, 487–497, 1976.

Landau, E. *Elementary Number Theory.* New York: Chelsea, 1958.

Sloane, N. J. A. Sequence A003112/M2509 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Vardi, I. *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, pp. 119–122 and 124, 1991.

Schur Multiplier

A property of FINITE SIMPLE GROUPS which is known for all such GROUPS.

see also FINITE GROUP, SIMPLE GROUP

Schur Number

The Schur numbers are the numbers in the partitioning of a set which are guaranteed to exist by SCHUR'S LEMMA. Schur numbers satisfy the inequality

$$s(k) \geq c(315)^{k/5}$$

for $k > 5$ and some constant c. SCHUR'S THEOREM also shows that

$$s(n) \leq R(n),$$

where $R(n)$ is a RAMSEY NUMBER. The first few Schur numbers are 1, 4, 13, 44, (≥ 157), ... (Sloane's A045652).

see also RAMSEY NUMBER, RAMSEY'S THEOREM, SCHUR'S LEMMA, SCHUR'S THEOREM

References

Fredericksson, H. "Schur Numbers and the Ramsey Numbers $N(3, 3, \ldots, 3; 2)$." *J. Combin. Theory Ser. A* **27**, 376–377, 1979.

Guy, R. K. "Schur's Problem. Partitioning Integers into Sum-Free Classes" and "The Modular Version of Schur's Problem." §E11 and E12 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 209–212, 1994.
Sloane, N. J. A. Sequence A045652 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Schur's Problem

see SCHUR'S LEMMA

Schur's Representation Lemma

If π on V and π' on V' are irreducible representations and $E : V \mapsto V'$ is a linear map such that $\pi'(g)E = E\pi(g)$ for all $g \in$ and group G, then $E = 0$ or E is invertible. Furthermore, if $V = V'$, then E is a SCALAR.

References

Knapp, A. W. "Group Representations and Harmonic Analysis, Part II." *Not. Amer. Math. Soc.* **43**, 537–549, 1996.

Schur's Theorem

As shown by Schur in 1916, the SCHUR NUMBER $s(n)$ satisfies

$$s(n) \leq R(n)$$

for $n = 1, 2, \ldots$, where $R(n)$ is a RAMSEY NUMBER.

see also RAMSEY NUMBER, SCHUR'S LEMMA, SCHUR NUMBER

Schwarz's Inequality

$$|\langle\psi_1|\psi_2\rangle|^2 \leq \langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle. \tag{1}$$

Written out explicitly

$$\left[\int_a^b \psi_1(x)\psi_2(x)\,dx\right]^2 \leq \int_a^b [\psi_1(x)]^2\,dx \int_a^b [\psi_2(x)]^2\,dx, \tag{2}$$

with equality IFF $g(x) = \alpha f(x)$ with α a constant. To derive, let $\psi(x)$ be a COMPLEX function and λ a COMPLEX constant such that $\psi(x) \equiv f(x) + \lambda g(x)$ for some f and g. Then

$$\int \psi^*\psi\,dx = \int f^*f\,dx + \lambda\int f^*g\,dx + \lambda^*\int g^*f\,dx + \lambda\lambda^*\int g^*g\,dx \geq 0, \tag{3}$$

with equality when $\psi(x) = 0$. Now, note that λ and λ^* are LINEARLY INDEPENDENT (they are ORTHOGONAL), so differentiate with respect to one of them (say λ^*) and set to zero to minimize $\int \psi^*\psi\,dx$.

$$\frac{\partial}{\partial}\int \psi^*\psi\,dx = \int g^*f\,dx + \lambda\int g^*g\,dx = 0 \tag{4}$$

$$\lambda = -\frac{\int g^*f\,dx}{\int g^*g\,dx}, \tag{5}$$

which means that

$$\lambda^* = -\frac{\int f^*g\,dx}{\int g^*g\,dx}. \tag{6}$$

Plugging back in,

$$\int \psi^*\psi\,dx = \int f^*f\,dx - \frac{\int g^*f\,dx}{\int g^*g\,dx}\int f^*g\,dx - \frac{\int f^*g\,dx}{\int g^*g\,dx}\int g^*f\,dx + \frac{\int g^*f\,dx\int f^*g\,dx}{(\int g^*g\,dx)^2}\int g^*g\,dx \geq 0. \tag{7}$$

Multiplying through by $\int g^*g\,dx$ gives

$$\int f^*f\,dx\int g^*g\,dx - \int g^*f\,dx\int f^*g\,dx - \int f^*g\,dx\int g^*f\,dx + \int g^*f\,dx\int f^*g\,dx \geq 0 \tag{8}$$

$$\int g^*f\,dx\int f^*g\,dx \leq \int f^*f\,dx\int g^*g\,dx \tag{9}$$

$$\left|\int g^*f\,dx\right| = \left|\int f^*g\,dx\right| \leq \int f^*f\,dx\int g^*g\,dx \tag{10}$$

or

$$|\langle f|g\rangle|^2 \leq \langle f|f\rangle\langle g|g\rangle. \tag{11}$$

BESSEL'S INEQUALITY can be derived from this.

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 11, 1972.
Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 527–529, 1985.

Schwarz-Pick Lemma

If f is an analytic map of the DISK $\mathbb{D}$ into $\mathbb{D}$ and f preserves the hyperbolic distance between any two points, then f is a disk map and preserves all distance.

References

Busemann, H. *The Geometry of Geodesics.* New York: Academic Press, p. 41, 1955.

Schwarz Reflection Principle

Let

$$g(z) \equiv \sum_{n=0}^{\infty} (z-z_0)^n \frac{f^{(n)}(z_0)}{n!}, \qquad (1)$$

then

$$g^*(z) = \left[\sum_{n=0}^{\infty} (z-z_0)^n \frac{f^{(n)}(z_0)}{n!} \right]^* = \sum_{n=0}^{\infty} (z^*-z_0{}^*) n \frac{f^{(n)}(z_0{}^*)}{n!}. \qquad (2)$$

If z_0 is pure real, then $z_0 = z_0{}^*$, so

$$g^*(z) = \sum_{n=0}^{\infty} (z^*-z_0)^n \frac{f^{(n)}(z_0)}{n!} = g(z^*). \qquad (3)$$

Therefore, if a function $f(z)$ is ANALYTIC over some region including the REAL LINE and $f(z)$ is REAL when z is real, then $f^*(z) = f(z^*)$.

Schwarz Triangle

The Schwarz triangles are SPHERICAL TRIANGLES which, by repeated reflection in their indices, lead to a set of congruent SPHERICAL TRIANGLES covering the SPHERE a finite number of times.

Schwarz triangles are specified by triples of numbers (p,q,r). There are four "families" of Schwarz triangles, and the largest triangles from each of these families are

$$(2\ 2\ n'), (\tfrac{3}{2}\ \tfrac{3}{2}\ \tfrac{3}{2}), (\tfrac{3}{2}\ \tfrac{4}{3}\ \tfrac{4}{3}), (\tfrac{5}{4}\ \tfrac{5}{4}\ \tfrac{5}{4}).$$

The others can be derived from

$$(p\ q\ r) = (p\ x\ r_1) + (x\ q\ r_2),$$

where

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}$$

and

$$\cos\left(\frac{\pi}{x}\right) = -\cos\left(\frac{\pi}{x'}\right) = \frac{\cos\left(\frac{\pi}{q}\right)\sin\left(\frac{\pi}{r_1}\right) - \cos\left(\frac{\pi}{p}\right)\sin\left(\frac{\pi}{r_2}\right)}{\sin\left(\frac{\pi}{r}\right)}.$$

see also COLUNAR TRIANGLE, SPHERICAL TRIANGLE

References

Coxeter, H. S. M. *Regular Polytopes, 3rd ed.* New York: Dover, pp. 112–113 and 296, 1973.

Schwarz, H. A. "Zur Theorie der hypergeometrischen Reihe." *J. reine angew. Math.* **75**, 292–335, 1873.

Schwarz's Triangle Problem

see FAGNANO'S PROBLEM

Schwarzian Derivative

The Schwarzian derivative is defined by

$$D_{\text{Schwarzian}} \equiv \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2.$$

The FEIGENBAUM CONSTANT is universal for 1-D MAPS if its Schwarzian derivative is NEGATIVE in the bounded interval (Tabor 1989, p. 220).

see also FEIGENBAUM CONSTANT

References

Tabor, M. *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, 1989.

Schwenk's Formula

Let $R+B$ be the number of MONOCHROMATIC FORCED TRIANGLES (where R and B are the number of red and blue TRIANGLES) in an EXTREMAL GRAPH. Then

$$R+B = \binom{n}{3} - \left\lfloor \tfrac{1}{2} n \left\lfloor \tfrac{1}{4}(n-1)^2 \right\rfloor \right\rfloor,$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT and $\lfloor x \rfloor$ is the FLOOR FUNCTION (Schwenk 1972).

see also EXTREMAL GRAPH, MONOCHROMATIC FORCED TRIANGLE

References

Schwenk, A. J. "Acquaintance Party Problem." *Amer. Math. Monthly* **79**, 1113–1117, 1972.

Scientific Notation

Scientific notation is the expression of a number n in the form $a \times 10^p$, where

$$p \equiv \lfloor \log_{10} |n| \rfloor$$

is the FLOOR of the base-10 LOGARITHM of n (the "order of magnitude"), and

$$a \equiv \frac{n}{10^p}$$

is a REAL NUMBER satisfying $1 \leq |a| < 10$. For example, in scientific notation, the number $n = 101,325$ has order of magnitude

$$p = \lfloor \log_{10} 101,325 \rfloor = \lfloor 5.00572 \rfloor = 5,$$

so n would be written 1.01325×10^5. The special case of 0 does not have a unique representation in scientific notation, i.e., $0 = 0 \times 10^0 = 0 \times 10^1 = \ldots$.

see also CHARACTERISTIC (REAL NUMBER), FIGURES, MANTISSA, SIGNIFICANT FIGURES

Score Sequence

The score sequence of a TOURNAMENT is a monotonic nondecreasing sequence of the OUTDEGREES of the VERTICES. The score sequences for $n = 1, 2, \ldots$ are 1, 1, 2,4, 9, 22, 59, 167, ... (Sloane's A000571).

see also TOURNAMENT

References

Ruskey, F. "Information on Score Sequences." `http://sue.csc.uvic.ca/~cos/inf/nump/ScoreSequence.html`.

Ruskey, F.; Cohen, R.; Eades, P.; and Scott, A. "Alley CATs in Search of Good Homes." *Congres. Numer.* **102**, 97–110, 1994.

Sloane, N. J. A. Sequence A000571/M1189 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Screw

A TRANSLATION along a straight line L and a ROTATION about L such that the angle of ROTATION is proportional to the TRANSLATION at each instant. Also known as a TWIST.

see also DINI'S SURFACE, HELICOID, ROTATION, SCREW THEOREM, SEASHELL, TRANSLATION

Screw Theorem

Any motion of a rigid body in space at every instant is a SCREW motion. This theorem was proved by Mozzi and Cauchy.

see also SCREW

Scruple

An archaic UNIT FRACTION variously defined as 1/200 (of an hour), 1/10 or 1/12 (of an inch), 1/12 (of a celestial body's angular diameter), or 1/60 (of an hour or DEGREE).

see also CALCUS, UNCIA

Sea Horse Valley

A portion of the MANDELBROT SET centered around $-1.25+0.047i$ with width approximately $0.009+0.005i$.

see also MANDELBROT SET

Searching

Searching refers to locating a given element or an element satisfying certain conditions from some (usually ordered or partially ordered) table, list, TREE, etc.

see also SORTING, TABU SEARCH, TREE SEARCHING

References

Knuth, D. E. *The Art of Computer Programming, 2nd ed, Vol. 3: Sorting and Searching.* Reading, MA: Addison-Wesley, 1973.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "How to Search an Ordered Table." §3.4 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 110–113, 1992.

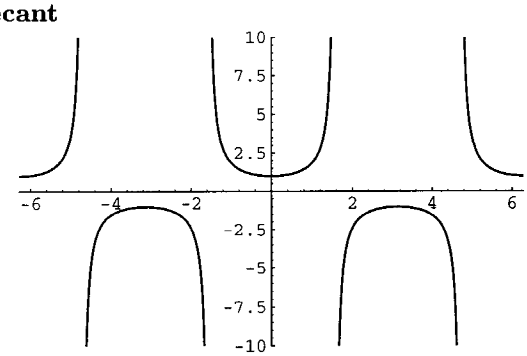
Search Tree

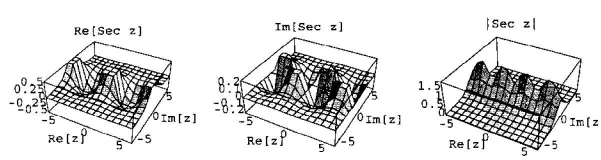
see TREE SEARCHING

Seashell

see CONICAL SPIRAL

Secant





The function defined by $\sec x \equiv 1/\cos x$, where $\cos x$ is the COSINE. The MACLAURIN SERIES of the secant is

$$\begin{aligned}\sec x &= \frac{(-1)^n E_{2n}}{(2n)!}x^{2n}\\ &= 1 + \tfrac{1}{2}x^2 + \tfrac{5}{24}x^4 + \tfrac{61}{720}x^6 + \tfrac{277}{8064}x^8 + \ldots,\end{aligned}$$

where E_{2n} is an EULER NUMBER.

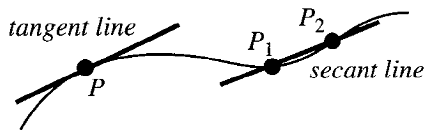
see also ALTERNATING PERMUTATION, COSECANT, COSINE, EULER NUMBER, EXSECANT, INVERSE SECANT

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Circular Functions." §4.3 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 71–79, 1972.

Spanier, J. and Oldham, K. B. "The Secant sec(x) and Cosecant csc(x) Functions." Ch. 33 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 311–318, 1987.

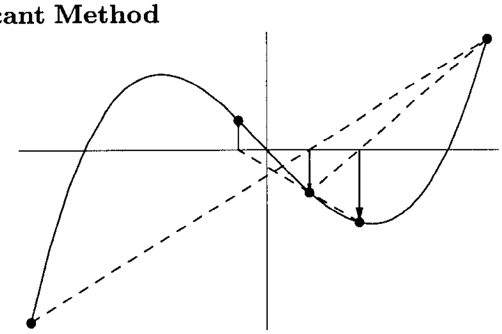
Secant Line



A line joining two points of a curve. In abstract mathematics, the points which a secant line connects can be either REAL or COMPLEX CONJUGATE IMAGINARY.

see also BITANGENT, TANGENT LINE, TRANSVERSAL LINE

Secant Method



A ROOT-finding algorithm which assumes a function to be approximately linear in the region of interest. Each improvement is taken as the point where the approximating line crosses the axis. The secant method retains only the most recent estimate, so the root does not necessarily remain bracketed. When the ALGORITHM does converge, its order of convergence is

$$\lim_{k\to\infty} |\epsilon_{k+1}| \approx C|\epsilon|^{\phi}, \tag{1}$$

where C is a constant and ϕ is the GOLDEN MEAN.

$$f'(x_{n-1}) \approx \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}} \tag{2}$$

$$f(x_n) \approx f(x_{n-1}) + f'(x_n)(x_n - x_{n-1}) = 0 \tag{3}$$

$$f(x_{n-1}) + \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}(x_n - x_{n-1}) = 0, \tag{4}$$

so

$$x_n = x_{n-1} - \frac{f(x_{n-1})(x_{n-1} - x_{n-2})}{f(x_{n-1}) - f(x_{n-2})}. \tag{5}$$

see also FALSE POSITION METHOD

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Secant Method, False Position Method, and Ridders' Method." §9.2 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 347–352, 1992.

Secant Number

A number, more commonly called an EULER NUMBER, giving the number of ODD ALTERNATING PERMUTATIONS. The term ZAG NUMBER is sometimes also used.

see also ALTERNATING PERMUTATION, EULER NUMBER, EULER ZIGZAG NUMBER, TANGENT NUMBER

Sech

see HYPERBOLIC SECANT

Second

see ARC SECOND

Second Curvature

see TORSION (DIFFERENTIAL GEOMETRY)

Second Derivative Test

Suppose $f(x)$ is a FUNCTION of x which is twice DIFFERENTIABLE at a STATIONARY POINT x_0.

1. If $f''(x_0) > 0$, then f has a RELATIVE MINIMUM at x_0.
2. If $f''(x_0) < 0$, then f has a RELATIVE MAXIMUM at x_0.

The EXTREMUM TEST gives slightly more general conditions under which functions with $f''(x_0) = 0$.

If $f(x, y)$ is a 2-D FUNCTION which has a RELATIVE EXTREMUM at a point (x_0, y_0) and has CONTINUOUS PARTIAL DERIVATIVES at this point, then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. The second PARTIAL DERIVATIVES test classifies the point as a MAXIMUM or MINIMUM. Define the DISCRIMINANT as

$$D \equiv f_{xx}f_{yy} - f_{xy}f_{yx} = f_{xx}f_{yy} - {f_{xy}}^2.$$

1. If $D > 0$, $f_{xx}(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) + f_{yy}(x_0, y_0) > 0$, the point is a RELATIVE MINIMUM.
2. If $D > 0$, $f_{xx}(x_0, y_0) < 0$, and $f_{xx}(x_0, y_0) + f_{yy}(x_0, y_0) < 0$, the point is a RELATIVE MAXIMUM.
3. If $D < 0$, the point is a SADDLE POINT.
4. If $D = 0$, higher order tests must be used.

see also DISCRIMINANT (SECOND DERIVATIVE TEST), EXTREMUM, EXTREMUM TEST, FIRST DERIVATIVE TEST, GLOBAL MAXIMUM, GLOBAL MINIMUM, HESSIAN DETERMINANT, MAXIMUM, MINIMUM, RELATIVE MAXIMUM, RELATIVE MINIMUM, SADDLE POINT (FUNCTION)

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 14, 1972.

Second Fundamental Tensor

see WEINGARTEN MAP

Section (Graph)

A section of a GRAPH is obtained by finding its intersection with a PLANE.

Section (Pencil)

The lines of a PENCIL joining the points of a RANGE to another POINT.

see also PENCIL, RANGE (LINE SEGMENT)

Section (Tangent Bundle)

A VECTOR FIELD is a section of its TANGENT BUNDLE, meaning that to every point x in a MANIFOLD M, a VECTOR $X(x) \in T_x M$ is associated, where T_x is the TANGENT SPACE.

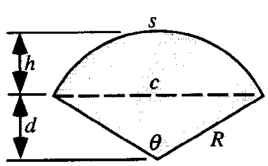
see also TANGENT BUNDLE, TANGENT SPACE

Sectional Curvature

The mathematical object κ which controls the rate of geodesic deviation.

see also BISHOP'S INEQUALITY, CHEEGER'S FINITENESS THEOREM, GEODESIC

Sector



A WEDGE obtained by taking a portion of a CIRCLE with CENTRAL ANGLE $\theta < \pi$ radians (180°), illustrated above as the shaded region. A sector of π radians would be a SEMICIRCLE. Let R be the radius of the CIRCLE, c the CHORD length, s the ARC LENGTH, h the height of the arced portion, and d the height of the triangular portion. Then

$$\begin{aligned} R &= h + d & (1)\\ s &= R\theta & (2)\\ d &= R\cos(\tfrac{1}{2}\theta) & (3)\\ &= \tfrac{1}{2}c\cot(\tfrac{1}{2}\theta) & (4)\\ &= \tfrac{1}{2}\sqrt{4R^2 - c^2} & (5)\\ c &= 2R\sin(\tfrac{1}{2}\theta) & (6)\\ &= 2d\tan(\tfrac{1}{2}\theta) & (7)\\ &= 2\sqrt{R^2 - d^2} & (8)\\ &= 2\sqrt{h(2R - h)}. & (9) \end{aligned}$$

The ANGLE θ obeys the relationships

$$\begin{aligned} \theta &= \frac{s}{R} = 2\cos^{-1}\left(\frac{d}{R}\right) = 2\tan^{-1}\left(\frac{c}{2d}\right)\\ &= 2\sin^{-1}\left(\frac{c}{2R}\right). \qquad (10) \end{aligned}$$

The AREA of the sector is

$$A = \tfrac{1}{2}Rs = \tfrac{1}{2}R^2\theta \qquad (11)$$

(Beyer 1987).

see also CIRCLE-CIRCLE INTERSECTION, LENS, OBTUSE TRIANGLE, SEGMENT

References

Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 125, 1987.

Sectorial Harmonic

A SPHERICAL HARMONIC of the form

$$\sin(m\theta)P_m^m(\cos\phi)$$

or

$$\cos(m\theta)P_m^m(\cos\phi).$$

see also SPHERICAL HARMONIC

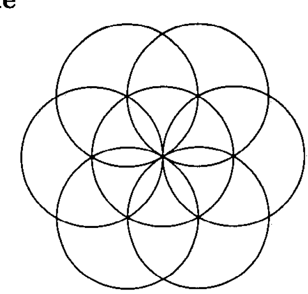
Secular Equation

see CHARACTERISTIC EQUATION

Seed

The initial number used as the starting point in a RANDOM NUMBER generating ALGORITHM.

Seed of Life



One of the beautiful arrangements of CIRCLES found at the Temple of Osiris at Abydos, Egypt (Rawles 1997). The CIRCLES are placed with 6-fold symmetry, forming a mesmerizing pattern of CIRCLES and LENSES.

see also CIRCLE, FIVE DISKS PROBLEM, FLOWER OF LIFE, VENN DIAGRAM

References

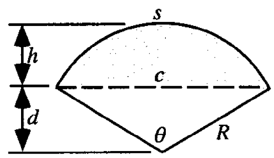
Rawles, B. *Sacred Geometry Design Sourcebook: Universal Dimensional Patterns.* Nevada City, CA: Elysian Pub., p. 15, 1997.

Weisstein, E. W. "Flower of Life." `http://www.astro.virginia.edu/~eww6n/math/notebooks/FlowerOfLife.m`.

Seek Time

see POINT-POINT DISTANCE—1-D

Segment



A portion of a CIRCLE whose upper boundary is a circular ARC and whose lower boundary is a CHORD making a CENTRAL ANGLE $\theta < \pi$ radians (180°), illustrated above as the shaded region. Let R be the radius of the CIRCLE, c the CHORD length, s the ARC LENGTH, h the height of the arced portion, and d the height of the triangular portion. Then

$$R = h + d \quad (1)$$
$$s = R\theta \quad (2)$$
$$d = R\cos(\tfrac{1}{2}\theta) \quad (3)$$
$$= \tfrac{1}{2}c\cot(\tfrac{1}{2}\theta) \quad (4)$$
$$= \tfrac{1}{2}\sqrt{4R^2 - c^2} \quad (5)$$
$$c = 2R\sin(\tfrac{1}{2}\theta) \quad (6)$$
$$= 2d\tan(\tfrac{1}{2}\theta) \quad (7)$$
$$= 2\sqrt{R^2 - d^2} \quad (8)$$
$$= 2\sqrt{h(2R - h)}. \quad (9)$$

The ANGLE θ obeys the relationships

$$\theta = \frac{s}{R} = 2\cos^{-1}\left(\frac{d}{R}\right) = 2\tan^{-1}\left(\frac{c}{2d}\right) = 2\sin^{-1}\left(\frac{c}{2R}\right). \quad (10)$$

The AREA of the segment is then

$$A = A_{\text{sector}} - A_{\text{isosceles triangle}} \quad (11)$$
$$= \tfrac{1}{2}R^2(\theta - \sin\theta) \quad (12)$$
$$= \tfrac{1}{2}(Rs - cd) \quad (13)$$
$$= R^2\cos^{-1}\left(\frac{d}{R}\right) - d\sqrt{R^2 - d^2} \quad (14)$$
$$= R^2\cos^{-1}\left(\frac{R-h}{R}\right) - (R-h)\sqrt{2Rh - h^2}, \quad (15)$$

where the formula for the ISOSCELES TRIANGLE in terms of the VERTEX angle has been used (Beyer 1987).

see also CHORD, CIRCLE-CIRCLE INTERSECTION, CYLINDRICAL SEGMENT, LENS, PARABOLIC SEGMENT, SAGITTA, SECTOR, SPHERICAL SEGMENT

References

Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 125, 1987.

Segmented Number

see PRIME NUMBER OF MEASUREMENT

Segner's Recurrence Formula

The recurrence FORMULA

$$E_n = E_2E_{n-1} + E_3E_{n-2} + \ldots + E_{n-1}E_2$$

which gives the solution to EULER'S POLYGON DIVISION PROBLEM.

see also CATALAN NUMBER, EULER'S POLYGON DIVISION PROBLEM

Segre's Theorem

For any REAL NUMBER $r \geq 0$, an IRRATIONAL number α can be approximated by infinitely many RATIONAL fractions p/q in such a way that

$$-\frac{1}{\sqrt{1+4r}\,q^2} < \frac{p}{q} - \alpha < \frac{r}{\sqrt{1+4r}\,q^2}.$$

If $r = 1$, this becomes HURWITZ'S IRRATIONAL NUMBER THEOREM.

see also HURWITZ'S IRRATIONAL NUMBER THEOREM

Seiberg-Witten Equations

$$D_A\psi = 0$$
$$F_A^+ = -\tau(\psi, \psi),$$

where τ is the sesquilinear map $\tau : W^+ \times W^+ \to \Lambda^+ \otimes \mathbb{C}$.

see also WITTEN'S EQUATIONS

References

Donaldson, S. K. "The Seiberg-Witten Equations and 4-Manifold Topology." *Bull. Amer. Math. Soc.* **33**, 45–70, 1996.

Morgan, J. W. *The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds.* Princeton, NJ: Princeton University Press, 1996.

Seiberg-Witten Invariants

see WITTEN'S EQUATIONS

Seidel-Entringer-Arnold Triangle

The NUMBER TRIANGLE consisting of the ENTRINGER NUMBERS $E_{n,k}$ arranged in "ox-plowing" order,

$$\begin{array}{c} E_{00} \\ E_{10} \to E_{11} \\ E_{22} \leftarrow E_{21} \leftarrow E_{20} \\ E_{30} \to E_{31} \to E_{32} \to E_{33} \\ E_{44} \leftarrow E_{43} \leftarrow E_{42} \leftarrow E_{41} \leftarrow E_{40} \end{array}$$

giving

$$\begin{array}{c} 1 \\ 0 \to 1 \\ 1 \leftarrow 1 \leftarrow 0 \\ 0 \to 1 \to 2 \to 2 \\ 5 \leftarrow 5 \leftarrow 4 \leftarrow 2 \leftarrow 0 \end{array}$$

see also BELL NUMBER, BOUSTROPHEDON TRANSFORM, CLARK'S TRIANGLE, ENTRINGER NUMBER, EULER'S TRIANGLE, LEIBNIZ HARMONIC TRIANGLE, NUMBER TRIANGLE, PASCAL'S TRIANGLE

References

Arnold, V. I. "Bernoulli-Euler Updown Numbers Associated with Function Singularities, Their Combinatorics, and Arithmetics." *Duke Math. J.* **63**, 537–555, 1991.

Arnold, V. I. "Snake Calculus and Combinatorics of Bernoulli, Euler, and Springer Numbers for Coxeter Groups." *Russian Math. Surveys* **47**, 3–45, 1992.

Conway, J. H. and Guy, R. K. In *The Book of Numbers.* New York: Springer-Verlag, 1996.

Dumont, D. "Further Triangles of Seidel-Arnold Type and Continued Fractions Related to Euler and Springer Numbers." *Adv. Appl. Math.* **16**, 275–296, 1995.

Entringer, R. C. "A Combinatorial Interpretation of the Euler and Bernoulli Numbers." *Nieuw. Arch. Wisk.* **14**, 241–246, 1966.

Millar, J.; Sloane, N. J. A.; and Young, N. E. "A New Operation on Sequences: The Boustrophedon Transform." *J. Combin. Th. Ser. A* **76**, 44–54, 1996.

Seidel, I. "Über eine einfache Entstehungsweise der Bernoullischen Zahlen und einiger verwandten Reihen." *Sitzungsber. Münch. Akad.* **4**, 157–187, 1877.

Seifert Circle

Eliminate each knot crossing by connecting each of the strands coming into the crossing to the adjacent strand leaving the crossing. The resulting strands no longer cross but form instead a set of nonintersecting CIRCLES called Seifert circles.

References

Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, p. 96, 1994.

Seifert Conjecture

Every smooth NONZERO VECTOR FIELD on the 3-SPHERE has at least one closed orbit. The conjecture was proposed in 1950, proved true for Hopf fibrations, but proved false in general by Kuperberg (1994).

References

Kuperberg, G. "A Volume-Preserving Counterexample to the Seifert Conjecture." *Comment. Math. Helv.* **71**, 70–97, 1996.

Kuperberg, G. and Kuperberg, K. "Generalized counterexamples to the Seifert Conjecture." *Ann. Math.* **143**, 547–576, 1996.

Kuperberg, G. and Kuperberg, K. "Generalized Counterexamples to the Seifert Conjecture." *Ann. Math.* **144**, 239–268, 1996.

Kuperberg, K. "A Smooth Counterexample to the Seifert Conjecture." *Ann. Math.* **140**, 723–732, 1994.

Seifert Form

For K a given KNOT in $\mathbb{S}^3$, choose a SEIFERT SURFACE M^2 in $\mathbb{S}^3$ for K and a bicollar $\hat{M} \times [-1,1]$ in $\mathbb{S}^3 - K$. If $x \in H_1(\hat{M})$ is represented by a 1-cycle in $\hat{M}$, let x^+ denote the homology cycle carried by $x \times 1$ in the bicollar. Similarly, let x^- denote $x \times -1$. The function $f : H_1(\hat{M}) \times H_1(\hat{M}) \to Z$ defined by

$$f(x,y) = \mathrm{lk}(x, y^+),$$

where lk denotes the LINKING NUMBER, is called a Seifert form for K.

see also SEIFERT MATRIX

References

Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, pp. 200–201, 1976.

Seifert Matrix

Given a SEIFERT FORM $f(x,y)$, choose a basis e_1, $\ldots$, e_{2g} for $H_1(\hat{M})$ as a Z-module so every element is uniquely expressible as

$$n_1 e_1 + \ldots + n_{2g} e_{2g}$$

with n_i integer, define the Seifert matrix V as the $2g \times 2g$ integral MATRIX with entries

$$v_{ij} = \mathrm{lk}(e_i, e_j^+).$$

The right-hand TREFOIL KNOT has Seifert matrix

$$V = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

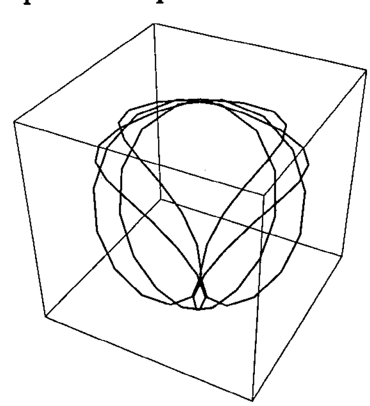
A Seifert matrix is not a knot invariant, but it can be used to distinguish between different SEIFERT SURFACES for a given knot.

see also ALEXANDER MATRIX

References

Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, pp. 200–203, 1976.

Seifert's Spherical Spiral



Is given by the CYLINDRICAL COORDINATES parametric equation

$$r = \mathrm{sn}(s)$$
$$\theta = ks$$
$$z = \mathrm{cn}(s),$$

where k is a POSITIVE constant and $\mathrm{sn}(s)$ and $\mathrm{cn}(s)$ are JACOBI ELLIPTIC FUNCTIONS (Whittaker and Watson 1990, pp. 527–528).

References
Bowman, F. *Introduction to Elliptic Functions, with Applications.* New York: Dover, p. 34, 1961.
Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis, 4th ed.* Cambridge, England: Cambridge University Press, 1990.

Seifert Surface

An orientable surface with one boundary component such that the boundary component of the surface is a given KNOT K. In 1934, Seifert proved that such a surface can be constructed for any KNOT. The process of generating this surface is known as Seifert's algorithm. Applying Seifert's algorithm to an alternating projection of an alternating knot yields a Seifert surface of minimal GENUS.

There are KNOTS for which the minimal genus Seifert surface cannot be obtained by applying Seifert's algorithm to any projection of that KNOT, as proved by Morton in 1986 (Adams 1994, p. 105).

see also GENUS (KNOT), SEIFERT MATRIX

References
Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, pp. 95–106, 1994.
Seifert, H. "Über das Geschlecht von Knotten." *Math. Ann.* **110**, 571–592, 1934.

Self-Adjoint Matrix

A MATRIX A for which

$$\mathsf{A}^\dagger \equiv (\mathsf{A}^{\mathrm{T}})^* = \mathsf{A},$$

where the ADJOINT OPERATOR is denoted $\mathsf{A}^\dagger$, A^{T} is the MATRIX TRANSPOSE, and $*$ is the COMPLEX CONJUGATE. If a MATRIX is self-adjoint, it is said to be HERMITIAN.

see also ADJOINT OPERATOR, HERMITIAN MATRIX, MATRIX TRANSPOSE

Self-Adjoint Operator

Given a differential equation

$$\tilde{\mathcal{L}}u(x) \equiv p_0\frac{du^2}{dx^2} + p_1\frac{du}{dx} + p_2 u, \tag{1}$$

where $p_i \equiv p_i(x)$ and $u \equiv u(x)$, the ADJOINT OPERATOR $\tilde{\mathcal{L}}^\dagger$ is defined by

$$\tilde{\mathcal{L}}^\dagger u \equiv \frac{d}{dx^2}(p_0 u) - \frac{d}{dx}(p_1 u) + p_2 u \tag{2}$$

$$= p_0\frac{d^2u}{dx^2} + (2p_0{}' - p_1)\frac{du}{dx} + (p_0{}'' - p_1{}' + p_2)u. \tag{3}$$

In order for the operator to be self-adjoint, i.e.,

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}^\dagger, \tag{4}$$

the second terms in (1) and (3) must be equal, so

$$p_0{}'(x) = p_1(x). \tag{5}$$

This also guarantees that the third terms are equal, since

$$p_0{}'(x) = p_1(x) \Rightarrow p_0{}''(x) = p_1{}'(x), \tag{6}$$

so (3) becomes

$$\tilde{\mathcal{L}}u = \tilde{\mathcal{L}}^\dagger u = p_0\frac{d^2}{dx^2} + p_0{}'\frac{du}{dx} + p_2 u \tag{7}$$

$$= \frac{d}{dx}\left(p_0\frac{du}{dx}\right) + p_2 u = 0. \tag{8}$$

The LEGENDRE DIFFERENTIAL EQUATION and the equation of SIMPLE HARMONIC MOTION are self-adjoint, but the LAGUERRE DIFFERENTIAL EQUATION and HERMITE DIFFERENTIAL EQUATION are not.

A nonself-adjoint second-order linear differential operator can always be transformed into a self-adjoint one using STURM-LIOUVILLE THEORY. In the special case $p_2(x) = 0$, (8) gives

$$\frac{d}{dx}\left[p_0(x)\frac{du}{dx}\right] = 0 \tag{9}$$

$$p_0(x)\frac{du}{dx} = C \tag{10}$$

$$du = C\frac{dx}{p_0(x)} \tag{11}$$

$$u = C\int\frac{dx}{p_0(x)}, \tag{12}$$

where C is a constant of integration.

A self-adjoint operator which satisfies the BOUNDARY CONDITIONS

$$v^* pU'|_{x=a} = v^* pU'|_{x=b} \tag{13}$$

is automatically a HERMITIAN OPERATOR.

see also ADJOINT OPERATOR, HERMITIAN OPERATOR, STURM-LIOUVILLE THEORY

References
Arfken, G. "Self-Adjoint Differential Equations." §9.1 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 497–509, 1985.

Self-Avoiding Walk

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let the number of RANDOM WALKS on a d-D lattice starting at the ORIGIN which never land on the same lattice point twice in n steps be denoted $c(n)$. The first few values are

$$c_d(0) = 1 \tag{1}$$
$$c_d(1) = 2d \tag{2}$$
$$c_d(2) = 2d(2d-1). \tag{3}$$

The connective constant

$$\mu_d \equiv \lim_{n\to\infty} [c_d(n)]^{1/n} \tag{4}$$

is known to exist and be FINITE. The best ranges for these constants are

$$\mu_2 \in [2.62002, 2.6939] \tag{5}$$
$$\mu_3 \in [4.572140, 4.7476] \tag{6}$$
$$\mu_4 \in [6.742945, 6.8179] \tag{7}$$
$$\mu_5 \in [8.828529, 8.8602] \tag{8}$$
$$\mu_6 \in [10.874038, 10.8886] \tag{9}$$

(Finch).

For the triangular lattice in the plane, $\mu < 4.278$ (Alm 1993), and for the hexagonal planar lattice, it is conjectured that

$$\mu = \sqrt{2+\sqrt{2}} \tag{10}$$

(Madras and Slade 1993).

The following limits are also believed to exist and to be FINITE:

$$\begin{cases} \lim_{n\to\infty} \frac{c(n)}{\mu^n n^{\gamma-1}} & \text{for } d \neq 4 \\ \lim_{n\to\infty} \frac{c(n)}{\mu^n n^{\gamma-1}(\ln n)^{1/4}} & \text{for } d = 4, \end{cases} \tag{11}$$

where the critical exponent $\gamma = 1$ for $d > 4$ (Madras and Slade 1993) and it has been conjectured that

$$\gamma = \begin{cases} \frac{43}{32} & \text{for } d = 2 \\ 1.162\ldots & \text{for } d = 3 \\ 1 & \text{for } d = 4. \end{cases} \tag{12}$$

Define the mean square displacement over all n-step self-avoiding walks ω as

$$s(n) \equiv \left\langle |\omega(n)|^2 \right\rangle = \frac{1}{c(n)} \sum_\omega |\omega(n)|^2. \tag{13}$$

The following limits are believed to exist and be FINITE:

$$\begin{cases} \lim_{n\to\infty} \frac{s(n)}{n^{2\nu}} & \text{for } d \neq 4 \\ \lim_{n\to\infty} \frac{s(n)}{n^{2\nu}(\ln n)^{1/4}} & \text{for } d = 4, \end{cases} \tag{14}$$

where the critical exponent $\nu = 1/2$ for $d > 4$ (Madras and Slade 1993), and it has been conjectured that

$$\nu = \begin{cases} \frac{3}{4} & \text{for } d = 2 \\ 0.59\ldots & \text{for } d = 3 \\ \frac{1}{2} & \text{for } d = 4. \end{cases} \tag{15}$$

see also RANDOM WALK

References

Alm, S. E. "Upper Bounds for the Connective Constant of Self-Avoiding Walks." *Combin. Prob. Comput.* **2**, 115–136, 1993.

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/cnntv/cnntv.html`.

Madras, N. and Slade, G. *The Self-Avoiding Walk.* Boston, MA: Birkhäuser, 1993.

Self-Conjugate Subgroup

see INVARIANT SUBGROUP

Self-Descriptive Number

A 10-DIGIT number satisfying the following property. Number the DIGITS 0 to 9, and let DIGIT n be the number of ns in the number. There is exactly one such number: 6210001000.

References

Pickover, C. A. "Chaos in Ontario." Ch. 28 in *Keys to Infinity.* New York: W. H. Freeman, pp. 217–219, 1995.

Self-Homologous Point

see SIMILITUDE CENTER

Self Number

A number (usually base 10 unless specified otherwise) which has no GENERATOR. Such numbers were originally called COLUMBIAN NUMBERS (S. 1974). There are infinitely many such numbers, since an infinite sequence of self numbers can be generated from the RECURRENCE RELATION

$$C_k = 8 \cdot 10^{k-1} + C_{k-1} + 8, \tag{1}$$

for $k = 2, 3, \ldots$, where $C_1 = 9$. The first few self numbers are 1, 3, 5, 7, 9, 20, 31, 42, 53, 64, 75, 86, 97, ... (Sloane's A003052).

An infinite number of 2-self numbers (i.e., base-2 self numbers) can be generated by the sequence

$$C_k = 2^j + C_{k-1} + 1 \tag{2}$$

for $k = 1, 2, \ldots$, where $C_1 = 1$ and j is the number of digits in C_{k-1}. An infinite number of n-self numbers can be generated from the sequence

$$C_k = (n-2)n^{k-1} + C_{k-1} + (n-2) \tag{3}$$

for $k = 2, 3, \ldots$, and

$$C_1 = \begin{cases} n-1 & \text{for } n \text{ even} \\ n-2 & \text{for } n \text{ odd.} \end{cases} \tag{4}$$

Joshi (1973) proved that if k is ODD, then m is a k-self number IFF m is ODD. Patel (1991) proved that $2k$, $4k+2$, and k^2+2k+1 are k-self numbers in every EVEN base $k \geq 4$.

see also DIGITADITION

References

Cai, T. "On k-Self Numbers and Universal Generated Numbers." *Fib. Quart.* **34**, 144–146, 1996.

Gardner, M. *Time Travel and Other Mathematical Bewilderments.* New York: W. H. Freeman, pp. 115–117, 122, 1988.

Joshi, V. S. Ph.D. dissertation. Gujarat University, Ahmadabad, 1973.

Kaprekar, D. R. *The Mathematics of New Self-Numbers.* Devaiali, pp. 19–20, 1963.

Patel, R. B. "Some Tests for k-Self Numbers." *Math. Student* **56**, 206–210, 1991.

S., B. R. Solution to Problem E 2048. *Amer. Math. Monthly* **81**, 407, 1974.

Sloane, N. J. A. Sequence A003052/M2404 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Self-Reciprocating Property

Let h be the number of sides of certain skew POLYGONS (Coxeter 1973, p. 15). Then

$$h = \frac{2(p+q+2)}{10-p-q}.$$

References

Coxeter, H. S. M. *Regular Polytopes, 3rd ed.* New York: Dover, 1973.

Self-Recursion

Self-recursion is a RECURSION which is defined in terms of itself, resulting in an ill-defined infinite regress.

see SELF-RECURSION

Self-Similarity

An object is said to be self-similar if it looks "roughly" the same on any scale. FRACTALS are a particularly interesting class of self-similar objects.

see also FRACTAL

References

Hutchinson, J. "Fractals and Self-Similarity." *Indiana Univ. J. Math.* **30**, 713–747, 1981.

Self-Transversality Theorem

Let j, r, and s be distinct INTEGERS (mod n), and let W_i be the point of intersection of the side or diagonal V_iV_{i+j} of the n-gon $P = [V_1, \ldots, V_n]$ with the transversal $V_{i+r}V_{i+s}$. Then a NECESSARY and SUFFICIENT condition for

$$\prod_{i=1}^{n} \left[\frac{V_iW_i}{W_iV_{i+j}}\right] = (-1)^n,$$

where $AB||CD$ and

$$\left[\frac{AB}{CD}\right],$$

is the ratio of the lengths $[A, B]$ and $[C, D]$ with a plus or minus sign depending on whether these segments have the same or opposite direction, is that

1. $n = 2m$ is EVEN with $j \equiv m \pmod{n}$ and $s \equiv r + m \pmod{n}$,
2. n is arbitrary and either $s \equiv 2r$ and $j \equiv 3r$, or
3. $r \equiv 2s \pmod{n}$ and $j \equiv 3s \pmod{n}$.

References

Grünbaum, B. and Shepard, G. C. "Ceva, Menelaus, and the Area Principle." *Math. Mag.* **68**, 254–268, 1995.

Selfridge's Conjecture

There exist infinitely many $n > 0$ with ${p_n}^2 > p_{n-i}p_{n+i}$ for all $i < n$. Also, there exist infinitely many $n > 0$ such that $2p_n < p_{n-i} + p_{n-i}$ for all $i < n$.

Selfridge-Hurwitz Residue

Let the RESIDUE from PÉPIN'S THEOREM be

$$R_n \equiv 3^{(F_n-1)/2} \pmod{F_n},$$

where F_n is a FERMAT NUMBER. Selfridge and Hurwitz use

$$R_n (\text{mod } 2^{35}-1, 2^{36}, 2^{36}-1).$$

A nonvanishing $R_n \pmod{2^{36}}$ indicates that F_n is COMPOSITE for $n > 5$.

see also FERMAT NUMBER, PÉPIN'S THEOREM

References

Crandall, R.; Doenias, J.; Norrie, C.; and Young, J. "The Twenty-Second Fermat Number is Composite." *Math. Comput.* **64**, 863–868, 1995.

Selmer Group

A GROUP which is related to the TANIYAMA-SHIMURA CONJECTURE.

see also TANIYAMA-SHIMURA CONJECTURE

Semi-Integral

An INTEGRAL of order 1/2. The semi-integral of the CONSTANT FUNCTION $f(x) = c$ is

$$\frac{d^{-1/2}c}{dx^{-1/2}} = 2c\sqrt{\frac{x}{\pi}}.$$

see also SEMIDERIVATIVE

References

Spanier, J. and Oldham, K. B. *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 8 and 14, 1987.

Semialgebraic Number

A subset of $\mathbb{R}^n$ which is a finite Boolean combination of sets of the form $\{\bar{x} = (x_1, \ldots, x_m) : f(\bar{x}) > 0\}$ and $\{\bar{x} : g(\bar{x}) = 0\}$, where $f, g \in \mathbb{R}[X_1, \ldots, X_n]$.

References

Bierstone, E. and Milman, P. "Semialgebraic and Subanalytic Sets." *IHES Pub. Math.* **67**, 5–42, 1988.

Marker, D. "Model Theory and Exponentiation." *Not. Amer. Math. Soc.* **43**, 753–759, 1996.

Semianalytic

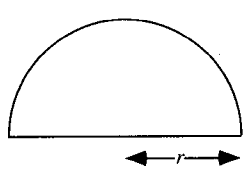
$X \subseteq \mathbb{R}^n$ is semianalytic if, for all $x \in \mathbb{R}^n$, there is an open neighborhood U of x such that $X \cap U$ is a finite Boolean combination of sets $\{\bar{x} \in U : f(\bar{x}) = 0\}$ and $\{\bar{x} \in U : g(\bar{x}) > 0\}$, where $f, g : U \to \mathbb{R}$ are ANALYTIC.

see also ANALYTIC FUNCTION, PSEUDOANALYTIC FUNCTION, SUBANALYTIC

References

Marker, D. "Model Theory and Exponentiation." *Not. Amer. Math. Soc.* **43**, 753–759, 1996.

Semicircle



Half a CIRCLE. The PERIMETER of the semicircle of RADIUS r is

$$L = 2r + \pi r = r(2 + \pi), \tag{1}$$

and the AREA is

$$A = 2\int_0^r \sqrt{r^2 - y^2}\, dy = \tfrac{1}{2}\pi r^2. \tag{2}$$

The weighted mean of y is

$$\langle y \rangle = 2\int_0^r y\sqrt{r^2 - y^2}\, dy = \tfrac{2}{3}r^3. \tag{3}$$

The CENTROID is then given by

$$\bar{y} = \frac{\langle y \rangle}{A} = \frac{4r}{3\pi}. \tag{4}$$

The semicircle is the CROSS-SECTION of a HEMISPHERE for any PLANE through the z-AXIS.

see also ARBELOS, ARC, CIRCLE, DISK, HEMISPHERE, LENS, RIGHT ANGLE, SALINON, THALES' THEOREM, YIN-YANG

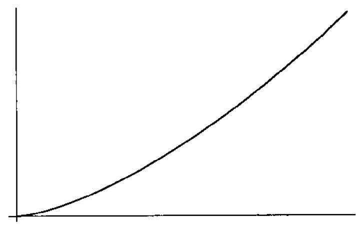
Semicolon Derivative

see COVARIANT DERIVATIVE

Semiconvergent Series

see ASYMPTOTIC SERIES

Semicubical Parabola



A PARABOLA-like curve with Cartesian equation

$$y = ax^{3/2}, \tag{1}$$

parametric equations

$$x = t^2 \tag{2}$$

$$y = at^3, \tag{3}$$

and POLAR COORDINATES,

$$r = \frac{\tan^2\theta \sec\theta}{a}. \tag{4}$$

The semicubical parabola is the curve along which a particle descending under gravity describes equal vertical spacings within equal times, making it an ISOCHRONOUS CURVE. The problem of finding the curve having this property was posed by Leibniz in 1687 and solved by Huygens (MacTutor Archive).

The ARC LENGTH, CURVATURE, and TANGENTIAL ANGLE are

$$s(t) = \tfrac{1}{27}(4 + 9t^2)^{3/2} - \tfrac{8}{27} \tag{5}$$

$$\kappa(t) = \frac{6}{t(4 + 9t^2)^{3/2}} \tag{6}$$

$$\phi(t) = \tan^{-1}(\tfrac{3}{2}t). \tag{7}$$

see also NEILE'S PARABOLA, PARABOLA INVOLUTE

References

Gray, A. "The Semicubical Parabola." §1.7 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 15–16, 1993.

Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 85–87, 1972.

Lee, X. "Semicubic Parabola." `http://www.best.com/~xah/Special Plane Curves _ dir / Semicubic Parabola _ dir / semicubicParabola.html`.

MacTutor History of Mathematics Archive. "Neile's Parabola." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Neiles.html`.

Yates, R. C. "Semi-Cubic Parabola." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 186–187, 1952.

Semiderivative

A DERIVATIVE of order 1/2. The semiderivative of the CONSTANT FUNCTION $f(x) = c$ is

$$\frac{d^{1/2}c}{dx^{1/2}} = \frac{c}{\sqrt{\pi x}}.$$

see also DERIVATIVE, SEMI-INTEGRAL

References

Spanier, J. and Oldham, K. B. *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 8 and 14, 1987.

Semidirect Product

The "split" extension G of GROUPS N and F which contains a SUBGROUP $\bar{F}$ isomorphic to F with $G = \bar{F}\bar{N}$ and $\bar{F} \cap \bar{N} = \{e\}$.

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 613, 1980.

Semiflow

An ACTION with $G = \mathbb{R}^+$.

see also FLOW

Semigroup

A mathematical object defined for a set and a BINARY OPERATOR in which the multiplication operation is ASSOCIATIVE. No other restrictions are placed on a semigroup; thus a semigroup need not have an IDENTITY ELEMENT and its elements need not have inverses within the semigroup. A semigroup is an ASSOCIATIVE GROUPOID.

A semigroup can be empty. The total number of semigroups of order n are 1, 4, 18, 126, 1160, 15973, 836021, ... (Sloane's A001423). The number of semigroups of order n with one IDEMPOTENT are 1, 2, 5, 19, 132, 3107, 623615, ... (Sloane's A002786), and with two IDEMPOTENTS are 2, 7, 37, 216, 1780, 32652, ... (Sloane's A002787). The number $a(n)$ of semigroups having n IDEMPOTENTS are 1, 2, 6, 26, 135, 875, ... (Sloane's A002788).

see also ASSOCIATIVE, BINARY OPERATOR, FREE SEMIGROUP, GROUPOID, INVERSE SEMIGROUP, MONOID, QUASIGROUP

References

Clifford, A. H. and Preston, G. B. *The Algebraic Theory of Semigroups.* Providence, RI: Amer. Math. Soc., 1961.
Sloane, N. J. A. Sequences A001423/M3550, A002786/M1522, A002787/M1802, and A002788/M1679 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Semilatus Rectum

Given an ELLIPSE, the semilatus rectum is defined as the distance L measured from a FOCUS such that

$$\frac{1}{L} \equiv \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right), \tag{1}$$

where $r_+ = a(1+e)$ and $r_- = a(1-e)$ are the APOAPSIS and PERIAPSIS, and e is the ELLIPSE's ECCENTRICITY. Plugging in for r_+ and r_- then gives

$$\begin{aligned}\frac{1}{L} &= \frac{1}{2a}\left(\frac{1}{1-e} + \frac{1}{1+e}\right) = \frac{1}{2a}\frac{(1+e)+(1-e)}{1-e^2} \\ &= \frac{1}{a}\frac{1}{1-e^2},\end{aligned} \tag{2}$$

so

$$L = a(1-e^2). \tag{3}$$

see also ECCENTRICITY, ELLIPSE, FOCUS, LATUS RECTUM, SEMIMAJOR AXIS, SEMIMINOR AXIS

Semimagic Square

A square that fails to be a MAGIC SQUARE only because one or both of the main diagonal sums do not equal the MAGIC CONSTANT is called a SEMIMAGIC SQUARE.

see also MAGIC SQUARE

Semimajor Axis

HALF the distance across an ELLIPSE along its long principal axis.

see also ELLIPSE, SEMIMINOR AXIS

Semiminor Axis

Half the distance across an ELLIPSE along its short principal axis.

see also ELLIPSE, SEMIMAJOR AXIS

Semiperfect Magic Cube

A semiperfect magic cube, also called an ANDREWS CUBE, is a MAGIC CUBE for which the cross-section diagonals do not sum to the MAGIC CONSTANT.

see also MAGIC CUBE, PERFECT MAGIC CUBE

References

Gardner, M. "Magic Squares and Cubes." Ch. 17 in *Time Travel and Other Mathematical Bewilderments.* New York: W. H. Freeman, pp. 213–225, 1988.

Semiperfect Number

A number such as $20 = 1 + 4 + 5 + 10$ which is the SUM of some (or all) its PROPER DIVISORS. A semiperfect number which is the SUM of *all* its PROPER DIVISORS is called a PERFECT NUMBER. The first few semiperfect numbers are 6, 12, 18, 20, 24, 28, 30, 36, 40, ... (Sloane's A005835). Every multiple of a semiperfect number is semiperfect, as are all numbers $2^m p$ for $m \geq 1$ and p a PRIME between 2^m and 2^{m+1} (Guy 1994, p. 47).

A semiperfect number cannot be DEFICIENT. Rare ABUNDANT NUMBERS which are not semiperfect are called WEIRD NUMBERS. Semiperfect numbers are sometimes also called PSEUDOPERFECT NUMBERS.

see also ABUNDANT NUMBER, DEFICIENT NUMBER, PERFECT NUMBER, PRIMITIVE SEMIPERFECT NUMBER, WEIRD NUMBER

References

Guy, R. K. "Almost Perfect, Quasi-Perfect, Pseudoperfect, Harmonic, Weird, Multiperfect and Hyperperfect Numbers." §B2 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 45–53, 1994.

Sloane, N. J. A. Sequence A005835/M4094 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Zachariou, A. and Zachariou, E. "Perfect, Semi-Perfect and Ore Numbers." *Bull. Soc. Math. Grèce (New Ser.)* **13**, 12–22, 1972.

Semiperimeter

The semiperimeter on a figure is defined as

$$s \equiv \tfrac{1}{2}p, \tag{1}$$

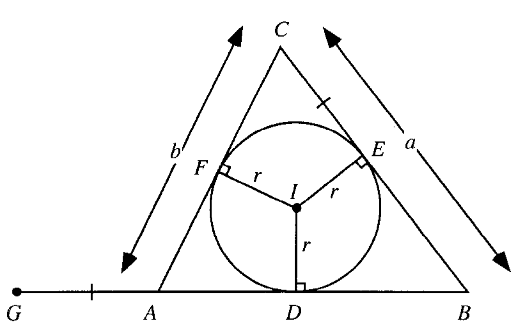
where p is the PERIMETER. The semiperimeter of POLYGONS appears in unexpected ways in the computation of their AREAS. The most notable cases are in the ALTITUDE, EXRADIUS, and INRADIUS of a TRIANGLE, the SODDY CIRCLES, HERON'S FORMULA for the AREA of a TRIANGLE in terms of the legs a, b, and c

$$A_\Delta = \sqrt{s(s-a)(s-b)(s-c)}, \tag{2}$$

and BRAHMAGUPTA'S FORMULA for the AREA of a QUADRILATERAL

$$A_{\text{quadrilateral}} = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{A+B}{2}\right)}. \tag{3}$$

The semiperimeter also appears in the beautiful L'HUILIER'S THEOREM about SPHERICAL TRIANGLES.



For a TRIANGLE, the following identities hold,

$$s-a = \tfrac{1}{2}(-a+b+c) \tag{4}$$
$$s-b = \tfrac{1}{2}(a+b-c) \tag{5}$$
$$s-c = \tfrac{1}{2}(a+b-c). \tag{6}$$

Now consider the above figure. Let I be the INCENTER of the TRIANGLE ΔABC, with D, E, and F the tangent points of the INCIRCLE. Extend the line BA with $GA = CE$. Note that the pairs of triangles (ADI, AFI), (BDI, BEI), (CFI, CEI) are congruent. Then

$$\begin{aligned} BG &= BD + AD + AG = BD + AD + CE \\ &= \tfrac{1}{2}(2BD + 2AD + 2CE) \\ &= \tfrac{1}{2}[(BD+BE) + (AD+AF) + (CE+CF)] \\ &= \tfrac{1}{2}[(BD+AD) + (BE+CE) + (AF+CF)] \\ &= \tfrac{1}{2}(AB+BC+AC) = \tfrac{1}{2}(a+b+c) = s. \end{aligned} \tag{7}$$

Furthermore,

$$\begin{aligned} s-a &= BG - BC \\ &= (BD+AD+AG) - (BE+CE) \\ &= (BD+AD+CE) - (BD+CE) = AC \quad (8) \\ s-b &= BG - AC \\ &= (BD+AD+AG) - (AF+CF) \\ &= (BD+AD+CE) - (AD+CE) = BD \quad (9) \\ s-c &= BG - AB = AG \quad (10) \end{aligned}$$

(Dunham 1990). These equations are some of the building blocks of Heron's derivation of HERON'S FORMULA.

References

Dunham, W. "Heron's Formula for Triangular Area." Ch. 5 in *Journey Through Genius: The Great Theorems of Mathematics.* New York: Wiley, pp. 113–132, 1990.

Semiprime

A COMPOSITE number which is the PRODUCT of two PRIMES (possibly equal). They correspond to the 2-ALMOST PRIMES. The first few are 4, 6, 9, 10, 14, 15, 21, 22, ... (Sloane's A001358).

see also ALMOST PRIME, CHEN'S THEOREM, COMPOSITE NUMBER, PRIME NUMBER

References

Sloane, N. J. A. Sequence A001358/M3274 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Semiprime Ring

Given an IDEAL A, a semiprime ring is one for which $A^n = 0$ IMPLIES $A = 0$ for any POSITIVE n. Every PRIME RING is semiprime.

see also PRIME RING

Semiregular Polyhedron

A POLYHEDRON or plane TESSELLATION is called semiregular if its faces are all REGULAR POLYGONS and its corners are alike (Walsh 1972; Coxeter 1973, pp. 4 and 58; Holden 1991, p. 41). The usual name for a semiregular polyhedron is an ARCHIMEDEAN SOLIDS, of which there are exactly 13.

see also ARCHIMEDEAN SOLID, POLYHEDRON, TESSELLATION

References
Coxeter, H. S. M. "Regular and Semi-Regular Polytopes I." *Math. Z.* **46**, 380–407, 1940.
Coxeter, H. S. M. *Regular Polytopes, 3rd ed.* New York: Dover, 1973.
Holden, A. *Shapes, Space, and Symmetry.* New York: Dover, 1991.
Walsh, T. R. S. "Characterizing the Vertex Neighbourhoods of Semi-Regular Polyhedra." *Geometriae Dedicata* **1**, 117–123, 1972.

Semiring

A semiring is a set together with two BINARY OPERATORS $S(+,*)$ satisfying the following conditions:

1. Additive associativity: For all $a, b, c \in S$, $(a+b)+c = a+(b+c)$,
2. Additive commutativity: For all $a, b \in S$, $a+b = b+a$,
3. Multiplicative associativity: For all $a, b, c \in S$, $(a*b)*c = a*(b*c)$,
4. Left and right distributivity: For all $a, b, c \in S$, $a*(b+c) = (a*b)+(a*c)$ and $(b+c)*a = (b*a)+(c*a)$.

Thus a semiring is therefore a commutative SEMIGROUP under addition and a SEMIGROUP under multiplication. A semiring can be empty.

see also BINARY OPERATOR, RING, RINGOID, SEMIGROUP

References
Rosenfeld, A. *An Introduction to Algebraic Structures.* New York: Holden-Day, 1968.

Semisecant

see TRANSVERSAL LINE

Semisimple

A p-ELEMENT x of a GROUP G is semisimple if $E(C_G(x)) \neq 1$, where $E(H)$ is the commuting product of all components of H and $C_G(x)$ is the CENTRALIZER of G.

see also CENTRALIZER, p-ELEMENT

Semisimple Algebra

An ALGEBRA with no nontrivial nilpotent IDEALS. In the 1890s, Cartan, Frobenius, and Molien independently proved that any finite-dimensional semisimple algebra over the REAL or COMPLEX numbers is a finite and unique DIRECT SUM of SIMPLE ALGEBRAS. This result was then extended to algebras over arbitrary fields by Wedderburn in 1907 (Kleiner 1996).

see also IDEAL, NILPOTENT ELEMENT, SIMPLE ALGEBRA

References
Kleiner, I. "The Genesis of the Abstract Ring Concept." *Amer. Math. Monthly* **103**, 417–424, 1996.

Semisimple Lie Group

A LIE GROUP which has a simply connected covering group HOMEOMORPHIC to $\mathbb{R}^n$. The prototype is any connected closed subgroup of upper TRIANGULAR COMPLEX MATRICES. The HEISENBERG GROUP is such a group.

see also HEISENBERG GROUP, LIE GROUP

References
Knapp, A. W. "Group Representations and Harmonic Analysis, Part II." *Not. Amer. Math. Soc.* **43**, 537–549, 1996.

Semisimple Ring

A SEMIPRIME RING which is also an ARTINIAN RING.

see also ARTINIAN RING

Semistable

When a PRIME l divides the DISCRIMINANT of a ELLIPTIC CURVE E, two or all three roots of E become congruent mod l. An ELLIPTIC CURVE is semistable if, for all such PRIMES l, only two roots become CONGRUENT mod l (with more complicated definitions for $p = 2$ or 3).

see also DISCRIMINANT (ELLIPTIC CURVE), ELLIPTIC CURVE

Sensitivity

The probability that a STATISTICAL TEST will be positive for a true statistic.

see also SPECIFICITY, STATISTICAL TEST, TYPE I ERROR, TYPE II ERROR

Sentence

A LOGIC FORMULA with no FREE variables.

Separating Edge

An EDGE of a GRAPH is separating if a path from a point A to a point B must pass over it. Separating EDGES can therefore be viewed as either bridges or dead ends.

see also EDGE (GRAPH)

Separating Family

A SEPARATING FAMILY is a SET of SUBSETS in which each pair of adjacent elements are found separated, each in one of two disjoint subsets. The 26 letters of the alphabet can be separated by a family of 9,

$$\begin{array}{ccc} (abcdefghi) & (jklmnopqr) & (stuvwxyz) \\ (abcjklstu) & (defmnovwx) & (ghipqryz) \\ (adgjmpsvy) & (behknqtwz) & (cfilorux) \end{array}.$$

The minimal size of the separating family for an n-set is 0, 2, 3, 4, 5, 5, 6, 6, 6, 7, 7, 7, ... (Sloane's A007600).

see also KATONA'S PROBLEM

References
Honsberger, R. "Cai Mao-Cheng's Solution to Katona's Problem on Families of Separating Subsets." Ch. 18 in *Mathematical Gems III.* Washington, DC: Math. Assoc. Amer., pp. 224–239, 1985.
Sloane, N. J. A. Sequence A007600/M0456 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Separation

Two distinct point pairs AC and BD separate each other if A, B, C, and D lie on a CIRCLE (or line) in such order that either of the arcs (or the line segment AC) contains one but not both of B and D. In addition, the point pairs separate each other if every CIRCLE through A and C intersects (or coincides with) every CIRCLE through B and D. If the point pairs separate each other, then the symbol $AC//BD$ is used.

Separation of Variables

A method of solving partial differential equations in a function Φ and variables x, y, ... by making a substitution of the form

$$\Phi(x, y, \ldots) \equiv X(x)Y(y)\cdots,$$

breaking the resulting equation into a set of independent ordinary differential equations, solving these for $X(x)$, $Y(y)$, ..., and then plugging them back into the original equation.

This technique works because if the product of functions of independent variables is a constant, each function must separately be a constant. Success requires choice of an appropriate coordinate system and may not be attainable at all depending on the equation. Separation of variables was first used by L'Hospital in 1750. It is especially useful in solving equations arising in mathematical physics, such as LAPLACE'S EQUATION, the HELMHOLTZ DIFFERENTIAL EQUATION, and the Schrödinger equation.

see also HELMHOLTZ DIFFERENTIAL EQUATION, LAPLACE'S EQUATION

References

Arfken, G. "Separation of Variables" and "Separation of Variables—Ordinary Differential Equations." §2.6 and §8.3 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 111–117 and 448–451, 1985.

Morse, P. M. and Feshbach, H. "Separable Coordinates" and "Table of Separable Coordinates in Three Dimensions." §5.1 in *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 464–523 and 655–666, 1953.

Separation Theorem

There exist numbers $y_1 < y_2 < \ldots < x_{n-1}$, $a < y_{n-1}$, $y_{n-1} < b$, such that

$$\lambda_\nu = \alpha(y_\nu) - \alpha(y_{\nu-1}),$$

where $\nu = 1, 2, \ldots, n$, $y_0 = a$ and $y_n = b$. Furthermore, the zeros $x_1, \ldots, x_n$, arranged in increasing order, alternate with the numbers $y_1, \ldots y_{n-1}$, so

$$x_\nu < y_\nu < x_{\nu+1}.$$

More precisely,

$$\begin{aligned}\alpha(x_\nu + \epsilon) - \alpha(a) &< \alpha(y_\nu) - \alpha(a)\\ &= \lambda_1 + \ldots + \lambda_\nu < \alpha(x_{\nu+1} - \epsilon) - \alpha(a)\end{aligned}$$

for $\nu = 1, \ldots, n-1$.

see also POINCARÉ SEPARATION THEOREM, STURMIAN SEPARATION THEOREM

References

Szegő, G. *Orthogonal Polynomials, 4th ed.* Providence, RI: Amer. Math. Soc., p. 50, 1975.

Separatrix

A phase curve (invariant MANIFOLD) which meets a HYPERBOLIC FIXED POINT (intersection of a stable and an unstable invariant MANIFOLD). A separatrix marks a boundary between phase curves with different properties. For example, the separatrix in the equation of motion for the pendulum occurs at the angular momentum where oscillation gives way to rotation.

Septendecillion

In the American system, 10^{54}.

see also LARGE NUMBER

Septillion

In the American system, 10^{24}.

see also LARGE NUMBER

Sequence

A sequence is an ordered set of mathematical objects which is denoted using braces. For example, the symbol $\{2n\}_{n=1}^\infty$ denotes the infinite sequence of EVEN NUMBERS $\{2, 4, \ldots, 2n, \ldots\}$.

see also 196-ALGORITHM, A-SEQUENCE, ALCUIN'S SEQUENCE, $B2$-SEQUENCE, BEATTY SEQUENCE, CARMICHAEL SEQUENCE, CAUCHY SEQUENCE, CONVERGENT SEQUENCE, DEGREE SEQUENCE, DENSITY (SEQUENCE), FRACTAL SEQUENCE, GIUGA SEQUENCE, INFINITIVE SEQUENCE, INTEGER SEQUENCE, ITERATION SEQUENCE, LIST, NONAVERAGING SEQUENCE, PRIMITIVE SEQUENCE, REVERSE-THEN-ADD SEQUENCE, SCORE SEQUENCE, SERIES, SIGNATURE SEQUENCE, SORT-THEN-ADD SEQUENCE, ULAM SEQUENCE

Sequency

The sequency k of a WALSH FUNCTION is defined as half the number of zero crossings in the time base.

see also WALSH FUNCTION

Sequency Function

see WALSH FUNCTION

Sequential Graph

A CONNECTED GRAPH having e EDGES is said to be sequential if it is possible to label the nodes i with distinct INTEGERS f_i in$\{0, 1, 2, \ldots, e-1\}$ such that when EDGE ij is labeled $f_i + f_j$, the set of EDGE labels is a block of e consecutive integers (Grace 1983, Gallian 1990). No HARMONIOUS GRAPH is known which cannot also be labeled sequentially.

see also CONNECTED GRAPH, HARMONIOUS GRAPH

References
Gallian, J. A. "Open Problems in Grid Labeling." *Amer. Math. Monthly* **97**, 133–135, 1990.
Grace, T. "On Sequential Labelings of Graphs." *J. Graph Th.* **7**, 195–201, 1983.

Series

A series is a sum of terms specified by some rule. If each term increases by a constant amount, it is said to be an ARITHMETIC SERIES. If each term equals the previous multiplied by a constant, it is said to be a GEOMETRIC SERIES. A series usually has an INFINITE number of terms, but the phrase INFINITE SERIES is sometimes used for emphasis or clarity.

If the sum of partial sequences comprising the first few terms of the series does not converge to a LIMIT (e.g., it oscillates or approaches $\pm\infty$), it is said to diverge. An example of a convergent series is the GEOMETRIC SERIES

$$\sum_{n=0}^{\infty} (\tfrac{1}{2})^n = 2,$$

and an example of a divergent series is the HARMONIC SERIES

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

A number of methods known as CONVERGENCE TESTS can be used to determine whether a given series converges. Although terms of a series can have either sign, convergence properties can often be computed in the "worst case" of all terms being POSITIVE, and then applied to the particular series at hand. A series of terms u_n is said to be ABSOLUTELY CONVERGENT if the series formed by taking the absolute values of the u_n,

$$\sum_n |u_n|,$$

converges.

An especially strong type of convergence is called UNIFORM CONVERGENCE, and series which are uniformly convergent have particularly "nice" properties. For example, the sum of a UNIFORMLY CONVERGENT series of continuous functions is continuous. A CONVERGENT SERIES can be DIFFERENTIATED term by term, provided that the functions of the series have continuous derivatives and that the series of DERIVATIVES is UNIFORMLY CONVERGENT. Finally, a UNIFORMLY CONVERGENT series of continuous functions can be INTEGRATED term by term.

For a table listing the COEFFICIENTS for various series operations, see Abramowitz and Stegun (1972, p. 15).

While it can be difficult to calculate analytical expressions for arbitrary convergent infinite series, many algorithms can handle a variety of common series types. The program *Mathematica®* (Wolfram Research, Champaign, IL) implements many of these algorithms. General techniques also exist for computing the numerical values to any but the most pathological series (Braden 1992).

see also ALTERNATING SERIES, ARITHMETIC SERIES, ARTISTIC SERIES, ASYMPTOTIC SERIES, BIAS (SERIES), CONVERGENCE IMPROVEMENT, CONVERGENCE TESTS, EULER-MACLAURIN INTEGRATION FORMULAS, GEOMETRIC SERIES, HARMONIC SERIES, INFINITE SERIES, MELODIC SERIES, q-SERIES, RIEMANN SERIES THEOREM, SEQUENCE, SERIES EXPANSION, SERIES REVERSION

References
Abramowitz, M. and Stegun, C. A. (Eds.). "Infinite Series." §3.6 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 14, 1972.
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Boas, R. P. Jr. "Partial Sums of Infinite Series, and How They Grow." *Amer. Math. Monthly* **84**, 237–258, 1977.
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Bromwich, T. J. I'a. and MacRobert, T. M. *An Introduction to the Theory of Infinite Series, 3rd ed.* New York: Chelsea, 1991.
Hansen, E. R. *A Table of Series and Products.* Englewood Cliffs, NJ: Prentice-Hall, 1975.
Hardy, G. H. *Divergent Series.* Oxford, England: Clarendon Press, 1949.
Jolley, L. B. W. *Summation of Series, 2nd rev. ed.* New York: Dover, 1961.
Knopp, K. *Theory and Application of Infinite Series.* New York: Dover, 1990.
Mangulis, V. *Handbook of Series for Scientists and Engineers.* New York: Academic Press, 1965.
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Series and Their Convergence." §5.1 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 159–163, 1992.
Rainville, E. D. *Infinite Series.* New York: Macmillan, 1967.

Series Expansion

see LAURENT SERIES, MACLAURIN SERIES, POWER SERIES, SERIES REVERSION, TAYLOR SERIES

Series Inversion

see SERIES REVERSION

Series Multisection

If

$$f(x) = f_0 + f_1 x + f_2 x^2 + \ldots + f_n x^n + \ldots,$$

then

$$S(n,j) = f_j x^j + f_{j+n} x^{j+n} + f_{j+2n} x^{j+2n} + \ldots$$

is given by

$$S(n,j) = \frac{1}{n}\sum_{t=0}^{n-1} w^{-jt} f(w^t x),$$

where $w = e^{2\pi i/n}$.

see also SERIES REVERSION

References

Honsberger, R. *Mathematical Gems III.* Washington, DC: Math. Assoc. Amer., pp. 210–214, 1985.

Series Reversion

Series reversion is the computation of the COEFFICIENTS of the inverse function given those of the forward function. For a function expressed in a series as

$$y = a_1x + a_2x^2 + a_3x^3 + \ldots, \tag{1}$$

the series expansion of the inverse series is given by

$$x = A_1y + A_2y^2 + A_3y^3 + \ldots. \tag{2}$$

By plugging (2) into (1), the following equation is obtained

$$\begin{aligned} y = {} & a_1A_1y + (a_2{A_1}^2 + a_1A_2)y^2 \\ & +(a_3{A_1}^3 + 2a_2A_1A_2 + a_1A_3)y^3 \\ & +(3a_3{A_1}^2A_2 + a_2{A_2}^2 + a_2A_1A_3) + \ldots. \end{aligned} \tag{3}$$

Equating COEFFICIENTS then gives

$$A_1 = {a_1}^{-1} \tag{4}$$

$$A_2 = -\frac{a_2}{a_1}{A_1}^2 = -{a_1}^{-3}a_2 \tag{5}$$

$$A_3 = {a_1}^{-5}(2{a_2}^2 - a_1a_3) \tag{6}$$

$$A_4 = {a_1}^{-7}(5a_1a_2a_3 - {a_1}^2a_4 - 5{a_2}^3) \tag{7}$$

$$\begin{aligned} A_5 = {} & {a_1}^{-9}(6{a_1}^2a_2a_4 + 3{a_1}^2a_2a_3 + 14{a_2}^4 - {a_1}^3a_5 \\ & - 21a_1{a_2}^2a_3) \end{aligned} \tag{8}$$

$$\begin{aligned} A_6 = {} & {a_1}^{-11}(7{a_1}^3a_2a_5 + 7{a_1}^3a_3a_4 + 84a_1{a_2}^3a_3 \\ & - {a_1}^4a_6 - 28{a_1}^2a_2{a_3}^2 - 42{a_2}^5 - 28{a_1}^2{a_2}^2a_4) \end{aligned} \tag{9}$$

$$\begin{aligned} A_7 = {} & {a_1}^{-13}(8{a_1}^4a_2a_6 + 8{a_1}^4a_3a_5 + 4{a_1}^4{a_4}^2 \\ & + 120{a_1}^2{a_2}^3a_4 + 180{a_1}^2{a_2}^2{a_3}^2 + 132{a_2}^6 \\ & - {a_1}^5a_7 - 36{a_1}^3{a_2}^2a_5 - 72{a_1}^3a_2a_3a_4 \\ & - 12{a_1}^3{a_3}^3 - 330a_1{a_2}^4a_3) \end{aligned} \tag{10}$$

(Dwight 1961, Abramowitz and Stegun 1972, p. 16). A derivation of the explicit formula for the nth term is given by Morse and Feshbach (1953),

$$\begin{aligned} A_n = {} & \frac{1}{n{a_1}^n}\sum_{s,t,u,\ldots}(-1)^{s+t+u+\ldots} \\ & \times\frac{n(n+1)\cdots(n-1+s+t+u+\ldots)}{s!t!u!\cdots}\left(\frac{a_2}{a_1}\right)^s\left(\frac{a_3}{a_1}\right)^t\cdots, \end{aligned} \tag{11}$$

where

$$s + 2t + 3u + \ldots = n - 1. \tag{12}$$

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, 1972.

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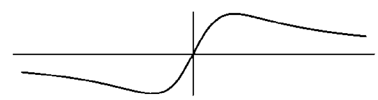
Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 411–413, 1953.

Serpentine Curve



A curve named and studied by Newton in 1701 and contained in his classification of CUBIC CURVES. It had been studied earlier by L'Hospital and Huygens in 1692 (MacTutor Archive).

The curve is given by the CARTESIAN equation

$$y(x) = \frac{abx}{x^2 + a^2} \tag{1}$$

and parametric equations

$$x(t) = a\cot t \tag{2}$$

$$y(t) = b\sin t\cos t. \tag{3}$$

The curve has a MAXIMUM at $x = a$ and a MINIMUM at $x = -a$, where

$$y'(x) = \frac{ab(a-x)(a+x)}{(a^2+x^2)^2} = 0, \tag{4}$$

and inflection points at $x = \pm\sqrt{3}\,a$, where

$$y''(x) = \frac{2abx(x^2 - 3a^2)}{(x^2+a^2)^3} = 0. \tag{5}$$

The CURVATURE is given by

$$\kappa(x) = \frac{2abx(x^2 - 3a^2)}{(x^2+a^2)^3\left[1 + \frac{(a^3b - abx^2)^2}{(x^2+a^2)^4}\right]^{3/2}} \tag{6}$$

$$\kappa(t) = -\frac{4\sqrt{2}\,ab[2\cos(2t) - 1]\cot t\csc^2 t}{\{b^2[1+\cos(4t)] + 2a^2\csc^4 t\}^{3/2}}. \tag{7}$$

References

Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 111–112, 1972.

MacTutor History of Mathematics Archive. "Serpentine." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Serpentine.html`.

Serret-Frenet Formulas

see FRENET FORMULAS

Set

A set is a FINITE or INFINITE collection of objects. Older words for set include AGGREGATE and CLASS. Russell also uses the term MANIFOLD to refer to a set. The study of sets and their properties is the object of SET THEORY. Symbols used to operate on sets include $\wedge$ (which denotes the EMPTY SET $\varnothing$), $\vee =$ (which denotes the POWER SET of a set), $\cap$ (which means "and" or INTERSECTION), and $\cup$ (which means "or" or UNION).

The NOTATION A^B, where A and B are arbitrary sets, is used to denote the set of MAPS from B to A. For example, an element of $X^{\mathbb{N}}$ would be a MAP from the NATURAL NUMBERS $\mathbb{N}$ to the set X. Call such a function f, then $f(1)$, $f(2)$, etc., are elements of X, so call them x_1, x_2, etc. This now looks like a SEQUENCE of elements of X, so sequences are really just functions from $\mathbb{N}$ to X. This NOTATION is standard in mathematics and is frequently used in symbolic dynamics to denote sequence spaces.

Let E, F, and G be sets. Then operation on these sets using the $\cap$ and $\cup$ operators is COMMUTATIVE

$$E \cap F = F \cap E \tag{1}$$

$$E \cup F = F \cup E, \tag{2}$$

ASSOCIATIVE

$$(E \cap F) \cap G = E \cap (F \cap G) \tag{3}$$

$$A \cap \left(\bigcup_{i=1}^{n} B_i \right) = \bigcup_{i=1}^{n} (A \cap B_i) \tag{4}$$

$$(E \cup F) \cup G = E \cup (F \cup G), \tag{5}$$

and DISTRIBUTIVE

$$(E \cap F) \cup G = (E \cup G) \cap (F \cup G) \tag{6}$$

$$(E \cup F) \cap G = (E \cap G) \cup (F \cap G). \tag{7}$$

The proofs follow trivially using VENN DIAGRAMS.

$$P\left(\bigcup_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} P(A_i). \tag{8}$$

The table below gives symbols for some common sets in mathematics.

Symbol	Set
$\mathbb{B}^n$	n-ball
$\mathbb{C}$	complex numbers
C^n, $C^{(n)}$	n-differentiable functions
$\mathbb{D}^n$	n-disk
$\mathbb{H}$	quaternions
$\mathbb{I}$	integers
$\mathbb{N}$	natural numbers
$\mathbb{Q}$	rational numbers
$\mathbb{R}^n$	real numbers in n-D
$\mathbb{S}^n$	n-sphere
$\mathbb{Z}$	integers
$\mathbb{Z}_n$	integers (mod n)
$\mathbb{Z}^-$	negative integers
$\mathbb{Z}^+$	positive integers
$\mathbb{Z}^*$	nonnegative integers

see also AGGREGATE, ANALYTIC SET, BOREL SET, $\mathbb{C}$, CLASS (SET), COANALYTIC SET, DEFINABLE SET, DERIVED SET, DOUBLE-FREE SET, EXTENSION, GROUND SET, $\mathbb{I}$, INTENSION, INTERSECTION, KINNEY'S SET, MANIFOLD, $\mathbb{N}$, PERFECT SET, POSET, $\mathbb{Q}$, $\mathbb{R}$, SET DIFFERENCE, SET THEORY, TRIPLE-FREE SET, UNION, VENN DIAGRAM, WELL-ORDERED SET, $\mathbb{Z}$, $\mathbb{Z}^-$, $\mathbb{Z}^+$

References

Courant, R. and Robbins, H. "The Algebra of Sets." Supplement to Ch. 2 in *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, pp. 108–116, 1996.

Set Difference

The set difference $A \backslash B$ is defined by

$$A \backslash B = \{x : x \in A \text{ and } x \notin B\}.$$

The same symbol is also used for QUOTIENT GROUPS.

Set Partition

A set partition of a SET S is a collection of disjoint SUBSETS $\mathbf{B}_0$, $\mathbf{B}_1$, ... of S whose UNION is S, where each $\mathbf{B}_i$ is called a BLOCK. The number of partitions of the SET $\{k\}_{k=1}^n$ is called a BELL NUMBER.

see also BELL NUMBER, BLOCK, RESTRICTED GROWTH STRING, STIRLING NUMBER OF THE SECOND KIND

References

Ruskey, F. "Info About Set Partitions." `http://sue.csc.uvic.ca/~cos/inf/setp/SetPartitions.html`.

Set Theory

The mathematical theory of SETS. Set theory is closely associated with the branch of mathematics known as LOGIC.

There are a number of different versions of set theory, each with its own rules and AXIOMS. In order of increasing CONSISTENCY STRENGTH, several versions of set theory include PEANO ARITHMETIC (ordinary ALGEBRA), second-order arithmetic (ANALYSIS), ZERMELO-FRAENKEL SET THEORY, Mahlo, weakly compact, hyper-Mahlo, ineffable, measurable, Ramsey, supercompact, huge, and n-huge set theory.

Given a set of REAL NUMBERS, there are 14 versions of set theory which can be obtained using only closure and complement (Beeler *et al.* 1972, Item 105).

see also AXIOMATIC SET THEORY, CONSISTENCY STRENGTH, CONTINUUM HYPOTHESIS, DESCRIPTIVE SET THEORY, IMPREDICATIVE, NAIVE SET THEORY, PEANO ARITHMETIC, SET, ZERMELO-FRAENKEL SET THEORY

References
Beeler, M.; Gosper, R. W.; and Schroeppel, R. *HAKMEM.* Cambridge, MA: MIT Artificial Intelligence Laboratory, Memo AIM-239, pp. 36–44, Feb. 1972.
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Halmos, P. R. *Naive Set Theory.* New York: Springer-Verlag, 1974.
MacTutor History of Mathematics Archive. "The Beginnings of Set Theory." http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Beginnings_of_set_theory.html.
Stewart, I. *The Problems of Mathematics, 2nd ed.* Oxford: Oxford University Press, p. 96, 1987.

Sexagesimal

The base-60 notational system for representing REAL NUMBERS. A base-60 number system was used by the Babylonians and is preserved in the modern measurement of time (hours, minutes, and seconds) and ANGLES (DEGREES, ARC MINUTES, and ARC SECONDS).

see also BASE (NUMBER), BINARY, DECIMAL, HEXADECIMAL, OCTAL, QUATERNARY, SCRUPLE, TERNARY, VIGESIMAL

References
Bergamini, D. *Mathematics.* New York: Time-Life Books, pp. 16–17, 1969.
Weisstein, E. W. "Bases." http://www.astro.virginia.edu/~eww6n/math/notebooks/Bases.m.

Sexdecillion

In the American system, 10^{51}.

see also LARGE NUMBER

Sextic Equation

The general sextic polynomial equation

$$x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

can be solved in terms of HYPERGEOMETRIC FUNCTIONS in one variable using Klein's approach to solving the QUINTIC EQUATION.

see also CUBIC EQUATION, QUADRATIC EQUATION, QUARTIC EQUATION, QUINTIC EQUATION

References
Coble, A. B. "The Reduction of the Sextic Equation to the Valentiner Form—Problem." *Math. Ann.* **70**, 337–350, 1911a.
Coble, A. B. "An Application of Moore's Cross-ratio Group to the Solution of the Sextic Equation." *Trans. Amer. Math. Soc.* **12**, 311–325, 1911b.
Cole, F. N. "A Contribution to the Theory of the General Equation of the Sixth Degree." *Amer. J. Math.* **8**, 265–286, 1886.

Sextic Surface

An ALGEBRAIC SURFACE which can be represented implicitly by a polynomial of degree six in x, y, and z. Examples are the BARTH SEXTIC and BOY SURFACE.

see also ALGEBRAIC SURFACE, BARTH SEXTIC, BOY SURFACE, CUBIC SURFACE, DECIC SURFACE, QUADRATIC SURFACE, QUARTIC SURFACE

References
Catanese, F. and Ceresa, G. "Constructing Sextic Surfaces with a Given Number of Nodes." *J. Pure Appl. Algebra* **23**, 1–12, 1982.
Hunt, B. "Algebraic Surfaces." http://www.mathematik.uni-kl.de/~wwwagag/Galerie.html.

Sextillion

In the American system, 10^{21}.

see also LARGE NUMBER

Sexy Primes

Since a PRIME NUMBER cannot be divisible by 2 or 3, it must be true that, for a PRIME p, $p \equiv 6 \pmod{1,5}$. This motivates the definition of sexy primes as a pair of primes (p, q) such that $p - q = 6$ ("sexy" since "sex" is the Latin word for "six."). The first few sexy prime pairs are (5, 11), (7, 13), (11, 17), (13, 19), (17, 23), (23, 29), (31, 37), (37, 43), (41, 47), (47, 53), ... (Sloane's A023201 and A046117).

Sexy constellations also exist. The first few sexy triplets (i.e., numbers such that each of $(p, p+6, p+12)$ is PRIME but $p+18$ is *not* PRIME) are (7, 13, 19), (17, 23, 29), (31, 37, 43), (47, 53, 59), ... (Sloane's A046118, A046119, and A046120). The first few sexy quadruplets are (11, 17, 23, 29), (41, 47, 53, 59), (61, 67, 73, 79), (251, 257, 263, 269), ... (Sloane's A046121, A046122, A046123, A046124). Sexy quadruplets can only begin with a PRIME ending in a "1." There is only a single sexy quintuplet, (5, 11, 17, 23, 29), since every fifth number of the form $6n \pm 1$ is divisible by 5, and therefore cannot be PRIME.

see also PRIME CONSTELLATION, PRIME QUADRUPLET, TWIN PRIMES

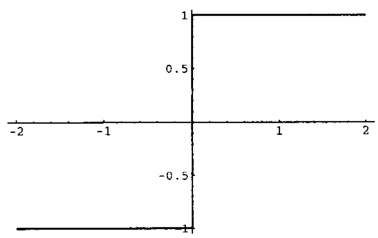
References
Sloane, N. J. A. Sequences A023201, A046117, A046118, A046119, A046120, A046121, A046122, A046123, and A046124 in "An On-Line Version of the Encyclopedia of Integer Sequences."
Trotter, T. "Sexy Primes." http://www.geocities.com/CapeCanaveral/Launchpad/8202/sexyprim.html.

Seydewitz's Theorem

If a TRIANGLE is inscribed in a CONIC SECTION, any line conjugate to one side meets the other two sides in conjugate points.

see also CONIC SECTION, TRIANGLE

Sgn



Also called SIGNUM. It can be defined as

$$\operatorname{sgn} \equiv \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} \tag{1}$$

or

$$\operatorname{sgn}(x) = 2H(x) - 1, \tag{2}$$

where $H(x)$ is the HEAVISIDE STEP FUNCTION. For $x \neq 0$, this can be written

$$\operatorname{sgn}(x) \equiv \frac{x}{|x|} \text{ for } x \neq 0. \tag{3}$$

see also HEAVISIDE STEP FUNCTION, RAMP FUNCTION

Shadow

The SURFACE corresponding to the region of obscuration when a solid is illuminated from a point light source (located at the RADIANT POINT). A DISK is the SHADOW of a SPHERE on a PLANE perpendicular to the SPHERE-RADIANT POINT line. If the PLANE is tilted, the shadow can be the interior of an ELLIPSE or a PARABOLA.

see also PROJECTIVE GEOMETRY

Shadowing Theorem

Although a numerically computed CHAOTIC trajectory diverges exponentially from the true trajectory with the same initial coordinates, there exists an errorless trajectory with a slightly different initial condition that stays near ("shadows") the numerically computed one. Therefore, the FRACTAL structure of chaotic trajectories seen in computer maps is real.

References

Ott, E. *Chaos in Dynamical Systems.* New York: Cambridge University Press, pp. 18–19, 1993.

Shafarevich Conjecture

A conjecture which implies the MORDELL CONJECTURE, as proved in 1968 by A. N. Parshin.

see also MORDELL CONJECTURE

References

Stewart, I. *The Problems of Mathematics, 2nd ed.* Oxford, England: Oxford University Press, p. 45, 1987.

Shah Function

$$\text{Ш}(x) \equiv \sum_{n=-\infty}^{\infty} \delta(x-n) \tag{1}$$

where $\delta(x)$ is the DELTA FUNCTION, so $\text{Ш}(x) = 0$ for $x \notin \mathbb{Z}$ (i.e., x not an INTEGER). The shah function obeys the identities

$$\text{Ш}(ax) = \frac{1}{a} \sum_{n=-\infty}^{\infty} \delta\left(x - \frac{n}{a}\right) \tag{2}$$

$$\text{Ш}(-x) = \text{Ш}(x) \tag{3}$$

$$\text{Ш}(x+n) = \text{Ш}(x), \tag{4}$$

for $2n \in \mathbb{Z}$ (i.e., n a half-integer).

It is normalized so that

$$\int_{n-1/2}^{n+1/2} \text{Ш}(x)\, dx = 1. \tag{5}$$

The "sampling property" is

$$\text{Ш}(x) f(x) = \sum_{n=-\infty}^{\infty} f(n)\delta(x-n) \tag{6}$$

and the "replicating property" is

$$\text{Ш}(x) * f(x) = \sum_{n=-\infty}^{\infty} f(x-n), \tag{7}$$

where $*$ denotes CONVOLUTION.

see also CONVOLUTION, DELTA FUNCTION, IMPULSE PAIR

Shah-Wilson Constant

see TWIN PRIMES CONSTANT

Shallit Constant

Define $f(x_1, x_2, \ldots, x_n)$ with x_i POSITIVE as

$$f(x_1, x_2, \ldots, x_n) \equiv \sum_{i=1}^{n} x_i + \sum_{1 \leq i \leq k \leq n} \prod_{j=i}^{k} \frac{1}{x_j}.$$

Then

$$\min f = 3n - C + o(1)$$

as n increases, where the Shallit constant is

$$C = 1.369451403937\ldots$$

(Shallit 1995). In their solution, Grosjean and De Meyer (quoted in Shallit 1995) reduced the complexity of the problem.

References

MacLeod, A. `http://www.mathsoft.com/asolve/constant/shapiro/macleod.html`.

Shallit, J. Solution by C. C. Grosjean and H. E. De Meyer. "A Minimization Problem." Problem 94-15 in *SIAM Review* **37**, 451–458, 1995.

Shallow Diagonal

see PASCAL'S TRIANGLE

Shanks' Algorithm

An ALGORITHM which finds the least NONNEGATIVE value of $\sqrt{a \pmod{p}}$ for given a and PRIME p.

Shanks' Conjecture

Let $p(g)$ be the first PRIME which follows a PRIME GAP of g between consecutive PRIMES. Shanks' conjecture holds that

$$\ln[p(g)] \sim \sqrt{g}.$$

see also PRIME DIFFERENCE FUNCTION, PRIME GAPS

References

Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 21, 1994.

Rivera, C. "Problems & Puzzles (Conjectures): Shanks' Conjecture." `http://www.sci.net.mx/~crivera/ppp/conj_009.htm`.

Shanks, D. "On Maximal Gaps Between Successive Primes." *Math. Comput.* **18**, 646–651, 1964.

Shannon Entropy

see ENTROPY

Shannon Sampling Theorem

see SAMPLING THEOREM

Shape Operator

The negative derivative

$$S(\mathbf{v}) = -D_{\mathbf{v}}\mathbf{N} \tag{1}$$

of the unit normal $\mathbf{N}$ vector field of a SURFACE is called the shape operator (or WEINGARTEN MAP or SECOND FUNDAMENTAL TENSOR). The shape operator S is an EXTRINSIC CURVATURE, and the GAUSSIAN CURVATURE is given by the DETERMINANT of S. If $\mathbf{x} : U \to \mathbb{R}^3$ is a REGULAR PATCH, then

$$S(\mathbf{x}_u) = -\mathbf{N}_u \tag{2}$$

$$S(\mathbf{x}_v) = -\mathbf{N}_v. \tag{3}$$

At each point $\mathbf{p}$ on a REGULAR SURFACE $M \subset \mathbb{R}^3$, the shape operator is a linear map

$$S : M_{\mathbf{p}} \to M_{\mathbf{p}}. \tag{4}$$

The shape operator for a surface is given by the WEINGARTEN EQUATIONS.

see also CURVATURE, FUNDAMENTAL FORMS, WEINGARTEN EQUATIONS

References

Gray, A. "The Shape Operator," "Calculation of the Shape Operator," and "The Eigenvalues of the Shape Operator." §14.1, 14.3, and 14.4 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 268–269, 274–279, 1993.

Reckziegel, H. In *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, p. 30, 1986.

Shapiro's Cyclic Sum Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Consider the sum

$$f_n(x_1, x_2, \ldots, x_n) = \frac{x_1}{x_2+x_3} + \frac{x_2}{x_3+x_4} + \ldots + \frac{x_{n-1}}{x_n+x_1} + \frac{x_n}{x_1+x_2}, \tag{1}$$

where the x_js are NONNEGATIVE and the DENOMINATORS are POSITIVE. Shapiro (1954) asked if

$$f_n(x_1, x_2, \ldots, x_n) \geq \tfrac{1}{2}n \tag{2}$$

for all n. It turns out (Mitrinovic *et al.* 1993) that this INEQUALITY is true for all EVEN $n \leq 12$ and ODD $n \leq 23$. Ranikin (1958) proved that for

$$f(n) = \inf_{x \geq 0} f_n(x_1, x_2, \ldots, x_n), \tag{3}$$

$$\lambda = \lim_{n\to\infty} \frac{f(n)}{n} = \inf_{n \geq 1} \frac{f(n)}{n} < \tfrac{1}{2} - 7 \times 10^{-8}. \tag{4}$$

λ can be computed by letting $\phi(x)$ be the CONVEX HULL of the functions

$$y_1 = e^{-x} \tag{5}$$

$$y_2 = \frac{2}{e^x + e^{x/2}}. \tag{6}$$

Then

$$\lambda = \tfrac{1}{2}\phi(0) = 0.4945668\ldots \tag{7}$$

(Drinfeljd 1971).

A modified sum was considered by Elbert (1973):

$$g_n(x_1, x_1, \ldots, x_n) = \frac{x_1+x_3}{x_1+x_2} + \frac{x_2+x_4}{x_2+x_3} + \ldots + \frac{x_{x-1}+x_1}{x_{n-1}+x_n} + \frac{x_n+x_2}{x_n+x_1}. \tag{8}$$

Consider

$$\mu = \lim_{n\to\infty} \frac{g(n)}{n}, \tag{9}$$

where

$$g(n) = \inf_{x \geq 0} g_n(x_1, x_2, \ldots, x_n), \tag{10}$$

and let $\psi(x)$ be the CONVEX HULL of

$$y_1 = \tfrac{1}{2}(1+e^x) \tag{11}$$

$$y_2 = \frac{1+e^x}{1+e^{x/2}}. \tag{12}$$

Then

$$\mu = \psi(0) = 0.978012\ldots. \tag{13}$$

see also CONVEX HULL

References

Drinfeljd, V. G. "A Cyclic Inequality." *Math. Notes. Acad. Sci. USSR* **9**, 68–71, 1971.

Elbert, A. "On a Cyclic Inequality." *Period. Math. Hungar.* **4**, 163–168, 1973.

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/shapiro/shapiro.html`.

Mitrinovic, D. S.; Pecaric, J. E.; and Fink, A. M. *Classical and New Inequalities in Analysis.* New York: Kluwer, 1993.

Sharing Problem

A problem also known as the POINTS PROBLEM or UNFINISHED GAME. Consider a tournament involving k players playing the same game repetitively. Each game has a single winner, and denote the number of games won by player i at some juncture w_i. The games are independent, and the probability of the ith player winning a game is p_i. The tournament is specified to continue until one player has won n games. If the tournament is discontinued before any player has won n games so that $w_i < n$ for $i = 1, \ldots, k$, how should the prize money be shared in order to distribute it proportionally to the players' chances of winning?

For player i, call the number of games left to win $r_i \equiv n - w_i > 0$ the "quota." For two players, let $p \equiv p_1$ and $q \equiv p_2 = 1 - p$ be the probabilities of winning a single game, and $a \equiv r_1 = n - w_1$ and $b \equiv r_2 = n - w_2$ be the number of games needed for each player to win the tournament. Then the stakes should be divided in the ratio $m : n$, where

$$m = p^a \left[1 + \frac{a}{1}q + \frac{a(a+1)}{2!}q^2 + \ldots + \frac{a(a+1)\cdots(a+b-2)}{(b-1)!}q^{b-1}\right] \quad (1)$$

$$n = q^b \left[1 + \frac{b}{1}p + \frac{b(b+1)}{2!}p^2 + \ldots + \frac{b(b+1)\cdots(b+a-2)}{(a-1)!}p^{a-1}\right] \quad (2)$$

(Kraitchik 1942).

If i players have equal probability of winning ("cell probability"), then the chance of player i winning for quotas $r_1, \ldots, r_k$ is

$$W_i = D_1^{k-1}(r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_k; r_i), \quad (3)$$

where D is the DIRICHLET INTEGRAL of type 2D. Similarly, the chance of player i losing is

$$L_i = C_1^{k-1}(r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_k; r_i), \quad (4)$$

where C is the DIRICHLET INTEGRAL of type 2C. If the cell quotas are not equal, the general Dirichlet integral $D_{\mathbf{a}}$ must be used, where

$$a_i = \frac{p_i}{1 - \sum_{i=1}^{k-1} p_i}. \quad (5)$$

If $r_i = r$ and $a_i = 1$, then W_i and L_i reduce to $1/k$ as they must. Let $P(r_1, \ldots, r_k)$ be the joint probability that the players would be RANKED in the order of the r_is in the argument list if the contest were completed. For $k = 3$,

$$P(r_1, r_2, r_3) = CD_1^{(1,1)}(r_1, r_2, r_3). \quad (6)$$

For $k = 4$ with quota vector $\mathbf{r} = (r_1, r_2, r_3, r_4)$ and $\Delta = p_2 + p_3 + p_4$,

$$P(\mathbf{r}) = \sum_{i=0}^{r_3-1} \sum_{j=0}^{r_4-1} \binom{r_2 - 1 + i + j}{r_2 - 1, i, j} \left(\frac{p_2}{\Delta}\right)^{r_2} \left(\frac{p_3}{\Delta}\right)^i \left(\frac{p_4}{\Delta}\right)^j \times C_{p_1/\Delta}^{(1)}(r_1, r_2 + i + j) D_{p_4/p_3}^{(1)}(r_4 - j, r_3 - i). \quad (7)$$

An expression for $k = 5$ is given by Sobel and Frankowski (1994, p. 838).

see also DIRICHLET INTEGRALS

References

Kraitchik, M. "The Unfinished Game." §6.1 in *Mathematical Recreations*. New York: W. W. Norton, pp. 117–118, 1942.

Sobel, M. and Frankowski, K. "The 500th Anniversary of the Sharing Problem (The Oldest Problem in the Theory of Probability)." *Amer. Math. Monthly* **101**, 833–847, 1994.

Sharkovsky's Theorem

see ŠARKOVSKII'S THEOREM

Sharpe's Differential Equation

A generalization of the BESSEL DIFFERENTIAL EQUATION for functions of order 0, given by

$$zy'' + y' + (z + A)y = 0.$$

Solutions are

$$y = e^{\pm iz} {}_1F_1\left(\tfrac{1}{2} \mp \tfrac{1}{2}iA; 1; \mp 2iz\right),$$

where ${}_1F_1(a; b; x)$ is a CONFLUENT HYPERGEOMETRIC FUNCTION.

see also BESSEL DIFFERENTIAL EQUATION, CONFLUENT HYPERGEOMETRIC FUNCTION

Sharpe Ratio

A risk-adjusted financial measure developed by Nobel Laureate William Sharpe. It uses a fund's standard deviation and excess return to determine the reward per unit of risk. The higher a fund's Sharpe ratio, the better the fund's "risk-adjusted" performance.

see also ALPHA, BETA

Sheaf (Geometry)

The set of all PLANES through a LINE.

see also LINE, PENCIL, PLANE

References

Woods, F. S. *Higher Geometry: An Introduction to Advanced Methods in Analytic Geometry*. New York: Dover, p. 12, 1961.

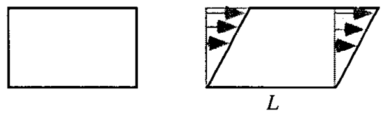
Sheaf (Topology)

A topological GADGET related to families of ABELIAN GROUPS and MAPS.

References

Iyanaga, S. and Kawada, Y. (Eds.). "Sheaves." §377 in *Encyclopedic Dictionary of Mathematics*. Cambridge, MA: MIT Press, p. 1171–1174, 1980.

Shear



A transformation in which all points along a given LINE L remain fixed while other points are shifted parallel to L by a distance proportional to their PERPENDICULAR distance from L. Shearing a plane figure does not change its AREA. The shear can also be generalized to 3-D, in which PLANES are translated instead of lines.

Shear Matrix

The shear matrix $\mathbf{e}_{ij}^{s}$ is obtained from the IDENTITY MATRIX by inserting s at (i,j), e.g.,

$$\mathbf{e}_{12}^{s}=\begin{bmatrix}1 & s & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

see also ELEMENTARY MATRIX

Shephard's Problem

Measurements of a centered convex body in Euclidean n-space (for $n \geq 3$) show that its brightness function (the volume of each projection) is smaller than that of another such body. Is it true that its VOLUME is also smaller? C. M. Petty and R. Schneider showed in 1967 that the answer is yes if the body with the larger brightness function is a projection body, but no in general for every n.

References

Gardner, R. J. "Geometric Tomography." *Not. Amer. Math. Soc.* **42**, 422–429, 1995.

Sheppard's Correction

A correction which must be applied to the MOMENTS computed from NORMALLY DISTRIBUTED data which have been binned. The corrected versions of the second, third, and fourth moments are

$$\mu_2 = {\mu_2}^{(0)} - \tfrac{1}{12}c^2 \tag{1}$$

$$\mu_3 = {\mu_3}^{(0)} \tag{2}$$

$$\mu_4 = {\mu_4}^{(0)} - \tfrac{1}{2}{\mu_2}^{(0)} + \tfrac{7}{240}c^2, \tag{3}$$

where c is the CLASS INTERVAL. If κ_r' is the rth CUMULANT of an ungrouped distribution and κ_r the rth CUMULANT of the grouped distribution with CLASS INTERVAL c, the corrected cumulants (under rather restrictive conditions) are

$$\kappa_r' = \begin{cases} \kappa_r & \text{for } r \text{ odd} \\ \kappa_r - \frac{B_r}{r}c^r & \text{for } r \text{ even,} \end{cases} \tag{4}$$

where B_r is the rth BERNOULLI NUMBER, giving

$$\kappa_1' = \kappa_1 \tag{5}$$

$$\kappa_2' = \kappa_2 - \tfrac{1}{12}c^2 \tag{6}$$

$$\kappa_3' = \kappa_3 \tag{7}$$

$$\kappa_4' = \kappa_4 + \tfrac{1}{120}c^4 \tag{8}$$

$$\kappa_5' = \kappa_5 \tag{9}$$

$$\kappa_6' = \kappa_6 - \tfrac{1}{252}c^6. \tag{10}$$

For a proof, see Kendall *et al.* (1987).

References

Kendall, M. G.; Stuart, A.; and Ord, J. K. *Kendall's Advanced Theory of Statistics, Vol. 1: Distribution Theory, 6th ed.* New York: Oxford University Press, 1987.

Kenney, J. F. and Keeping, E. S. "Sheppard's Correction." §4.12 in *Mathematics of Statistics, Pt. 2, 2nd ed.* Princeton, NJ: Van Nostrand, pp. 80–82, 1951.

Sherman-Morrison Formula

A formula which allows the new MATRIX to be computed for a small change to a MATRIX A. If the change can be written in the form

$$\mathbf{u}\otimes\mathbf{v}$$

for two vectors $\mathbf{u}$ and $\mathbf{v}$, then the Sherman-Morrison formula is

$$(\mathsf{A}+\mathbf{u}\otimes\mathbf{v})^{-1}=\mathsf{A}^{-1}-\frac{(\mathsf{A}^{-1}\mathbf{u})\otimes(\mathbf{v}\cdot\mathsf{A}^{-1})}{1+\lambda},$$

where

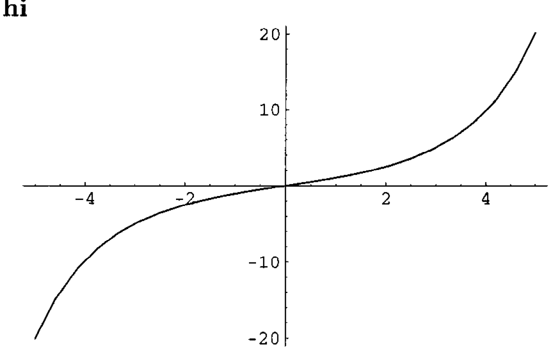
$$\lambda\equiv\mathbf{v}\cdot\mathsf{A}^{-1}\mathbf{u}.$$

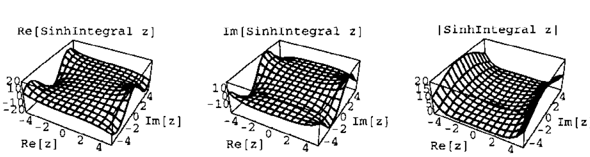
see also WOODBURY FORMULA

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Sherman-Morrison Formula." In *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 65–67, 1992.

Shi





$$\mathrm{Shi}(z) = \int_0^z \frac{\sinh t}{t}\,dt.$$

The function is given by the *Mathematica*® (Wolfram Research, Champaign, IL) command `SinhIntegral[z]`.

see also CHI, COSINE INTEGRAL, SINE INTEGRAL

References
Abramowitz, M. and Stegun, C. A. (Eds.). "Sine and Cosine Integrals." §5.2 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 231–233, 1972.

Shift

A TRANSLATION without ROTATION or distortion.

see also DILATION, EXPANSION, ROTATION, TRANSLATION, TWIRL

Shift Property

see DELTA FUNCTION

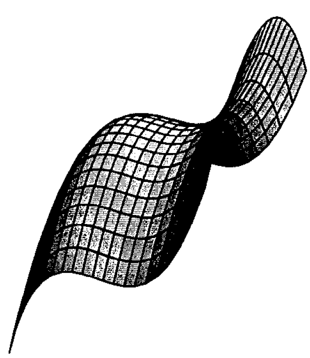
Shimura-Taniyama Conjecture

see TANIYAMA-SHIMURA CONJECTURE

Shimura-Taniyama-Weil Conjecture

see TANIYAMA-SHIMURA CONJECTURE

Shoe Surface



A surface given by the parametric equations

$$\begin{aligned} x(u,v) &= u \\ y(u,v) &= v \\ z(u,v) &= \tfrac{1}{3}u^3 - \tfrac{1}{2}v^2. \end{aligned}$$

References
Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 634, 1993.

Shoemaker's Knife

see ARBELOS

Shortening

A KNOT used to shorten a long rope.

see also BEND (KNOT)

References
Owen, P. *Knots.* Philadelphia, PA: Courage, p. 65, 1993.

Shuffle

The randomization of a deck of CARDS by repeated interleaving. More generally, a shuffle is a rearrangement of the elements in an ordered list. Shuffling by exactly interleaving two halves of a deck is called a RIFFLE SHUFFLE. Normal shuffling leaves gaps of different lengths between the two layers of cards and so randomizes the order of the cards.

A deck of 52 CARDS must be shuffled seven times for it to be randomized (Aldous and Diaconis 1986, Bayer and Diaconis 1992). This is intermediate between too few shuffles and the decreasing effectiveness of many shuffles. One of Bayer and Diaconis's randomness CRITERIA, however, gives $3\lg k/2$ shuffles for a k-card deck, yielding 11–12 shuffles for 52 CARDS. Keller (1995) shows that roughly $\ln k$ shuffles are needed just to randomize the bottom card.

see also BAYS' SHUFFLE, CARDS, FARO SHUFFLE, MONGE'S SHUFFLE, RIFFLE SHUFFLE

References
Aldous, D. and Diaconis, P. "Shuffling Cards and Stopping Times." *Amer. Math. Monthly* **93**, 333–348, 1986.
Bayer, D. and Diaconis, P. "Trailing the Dovetail Shuffle to Its Lair." *Ann. Appl. Probability* **2**, 294–313, 1992.
Keller, J. B. "How Many Shuffles to Mix a Deck?" *SIAM Review* **37**, 88–89, 1995.
Morris, S. B. "Practitioner's Commentary: Card Shuffling." *UMAP J.* **15**, 333–338, 1994.
Rosenthal, J. W. "Card Shuffling." *Math. Mag.* **54**, 64–67, 1981.

Siamese Dodecahedron

see SNUB DISPHENOID

Siamese Method

A method for constructing MAGIC SQUARES of ODD order, also called DE LA LOUBERE'S METHOD.

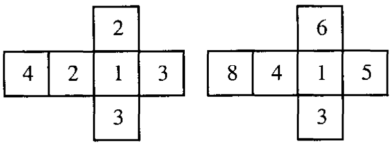
see also MAGIC SQUARE

Sibling

Two nodes connected to the same node in a ROOTED TREE are called siblings.

see also CHILD, ROOTED TREE

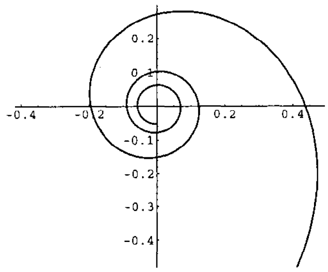
Sicherman Dice



A pair of DICE which have the same ODDS for throwing every number as a normal pair of 6-sided DICE. They are the only such alternate arrangement.

see also DICE, EFRON'S DICE

Sici Spiral



The spiral

$$x = c \operatorname{ci} t$$
$$y = c(\operatorname{si} t - \tfrac{1}{2}\pi),$$

where $\operatorname{ci}(t)$ and $\operatorname{si}(t)$ are the COSINE INTEGRAL and SINE INTEGRAL and c is a constant.

see also COSINE INTEGRAL, SINE INTEGRAL, SPIRAL

References
von Seggern, D. *CRC Standard Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 204 and 270, 1993.

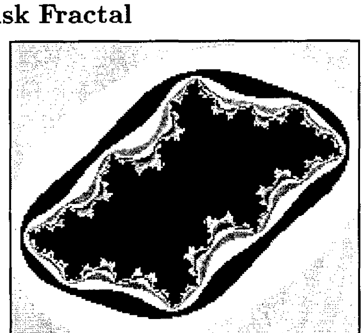
Side

The edge of a POLYGON and face of a POLYHEDRON are sometimes called sides.

Sidon Sequence

see $B2$-SEQUENCE

Siegel Disk Fractal



A JULIA SET with $c = -0.390541 - 0.586788i$. The FRACTAL somewhat resembles the better known MANDELBROT SET.

see also DOUADY'S RABBIT FRACTAL, JULIA SET, MANDELBROT SET, SAN MARCO FRACTAL

References
Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, p. 176, 1991.

Siegel Modular Function

A Γ_n-invariant meromorphic function on the space of all $n \times n$ complex symmetric matrices with POSITIVE IMAGINARY PART. In 1984, H. Umemura expressed the ROOTS of an arbitrary POLYNOMIAL in terms of elliptic Siegel functions.

References
Iyanaga, S. and Kawada, Y. (Eds.). "Siegel Modular Functions." §34F in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, pp. 131–132, 1980.

Siegel's Paradox

If a fixed FRACTION x of a given amount of money P is lost, and then the same FRACTION x of the remaining amount is gained, the result is less than the original and equal to the final amount if a FRACTION x is first gained, then lost. This can easily be seen from the fact that

$$[P(1-x)](1+x) = P(1-x^2) < P$$
$$[P(1+x)](1-x) = P(1-x^2) < P.$$

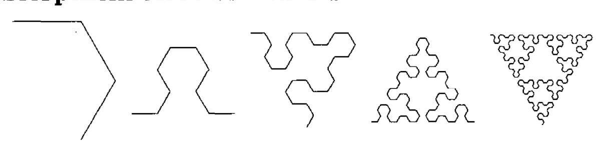
Siegel's Theorem

An ELLIPTIC CURVE can have only a finite number of points with INTEGER coordinates.

see also ELLIPTIC CURVE

References
Davenport, H. "Siegel's Theorem." Ch. 21 in *Multiplicative Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 126–125, 1980.

Sierpiński Arrowhead Curve

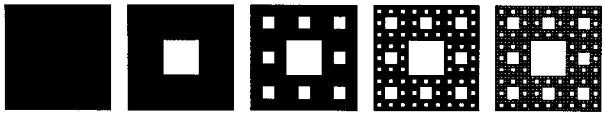


A FRACTAL which can be written as a LINDENMAYER SYSTEM with initial string "YF", STRING REWRITING rules "X" -> "YF+XF+Y", "Y" -> "XF-YF-X", and angle 60°.

see also DRAGON CURVE, HILBERT CURVE, KOCH SNOWFLAKE, LINDENMAYER SYSTEM, PEANO CURVE, PEANO-GOSPER CURVE, SIERPIŃSKI CURVE, SIERPIŃSKI SIEVE

References
Dickau, R. M. "Two-Dimensional L-Systems." `http://forum.swarthmore.edu/advanced/robertd/lsys2d.html`.

Sierpiński Carpet



A FRACTAL which is constructed analogously to the SIERPIŃSKI SIEVE, but using squares instead of triangles. Let N_n be the number of black boxes, L_n the length of a side of a white box, and A_n the fractional AREA of black boxes after the nth iteration. Then

$$N_n = 8^n \tag{1}$$
$$L_n = (\tfrac{1}{3})^n = 3^{-n} \tag{2}$$
$$A_n = {L_n}^2 N_n = (\tfrac{8}{9})^n. \tag{3}$$

The CAPACITY DIMENSION is therefore

$$d_{\rm cap} = -\lim_{n\to\infty} \frac{\ln N_n}{\ln L_n} = -\lim_{n\to\infty} \frac{\ln(8^n)}{\ln(3^{-n})} = \frac{\ln 8}{\ln 3} = \frac{3\ln 2}{\ln 3} = 1.892789261\ldots. \tag{4}$$

see also MENGER SPONGE, SIERPIŃSKI SIEVE

References
Dickau, R. M. "The Sierpinski Carpet." http://forum.swarthmore.edu/advanced/robertd/carpet.html.
Peitgen, H.-O.; Jürgens, H.; and Saupe, D. *Chaos and Fractals: New Frontiers of Science.* New York: Springer-Verlag, pp. 112–121, 1992.
Weisstein, E. W. "Fractals." http://www.astro.virginia.edu/~eww6n/math/notebooks/Fractal.m.

Sierpiński's Composite Number Theorem

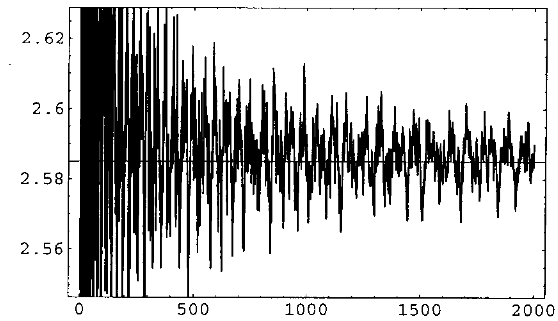
There exist infinitely many ODD INTEGERS k such that $k\cdot 2^n+1$ is COMPOSITE for every $n \geq 1$. Numbers k with this property are called SIERPIŃSKI NUMBERS OF THE SECOND KIND, and analogous numbers with the plus sign replaced by a minus are called RIESEL NUMBERS. It is conjectured that the smallest SIERPIŃSKI NUMBER OF THE SECOND KIND is $k = 78{,}557$ and the smallest RIESEL NUMBER is $k = 509{,}203$.

see also CUNNINGHAM NUMBER, SIERPIŃSKI NUMBER OF THE SECOND KIND

References
Buell, D. A. and Young, J. "Some Large Primes and the Sierpiński Problem." SRC Tech. Rep. 88004, Supercomputing Research Center, Lanham, MD, 1988.
Jaeschke, G. "On the Smallest k such that $k \cdot 2^N + 1$ are Composite." *Math. Comput.* **40**, 381–384, 1983.
Jaeschke, G. Corrigendum to "On the Smallest k such that $k \cdot 2^N + 1$ are Composite." *Math. Comput.* **45**, 637, 1985.
Keller, W. "Factors of Fermat Numbers and Large Primes of the Form $k \cdot 2^n + 1$." *Math. Comput.* **41**, 661–673, 1983.
Keller, W. "Factors of Fermat Numbers and Large Primes of the Form $k \cdot 2^n + 1$, II." In prep.
Ribenboim, P. *The New Book of Prime Number Records.* New York: Springer-Verlag, pp. 357–359, 1996.
Riesel, H. "Några stora primtal." *Elementa* **39**, 258–260, 1956.
Sierpiński, W. "Sur un problème concernant les nombres $k \cdot 2^n + 1$." *Elem. d. Math.* **15**, 73–74, 1960.

see also COMPOSITE NUMBER, SIERPIŃSKI NUMBERS OF THE SECOND KIND, SIERPIŃSKI'S PRIME SEQUENCE THEOREM

Sierpiński Constant



Let $r_k(n)$ denote the number of representations of n by k squares, then the SUMMATORY FUNCTION of $r_2(k)/k$ has the ASYMPTOTIC expansion

$$\sum_{k=1}^n \frac{r_2(k)}{k} = K + \pi \ln n + \mathcal{O}(n^{-1/2}),$$

where $K = 2.5849817596$ is the Sierpiński constant. The above plot shows

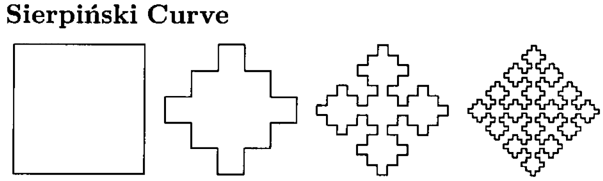
$$\left[\sum_{k=1}^n \frac{r_2(k)}{k}\right] - \pi \ln n,$$

with the value of K indicated as the solid horizontal line.

see also $r_k(n)$

References
Sierpiński, W. *Oeuvres Choiseies, Tome 1.* Editions Scientifiques de Pologne, 1974.

Sierpiński Curve



There are several FRACTAL curves associated with Sierpiński. The above curve is one example, and the SIERPIŃSKI ARROWHEAD CURVE is another. The limit of the curve illustrated above has AREA

$$A = \tfrac{5}{12}.$$

The AREA for a related curve illustrated by Cundy and Rollett (1989) is

$$A = \tfrac{1}{3}(7 - 4\sqrt{2}).$$

see also EXTERIOR SNOWFLAKE, GOSPER ISLAND, HILBERT CURVE, KOCH ANTISNOWFLAKE, KOCH SNOWFLAKE, PEANO CURVE, PEANO-GOSPER CURVE, SIERPIŃSKI ARROWHEAD CURVE

References
Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., pp. 67–68, 1989.
Dickau, R. M. "Two-Dimensional L-Systems." `http://forum.swarthmore.edu/advanced/robertd/lsys2d.html`.
Gardner, M. *Penrose Tiles and Trapdoor Ciphers... and the Return of Dr. Matrix, reissue ed.* New York: W. H. Freeman, p. 34, 1989.
Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, p. 207, 1991.

Sierpiński Gasket

see SIERPIŃSKI SIEVE

Sierpiński-Menger Sponge

see MENGER SPONGE

Sierpiński Number of the First Kind

Numbers of the form $S_n \equiv n^n + 1$. The first few are 2, 5, 28, 257, 3126, 46657, 823544, 16777217, ... (Sloane's A014566). Sierpiński proved that if S_n is PRIME with $n \geq 2$, then $S_n = F_{m+2^m}$, where F_m is a FERMAT NUMBER with $m \geq 0$. The first few such numbers are $F_1 = 5$, $F_3 = 257$, F_6, F_{11}, F_{20}, and F_{37}. Of these, 5 and 257 are PRIME, and the first unknown case is $F_{37} > 10^{3\times 10^{10}}$.

see also CULLEN NUMBER, CUNNINGHAM NUMBER, FERMAT NUMBER, WOODALL NUMBER

References
Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, p. 155, 1979.
Ribenboim, P. *The Book of Prime Number Records, 2nd ed.* New York: Springer-Verlag, p. 74, 1989.
Sloane, N. J. A. Sequence A014566 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Sierpiński Number of the Second Kind

A number k satisfying SIERPIŃSKI'S COMPOSITE NUMBER THEOREM, i.e., such that $k \cdot 2^n + 1$ is COMPOSITE for every $n \geq 1$. The smallest known is $k = 78,557$, but there remain 35 smaller candidates (the smallest of which is 4847) which are known to generate only composite numbers for $n \leq 18,000$ or more (Ribenboim 1996, p. 358).

Let $a(k)$ be smallest n for which $(2k-1) \cdot 2^n + 1$ is PRIME, then the first few values are 0, 1, 1, 2, 1, 1, 2, 1, 3, 6, 1, 1, 2, 2, 1, 8, 1, 1, 2, 1, 1, 2, 2, 583, ... (Sloane's A046067). The second smallest n are given by 1, 2, 3, 4, 2, 3, 8, 2, 15, 10, 4, 9, 4, 4, 3, 60, 6, 3, 4, 2, 11, 6, 9, 1483, ... (Sloane's A046068). Quite large n can be required to obtain the first prime even for small k. For example, the smallest prime of the form $383 \cdot 2^n + 1$ is $383 \cdot 2^{6393} + 1$. There are an infinite number of Sierpiński numbers which are PRIME.

The smallest odd k such that $k + 2^n$ is COMPOSITE for all $n < k$ are 773, 2131, 2491, 4471, 5101, ...

see also MERSENNE NUMBER, RIESEL NUMBER, SIERPIŃSKI'S COMPOSITE NUMBER THEOREM

References
Buell, D. A. and Young, J. "Some Large Primes and the Sierpiński Problem." SRC Tech. Rep. 88004, Supercomputing Research Center, Lanham, MD, 1988.
Jaeschke, G. "On the Smallest k such that $k \cdot 2^N + 1$ are Composite." *Math. Comput.* **40**, 381–384, 1983.
Jaeschke, G. Corrigendum to "On the Smallest k such that $k \cdot 2^N + 1$ are Composite." *Math. Comput.* **45**, 637, 1985.
Keller, W. "Factors of Fermat Numbers and Large Primes of the Form $k \cdot 2^n + 1$." *Math. Comput.* **41**, 661–673, 1983.
Keller, W. "Factors of Fermat Numbers and Large Primes of the Form $k \cdot 2^n + 1$, II." In prep.
Ribenboim, P. *The New Book of Prime Number Records.* New York: Springer-Verlag, pp. 357–359, 1996.
Sierpiński, W. "Sur un problème concernant les nombres $k \cdot 2^n + 1$." *Elem. d. Math.* **15**, 73–74, 1960.
Sloane, N. J. A. Sequence A046067 in "An On-Line Version of the Encyclopedia of Integer Sequences."046068

Sierpiński's Prime Sequence Theorem

For any M, there exists a t' such that the sequence

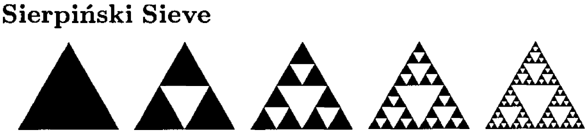
$$n^2 + t',$$

where $n = 1, 2, \ldots$ contains at least M PRIMES.

see also DIRICHLET'S THEOREM, FERMAT $4n+1$ THEOREM, SIERPIŃSKI'S COMPOSITE NUMBER THEOREM

References
Abel, U. and Siebert, H. "Sequences with Large Numbers of Prime Values." *Amer. Math. Monthly* **100**, 167–169, 1993.
Ageev, A. A. "Sierpinski's Theorem is Deducible from Euler and Dirichlet." *Amer. Math. Monthly* **101**, 659–660, 1994.
Forman, R. "Sequences with Many Primes." *Amer. Math. Monthly* **99**, 548–557, 1992.
Garrison, B. "Polynomials with Large Numbers of Prime Values." *Amer. Math. Monthly* **97**, 316–317, 1990.
Sierpiński, W. "Les binômes $x^2 + n$ et les nombres premiers." *Bull. Soc. Roy. Sci. Liege* **33**, 259–260, 1964.

Sierpiński Sieve



A FRACTAL described by Sierpiński in 1915. It is also called the SIERPIŃSKI GASKET or SIERPIŃSKI TRIANGLE. The curve can be written as a LINDENMAYER SYSTEM with initial string "FXF--FF--FF", STRING REWRITING rules "F" -> "FF", "X"->"--FXF++FXF++FXF--", and angle 60°.

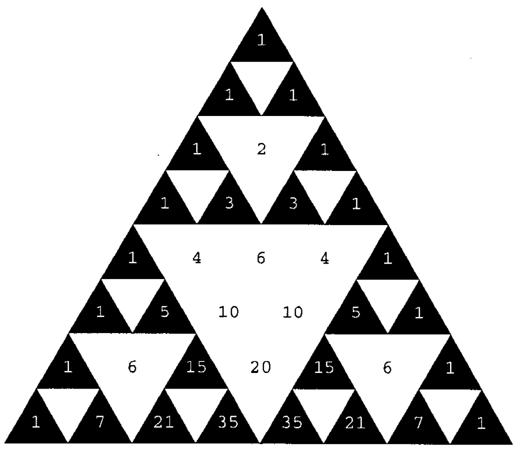
Let N_n be the number of black triangles after iteration n, L_n the length of a side of a triangle, and A_n the fractional AREA which is black after the nth iteration. Then

$$N_n = 3^n \tag{1}$$
$$L_n = (\tfrac{1}{2})^n = 2^{-n} \tag{2}$$
$$A_n = {L_n}^2 N_n = (\tfrac{3}{4})^n. \tag{3}$$

The CAPACITY DIMENSION is therefore

$$d_{\mathrm{cap}} = -\lim_{n\to\infty} \frac{\ln N_n}{\ln L_n} = -\lim_{n\to\infty} \frac{\ln(3^n)}{\ln(2^{-n})} = \frac{\ln 3}{\ln 2}$$
$$= 1.584962501\ldots. \quad (4)$$

In PASCAL'S TRIANGLE, coloring all ODD numbers black and EVEN numbers white produces a Sierpiński sieve.



see also LINDENMAYER SYSTEM, SIERPIŃSKI ARROWHEAD CURVE, SIERPIŃSKI CARPET, TETRIX

References
Crownover, R. M. *Introduction to Fractals and Chaos.* Sudbury, MA: Jones & Bartlett, 1995.
Dickau, R. M. "Two-Dimensional L-Systems." http://forum.swarthmore.edu/advanced/robertd/lsys2d.html.
Dickau, R. M. "Typeset Fractals." *Mathematica J.* **7**, 15, 1997.
Dickau, R. "Sierpinski-Menger Sponge Code and Graphic." http://www.mathsource.com/cgi-bin/MathSource/Applications/Graphics/0206-110.
Lauwerier, H. *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, pp. 13–14, 1991.
Peitgen, H.-O.; Jürgens, H.; and Saupe, D. *Chaos and Fractals: New Frontiers of Science.* New York: Springer-Verlag, pp. 78–88, 1992.
Peitgen, H.-O. and Saupe, D. (Eds.). *The Science of Fractal Images.* New York: Springer-Verlag, p. 282, 1988.
Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, pp. 108 and 151–153, 1991.
Wang, P. "Renderings." http://www.ugcs.caltech.edu/~peterw/portfolio/renderings/.
Weisstein, E. W. "Fractals." http://www.astro.virginia.edu/~eww6n/math/notebooks/Fractal.m.

Sierpiński Sponge

see TETRIX

Sierpiński Tetrahedron

see TETRIX

Sierpiński's Theorem

see SIERPIŃSKI'S COMPOSITE NUMBER THEOREM, SIERPIŃSKI'S PRIME SEQUENCE THEOREM

Sierpiński Triangle

see SIERPIŃSKI SIEVE

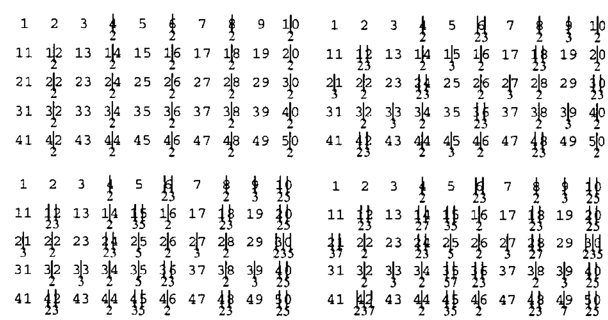
Sieve

A process of successively crossing out members of a list according to a set of rules such that only some remain. The best known sieve is the ERATOSTHENES SIEVE for generating PRIME NUMBERS. In fact, numbers generated by sieves seem to share a surprisingly large number of properties with the PRIME NUMBERS.

see also HAPPY NUMBER, NUMBER FIELD SIEVE FACTORIZATION METHOD, PRIME NUMBER, QUADRATIC SIEVE FACTORIZATION METHOD, SIERPIŃSKI SIEVE, SIEVE OF ERATOSTHENES, WALLIS SIEVE

References
Halberstam, H. and Richert, H.-E. *Sieve Methods.* New York: Academic Press, 1974.
Pomerance, C. "A Tale of Two Sieves." *Not. Amer. Math. Soc.* **43**, 1473–1485, 1996.

Sieve of Eratosthenes



An ALGORITHM for making tables of PRIMES. Sequentially write down the INTEGERS from 2 to the highest number n you wish to include in the table. Cross out all numbers > 2 which are divisible by 2 (every second number). Find the smallest remaining number > 2. It is 3. So cross out all numbers > 3 which are divisible by 3 (every third number). Find the smallest remaining number > 3. It is 5. So cross out all numbers > 5 which are divisible by 5 (every fifth number).

Continue until you have crossed out all numbers divisible by $\lfloor\sqrt{n}\rfloor$, where $\lfloor x\rfloor$ is the FLOOR FUNCTION. The numbers remaining are PRIME. This procedure is illustrated in the above diagram which sieves up to 50, and therefore crosses out PRIMES up to $\lfloor\sqrt{50}\rfloor = 7$. If the procedure is then continued up to n, then the number of cross-outs gives the number of distinct PRIME factors of each number.

References
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 127–130, 1996.
Pappas, T. *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, pp. 100–101, 1989.
Ribenboim, P. *The New Book of Prime Number Records.* New York: Springer-Verlag, pp. 20–21, 1996.

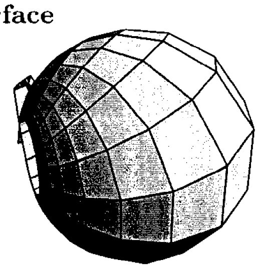
Sievert Integral

The integral

$$\int_0^\theta e^{-x\sec\phi}\,d\phi.$$

References
Abramowitz, M. and Stegun, C. A. (Eds.). "Sievert Integral." §27.4 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 1000–1001, 1972.

Sievert's Surface



A special case of ENNEPER'S SURFACES which can be given parametrically by

$$x = r\cos\phi \tag{1}$$

$$y = r\sin\phi \tag{2}$$

$$z = \frac{\ln[\tan(\frac{1}{2}v)] + a(C+1)\cos v}{\sqrt{C}}, \tag{3}$$

where

$$\phi \equiv -\frac{u}{\sqrt{C+1}} + \tan^{-1}(\tan u\sqrt{C+1}\,) \tag{4}$$

$$a \equiv \frac{2}{C+1-C\sin^2 v\cos^2 u} \tag{5}$$

$$r \equiv \frac{a\sqrt{(C+1)(1+C\sin^2 u)}\,\sin v}{\sqrt{C}}, \tag{6}$$

with $|u| < \pi/2$ and $0 < v < \pi$ (Reckziegel 1986).

see also ENNEPER'S SURFACES, KUEN SURFACE, REMBS' SURFACES

References
Fischer, G. (Ed.). Plate 87 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, p. 83, 1986.
Reckziegel, H. "Sievert's Surface." §3.4.4.3 in *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, pp. 38–39, 1986.
Sievert, H. *Über die Zentralflächen der Enneperschen Flačhen konstanten Krümmungsmaßes.* Dissertation, Tübingen, 1886.

Sifting Property

The property

$$\int f(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})\,d\mathbf{y} = f(\mathbf{x})$$

obeyed by the DELTA FUNCTION $\delta(\mathbf{x})$.

see also DELTA FUNCTION

Sigma Algebra

Let X be a SET. Then a σ-algebra F is a nonempty collection of SUBSETS of X such that the following hold:

1. The EMPTY SET is in F.
2. If A is in F, then so is the complement of A.
3. If A_n is a SEQUENCE of elements of F, then the UNION of the A_ns is in F.

If S is any collection of subsets of X, then we can always find a σ-algebra containing S, namely the POWER SET of X. By taking the INTERSECTION of all σ-algebras containing S, we obtain the smallest such σ-algebra. We call the smallest σ-algebra containing S the σ-algebra generated by S.

see also BOREL SIGMA ALGEBRA, BOREL SPACE, MEASURABLE SET, MEASURABLE SPACE, MEASURE ALGEBRA, STANDARD SPACE

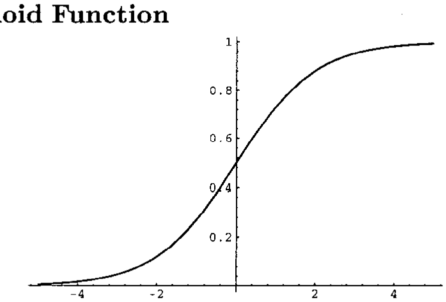
Sigma Function

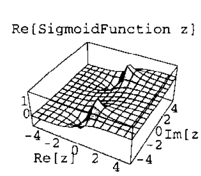
see DIVISOR FUNCTION

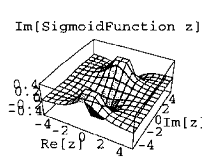
Sigmoid Curve

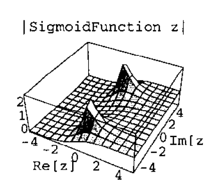
see SIGMOID FUNCTION

Sigmoid Function









The function

$$y = \frac{1}{1+e^{-x}}$$

which is the solution to the ORDINARY DIFFERENTIAL EQUATION

$$\frac{dy}{dx} = y(1-y).$$

It has an inflection point at $x = 0$, where

$$y''(x) = -\frac{e^x(e^x-1)}{(e^x+1)^3} = 0.$$

see also EXPONENTIAL FUNCTION, EXPONENTIAL RAMP

References
von Seggern, D. *CRC Standard Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 124, 1993.

Sign

The sign of a number, also called SGN, is -1 for a NEGATIVE number (i.e., one with a MINUS SIGN "$-$"), 0 for the number ZERO, or $+1$ for a POSITIVE number (i.e., one with a PLUS SIGN "$+$").

see also ABSOLUTE VALUE, MINUS SIGN, NEGATIVE, PLUS SIGN, POSITIVE, SGN, ZERO

Signalizer Functor Theorem

$$\Theta(G; A) = \langle \theta(a) : a \in A - 1 \rangle$$

is an A-invariant solvable p'-subgroup of G.

Signature (Knot)

The signature $s(K)$ of a KNOT K can be defined using the SKEIN RELATIONSHIP

$$s(\text{unknot}) = 0$$

$$s(K_+) - s(K_-) \in \{0, 2\},$$

and

$$4|s(K) \leftrightarrow \nabla(K)(2i) > 0,$$

where $\nabla(K)$ is the ALEXANDER-CONWAY POLYNOMIAL and $\nabla(K)(2i)$ is an ODD NUMBER.

Many UNKNOTTING NUMBERS can be determined using a knot's signature.

see also SKEIN RELATIONSHIP, UNKNOTTING NUMBER

References

Gordon, C. McA.; Litherland, R. A.; and Murasugi, K. "Signatures of Covering Links." *Canad. J. Math.* **33**, 381–394, 1981.

Murasugi, K. "On the Signature of Links." *Topology* **9**, 283–298, 1970.

Murasugi, K. "Signatures and Alexander Polynomials of Two-Bridge Knots." *C. R. Math. Rep. Acad. Sci. Canada* **5**, 133–136, 1983.

Murasugi, K. "On the Signature of a Graph." *C. R. Math. Rep. Acad. Sci. Canada* **10**, 107–111, 1988.

Murasugi, K. "On Invariants of Graphs with Applications to Knot Theory." *Trans. Amer. Math. Soc.* **314**, 1–49, 1989.

Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, 1976.

Stoimenow, A. "Signatures." `http://www.informatik.hu-berlin.de/~stoimeno/ptab/sig10.html`.

Signature (Quadratic Form)

The signature of the QUADRATIC FORM

$$Q = {y_1}^2 + {y_2}^2 + \ldots + {y_p}^2 - {y_{p+1}}^2 - {y_{p+2}}^2 - \ldots - {y_r}^2$$

is the number s of POSITIVE squared terms in the reduced form. (The signature is sometimes defined as $2s - r$.)

see also p-SIGNATURE, RANK (QUADRATIC FORM), SYLVESTER'S INERTIA LAW, SYLVESTER'S SIGNATURE

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1105, 1979.

Signature (Recurrence Relation)

Let a sequence be defined by

$$\begin{aligned} A_{-1} &= s \\ A_0 &= 3 \\ A_1 &= r \\ A_n &= rA_{n-1} - sA_{n-2} + A_{n-3}. \end{aligned}$$

Also define the associated POLYNOMIAL

$$f(x) = x^3 - rx^2 + sx + 1,$$

and let Δ be its discriminant. The PERRIN SEQUENCE is a special case corresponding to $A_n(0, -1)$. The signature mod m of an INTEGER n with respect to the sequence $A_k(r, s)$ is then defined as the 6-tuple (A_{-n-1}, A_{-n}, A_{-n+1}, A_{n-1}, A_n, A_{n+1}) (mod m).

1. An INTEGER n has an S-signature if its signature (mod n) is (A_{-2}, A_{-1}, A_0, A_1, A_2).
2. An INTEGER n has a Q-signature if its signature (mod n) is CONGRUENT to (A, s, B, B, r, C) where, for some INTEGER a with $f(a) \equiv 0 \pmod{n}$, $A \equiv a^{-2} + 2a$, $B \equiv -ra^2 + (r^2 - s)a$, and $C \equiv a^2 + 2a^{-1}$.
3. An INTEGER n has an I-signature if its signature (mod n) is CONGRUENT to (r, s, D', D, r, s), where $D' + D \equiv rs - 3$ and $(D' - D)^2 \equiv \Delta$.

see also PERRIN PSEUDOPRIME

References

Adams, W. and Shanks, D. "Strong Primality Tests that Are Not Sufficient." *Math. Comput.* **39**, 255–300, 1982.

Grantham, J. "Frobenius Pseudoprimes." `http://www.clark.net/pub/grantham/pseudo/pseudo.ps`

Signature Sequence

Let θ be an IRRATIONAL NUMBER, define $S(\theta) = \{c + d\theta : c, d \in \mathbb{N}\}$, and let $c_n(\theta) + d_n\theta(\theta)$ be the sequence obtained by arranging the elements of $S(\theta)$ in increasing order. A sequence x is said to be a signature sequence if there EXISTS a POSITIVE IRRATIONAL NUMBER θ such that $x = \{c_n(\theta)\}$, and x is called the signature of θ.

The signature of an IRRATIONAL NUMBER is a FRACTAL SEQUENCE. Also, if x is a signature sequence, then the LOWER-TRIMMED SUBSEQUENCE is $V(x) = x$.

References

Kimberling, C. "Fractal Sequences and Interspersions." *Ars Combin.* **45**, 157–168, 1997.

Signed Deviation

The signed deviation is defined by

$$\Delta u_i \equiv (u_i - \bar{u}),$$

so the average deviation is

$$\overline{\Delta u} = \overline{u_i - \bar{u}} = \overline{u_i} - \bar{u} = 0.$$

see also ABSOLUTE DEVIATION, DEVIATION, DISPERSION (STATISTICS), MEAN DEVIATION, QUARTILE DEVIATION, STANDARD DEVIATION, VARIANCE

Significance

Let $\delta \equiv z \leq z_{\text{observed}}$. A value $0 \leq \alpha \leq 1$ such that $P(\delta) \leq \alpha$ is considered "significant" (i.e., is not simply due to chance) is known as an ALPHA VALUE. The PROBABILITY that a variate would assume a value greater than or equal to the observed value strictly by chance, $P(\delta)$, is known as a P-VALUE.

Depending on the type of data and conventional practices of a given field of study, a variety of different alpha values may be used. One commonly used terminology takes $P(\delta) \geq 5\%$ as "not significant," $1\% < P(\delta) < 5\%$, as "significant" (sometimes denoted *), and $P(\delta) < 1\%$ as "highly significant" (sometimes denoted **). Some authors use the term "almost significant" to refer to $5\% < P(\delta) < 10\%$, although this practice is not recommended.

see also ALPHA VALUE, CONFIDENCE INTERVAL, P-VALUE, PROBABLE ERROR, SIGNIFICANCE TEST, STATISTICAL TEST

Significance Test

A test for determining the probability that a given result could not have occurred by chance (its SIGNIFICANCE).

see also SIGNIFICANCE, STATISTICAL TEST

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 491–492, 1987.

Significant Digits

When a number is expressed in SCIENTIFIC NOTATION, the number of significant figures is the number of DIGITS needed to express the number to within the uncertainty of measurement. For example, if a quantity had been measured to be 1.234 ± 0.002, four figures would be significant. No more figures should be given than are allowed by the uncertainty. For example, a quantity written as 1.234 ± 0.1 is incorrect; it should really be written as 1.2 ± 0.1.

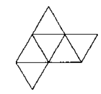
The number of significant figures of a MULTIPLICATION or DIVISION of two or more quantities is equal to the smallest number of significant figures for the quantities involved. For ADDITION or MULTIPLICATION, the number of significant figures is determined with the smallest significant figure of all the quantities involved. For example, the sum $10.234 + 5.2 + 100.3234$ is 115.7574, but should be written 115.8 (with rounding), since the quantity 5.2 is significant only to ± 0.1.

see also NINT, ROUND, TRUNCATE

Significant Figures

see SIGNIFICANT DIGITS

Signpost



A 6-POLYIAMOND.

References

Golomb, S. W. *Polyominoes: Puzzles, Patterns, Problems, and Packings, 2nd ed.* Princeton, NJ: Princeton University Press, p. 92, 1994.

Signum

see SGN

Silver Constant

The REAL ROOT of the equation

$$x^3 - 5x^2 + 6x - 1 = 0,$$

which is $3.2469\ldots$. It is the seventh BERAHA CONSTANT.

see also BERAHA CONSTANTS

References

Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, pp. 51 and 143, 1983.

Silver Mean

see SILVER RATIO

Silver Ratio

The quantity defined by the CONTINUED FRACTION

$$\delta_S \equiv [2, 2, 2, \ldots] = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}.$$

It follows that

$$(\delta_S - 1)^2 = 2,$$

so

$$\delta_S = \sqrt{2} + 1 = 2.41421\ldots.$$

see also GOLDEN RATIO, GOLDEN RATIO CONJUGATE

Silverman Constant

$$\sum_{n=1}^{\infty} \frac{1}{\phi(n)\sigma(n)} = \prod_{p \text{ prime}} \left(1 + \sum_{k=1}^{\infty} \frac{1}{p^{2k} - p^{k-1}}\right) = 1.786576459\ldots,$$

where $\phi(n)$ is the TOTIENT FUNCTION and $\sigma(n)$ is the DIVISOR FUNCTION.

References

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/totient/totient.html`.

Zimmerman, P. `http://www.mathsoft.com/asolve/constant/totient/zimmermn.html`.

Silverman's Sequence

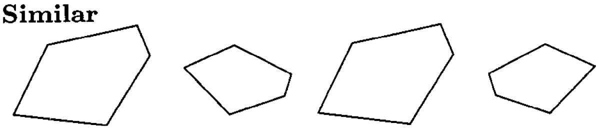
Let $f(1) = 1$, and let $f(n)$ be the number of occurrences of n in a nondecreasing sequence of INTEGERS. Then the first few values of $f(n)$ are 1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, ... (Sloane's A001462). The asymptotic value of the nth term is $\phi^{2-\phi}n^{\phi-1}$, where ϕ is the GOLDEN RATIO.

References

Guy, R. K. "Silverman's Sequences." §E25 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 225–226, 1994.

Sloane, N. J. A. Sequence A001462/M0257 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Similar



directly similar *inversely similar*

Two figures are said to be similar when all corresponding ANGLES are equal. Two figures are DIRECTLY SIMILAR when all corresponding ANGLES are equal and described in the same rotational sense. This relationship is written $A \sim B$. (The symbol $\sim$ is also used to mean "is the same order of magnitude as" and "is ASYMPTOTIC to.") Two figures are INVERSELY SIMILAR when all corresponding ANGLES are equal and described in the opposite rotational sense.

see also DIRECTLY SIMILAR, INVERSELY SIMILAR, SIMILARITY TRANSFORMATION

References

Project Mathematics! *Similarity.* Videotape (27 minutes). California Institute of Technology. Available from the Math. Assoc. Amer.

Similarity Axis

see D'ALEMBERT'S THEOREM

Similarity Dimension

To multiply the size of a d-D object by a factor a, $c \equiv a^d$ copies are required, and the quantity

$$d = \frac{\ln c}{\ln a}$$

is called the similarity dimension.

Similarity Point

External (or positive) and internal (or negative) similarity points of two CIRCLES with centers C and C' and RADII r and r' are the points E and I on the lines CC' such that

$$\frac{CE}{C'E} = \frac{r}{r'},$$

or

$$\frac{CI}{C'I} = -\frac{r}{r'}.$$

Similarity Transformation

An ANGLE-preserving transformation. A similarity transformation has a transformation MATRIX A' of the form

$$\mathsf{A}' \equiv \mathsf{BAB}^{-1}.$$

If A is an ANTISYMMETRIC MATRIX ($a_{ij} = -a_{ji}$) and B is an ORTHOGONAL MATRIX, then

$$\begin{aligned}(bab^{-1})_{ij} &= b_{ik}a_{kl}b^{-1}_{lj} = -b_{ik}a_{lk}b^{-1}_{lj} = -b^{\dagger}{}_{ki}a_{lk}(b^{\dagger})^{-1}{}_{jl}\\ &= -b^{-1}{}_{ki}a_{ki}b_{jl} = b_{jl}a_{lk}b^{-1}_{ki} = -(bab^{-1})_{ji}.\end{aligned}$$

Similarity transformations and the concept of SELF-SIMILARITY are important foundations of FRACTALS and ITERATED FUNCTION SYSTEMS.

see also CONFORMAL TRANSFORMATION

References

Lauwerier, H. *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, pp. 83–103, 1991.

Similitude Center

Also called a SELF-HOMOLOGOUS POINT. If two SIMILAR figures lie in the plane but do not have parallel sides (they are not HOMOTHETIC), there exists a center of similitude which occupies the same homologous position with respect to the two figures. The LOCUS of similitude centers of two nonconcentric circles is another circle having the line joining the two homothetic centers as its DIAMETER.

There are a number of interesting theorems regarding three CIRCLES (Johnson 1929, pp. 151–152).

1. The external similitude centers of three circles are COLLINEAR.
2. Any two internal similitude centers are COLLINEAR with the third external one.
3. If the center of each circle is connected with the internal similitude center of the other three [sic], the connectors are CONCURRENT.
4. If one center is connected with the internal similitude center of the other two, the others with the corresponding external centers, the connectors are CONCURRENT.

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 19–27 and 151–153, 1929.

Similitude Ratio

Two figures are HOMOTHETIC if they are related by a DILATION (a dilation is also known as a HOMOTHECY). This means that the connectors of corresponding points are CONCURRENT at a point which divides each connector in the same ratio k, known as the similitude ratio.

see also CONCURRENT, DILATION, HOMOTHECY, HOMOTHETIC

Simple Algebra

An ALGEBRA with no nontrivial IDEALS.

see also ALGEBRA, IDEAL, SEMISIMPLE ALGEBRA

Simple Continued Fraction

A CONTINUED FRACTION

$$\sigma = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ldots}}} \tag{1}$$

in which the b_is are all unity, leaving a continued fraction of the form

$$\sigma = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}. \tag{2}$$

A simple continued fraction can be written in a compact abbreviated NOTATION as

$$\sigma = [a_0, a_1, a_2, a_3, \ldots]. \tag{3}$$

Bach and Shallit (1996) show how to compute the JACOBI SYMBOL in terms of the simple continued fraction of a RATIONAL NUMBER a/b.

see also CONTINUED FRACTION

References

Bach, E. and Shallit, J. *Algorithmic Number Theory, Vol. 1: Efficient Algorithms.* Cambridge, MA: MIT Press, pp. 343–344, 1996.

Simple Curve

A curve is simple closed if it does not cross itself.

see also JORDAN CURVE

Simple Graph

A GRAPH for which at most one EDGE connects any two nodes.

see also ADJACENCY MATRIX, EDGE (GRAPH)

Simple Group

A simple group is a GROUP whose NORMAL SUBGROUPS (INVARIANT SUBGROUPS) are ORDER one or the whole of the original GROUP. Simple groups include ALTERNATING GROUPS, CYCLIC GROUPS, LIE-TYPE GROUPS (five varieties), and SPORADIC GROUPS (26 varieties, including the MONSTER GROUP). The CLASSIFICATION THEOREM of finite simple groups states that such groups can be classified completely into the three types:

1. CYCLIC GROUPS of PRIME ORDER,
2. ALTERNATING GROUPS of degree at least five
3. LIE-TYPE CHEVALLEY GROUPS,
4. LIE-TYPE (TWISTED CHEVALLEY GROUPS or the TITS GROUP), and
5. SPORADIC GROUPS.

BURNSIDE'S CONJECTURE states that every non-ABELIAN SIMPLE GROUP has EVEN ORDER.

see also ALTERNATING GROUP, BURNSIDE'S CONJECTURE, CHEVALLEY GROUPS, CLASSIFICATION THEOREM, CYCLIC GROUP, FEIT-THOMPSON THEOREM, FINITE GROUP, GROUP, INVARIANT SUBGROUP, LIE-TYPE GROUP, MONSTER GROUP, SCHUR MULTIPLIER, SPORADIC GROUP, TITS GROUP, TWISTED CHEVALLEY GROUPS

Simple Harmonic Motion

Simple harmonic motion refers to the periodic sinusoidal oscillation of an object or quantity. Simple harmonic motion is executed by any quantity obeying the DIFFERENTIAL EQUATION

$$\ddot{x} + {\omega_0}^2 x = 0, \tag{1}$$

where $\ddot{x}$ denotes the second DERIVATIVE of x with respect to t, and ω_0 is the angular frequency of oscillation. This ORDINARY DIFFERENTIAL EQUATION has an irregular SINGULARITY at ∞. The general solution is

$$x = A\sin(\omega_0 t) + B\cos(\omega_0 t) \tag{2}$$

$$= C\cos(\omega_0 t + \phi), \tag{3}$$

where the two constants A and B (or C and ϕ) are determined from the initial conditions.

Many physical systems undergoing small displacements, including any objects obeying Hooke's law, exhibit simple harmonic motion. This equation arises, for example, in the analysis of the flow of current in an electronic CL circuit (which contains a capacitor and an inductor). If a damping force such as Friction is present, an additional term $\beta\dot{x}$ must be added to the DIFFERENTIAL EQUATION and motion dies out over time.

Adding a damping force proportional to $\dot{x}$, the first derivative of x with respect to time, the equation of motion for *damped* simple harmonic motion is

$$\ddot{x} + \beta\dot{x} + {\omega_0}^2 x = 0, \tag{4}$$

where β is the damping constant. This equation arises, for example, in the analysis of the flow of current in an electronic CLR circuit, (which contains a capacitor, an inductor, and a resistor). This ORDINARY DIFFERENTIAL EQUATION can be solved by looking for trial solutions of the form $x = e^{rt}$. Plugging this into (4) gives

$$(r^2 + \beta r + {\omega_0}^2)e^{rt} = 0 \tag{5}$$

$$r^2 + \beta r + {\omega_0}^2 = 0. \tag{6}$$

This is a QUADRATIC EQUATION with solutions

$$r = \tfrac{1}{2}(-\beta \pm \sqrt{\beta^2 - 4\omega_0{}^2}). \tag{7}$$

There are therefore three solution regimes depending on the SIGN of the quantity inside the SQUARE ROOT,

$$\alpha \equiv \beta^2 - 4\omega_0{}^2. \tag{8}$$

The three regimes are

1. $\alpha > 0$ is POSITIVE: overdamped,
2. $\alpha = 0$ is ZERO: critically damped,
3. $\alpha < 0$ is NEGATIVE: underdamped.

If a periodic (sinusoidal) forcing term is added at angular frequency ω, the same three solution regimes are again obtained. Surprisingly, the resulting motion is still periodic (after an initial transient response, corresponding to the solution to the unforced case, has died out), but it has an amplitude different from the forcing amplitude.

The "particular" solution $x_p(t)$ to the forced second-order nonhomogeneous ORDINARY DIFFERENTIAL EQUATION

$$\ddot{x} + p(t)\dot{x} + q(t)x = A\cos(\omega t) \tag{9}$$

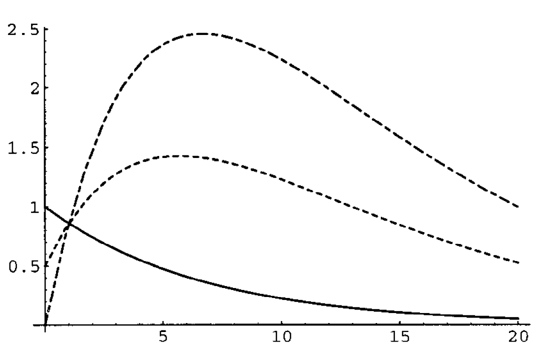
due to forcing is given by the equation

$$x_p(t) = -x_1(t)\int \frac{x_2(t)g(t)}{W(t)}\,dt + x_2(t)\int \frac{x_1(t)g(t)}{W(t)}\,dt, \tag{10}$$

where x_1 and x_2 are the homogeneous solutions to the unforced equation

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0 \tag{11}$$

and $W(t)$ is the WRONSKIAN of these two functions. Once the sinusoidal case of forcing is solved, it can be generalized to *any* periodic function by expressing the periodic function in a FOURIER SERIES.



Critical damping is a special case of damped simple harmonic motion in which

$$\alpha \equiv \beta^2 - 4\omega_0{}^2 = 0, \tag{12}$$

so

$$\beta = 2\omega_0. \tag{13}$$

The above plot shows an underdamped simple harmonic oscillator with $\omega = 0.3$, $\beta = 0.15$. The solid curve is for $(A, B) = (1, 0)$, the dot-dashed for $(0, 1)$, and the dotted for $(1/2, 1/2)$. In this case, $\alpha = 0$ so the solutions of the form $x = e^{rt}$ satisfy

$$r_\pm = \tfrac{1}{2}(-\beta) = -\tfrac{1}{2}\beta = -\omega_0. \tag{14}$$

One of the solutions is therefore

$$x_1 = e^{-\omega_0 t}. \tag{15}$$

In order to find the other linearly independent solution, we can make use of the identity

$$x_2(t) = x_1(t)\int \frac{e^{-\int p(t)\,dt}}{[x_1(t)]^2}\,dt. \tag{16}$$

Since we have $p(t) = 2\omega_0$, $e^{-\int p(t)\,dt}$ simplifies to $e^{-2\omega_0 t}$. Equation (16) therefore becomes

$$x_2(t) = e^{-\omega_0 t}\int \frac{e^{-2\omega_0 t}}{[e^{-\omega_0 t}]^2}\,dt = e^{-\omega_0 t}\int dt = te^{-\omega_0 t}. \tag{17}$$

The general solution is therefore

$$x = (A + Bt)e^{-\omega_0 t}. \tag{18}$$

In terms of the constants A and B, the initial values are

$$x(0) = A \tag{19}$$
$$x'(0) = B - A\omega, \tag{20}$$

so

$$A = x(0) \tag{21}$$
$$B = x'(0) + \omega_0 x(0). \tag{22}$$

For sinusoidally forced simple harmonic motion with critical damping, the equation of motion is

$$\ddot{x} + 2\omega_0\dot{x} + \omega_0{}^2 x = A\cos(\omega t), \tag{23}$$

and the WRONSKIAN is

$$\begin{aligned} W(t) &\equiv x_1\dot{x}_2 - \dot{x}_1 x_2 \\ &= e^{-\omega_0 t}(e^{-\omega_0 t} - \omega_0 te^{-\omega_0 t}) + \omega_0 e^{-\omega_0 t}te^{-\omega_0 t} \\ &= e^{-2\omega_0 t}(1 - \omega_0 t + \omega_0 t) = e^{-2\omega_0 t}. \end{aligned} \tag{24}$$

Plugging this into the equation for the particular solution gives

$$\begin{aligned} x_p(t) &= -e^{-\omega_0 t}\int \frac{te^{-\omega_0 t}A\cos(\omega t)}{e^{-2\omega_0 t}}\,dt \\ &\quad + te^{-\omega_0 t}\int \frac{e^{-\omega_0 t}A\cos(\omega t)}{e^{-2\omega_0 t}}\,dt \\ &= Ae^{-\omega_0 t}\left[-\int te^{\omega_0 t}\cos(\omega t)\,dt + t\int e^{\omega_0 t}\cos(\omega t)\,dt\right] \\ &= Ae^{-\omega_0 t}\left\{-\frac{e^{\omega_0 t}}{(\omega^2+{\omega_0}^2)^2}[(\omega^2 + t\omega^2\omega_0 - {\omega_0}^2 + t{\omega_0}^3) \right. \\ &\quad \times\cos(\omega t) + \omega(t\omega^2 - 2\omega_0 + t{\omega_0}^2)\sin(\omega t)] \\ &\quad \left. + t\frac{e^{\omega_0 t}}{\omega^2+{\omega_0}^2}[\omega_0\cos(\omega t) + \omega\sin(\omega t)]\right\} \\ &= \frac{A}{(\omega^2+{\omega_0}^2)^2}[({\omega_0}^2-\omega^2)\cos(\omega t) + 2\omega\omega_0\sin(\omega t)]. \end{aligned} \tag{25}$$

In order to put this in the desired form, note that we want to equate

$$\begin{aligned} C\cos\theta + S\sin\theta &= Q\cos(\theta+\delta) \\ &= Q(\cos\theta\cos\delta - \sin\theta\sin\delta). \end{aligned} \tag{26}$$

This means

$$C \equiv Q\cos\delta = {\omega_0}^2 - \omega^2 \tag{27}$$

$$S \equiv -Q\sin\delta = 2\omega\omega_0, \tag{28}$$

so

$$Q = \sqrt{C^2+S^2} \tag{29}$$

$$\delta = \tan^{-1}\left(-\frac{S}{C}\right). \tag{30}$$

Plugging in,

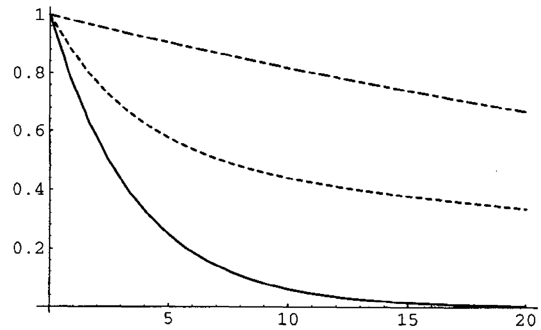
$$\begin{aligned} Q &= \sqrt{{\omega_0}^4 - 2{\omega_0}^2\omega^2 + \omega^4 + 4{\omega_0}^2\omega^2} \\ &= \sqrt{{\omega_0}^4 + 2{\omega_0}^2\omega^2 + \omega^4} = {\omega_0}^2 + \omega^2. \end{aligned} \tag{31}$$

$$\delta = \tan^{-1}\left(\frac{2\omega\omega_0}{\omega^2 - {\omega_0}^2}\right). \tag{32}$$

The solution in the requested form is therefore

$$\begin{aligned} x_p &= \frac{A}{(\omega^2+{\omega_0}^2)^2}({\omega_0}^2+\omega^2)\cos(\omega t+\delta) \\ &= \frac{A}{\omega^2+{\omega_0}^2}\cos(\omega t+\delta), \end{aligned} \tag{33}$$

where δ is defined by (32).



Overdamped simple harmonic motion occurs when

$$\beta^2 - 4{\omega_0}^2 > 0, \tag{34}$$

so

$$\alpha \equiv \beta^2 - 4{\omega_0}^2 > 0. \tag{35}$$

The above plot shows an overdamped simple harmonic oscillator with $\omega = 0.3$, $\beta = 0.075$. The solid curve is for $(A, B) = (1, 0)$, the dot-dashed for (0, 1), and the dotted for (1/2, 1/2). The solutions are

$$x_1 = e^{r_- t} \tag{36}$$

$$x_2 = e^{r_+ t}, \tag{37}$$

where

$$r_\pm \equiv \tfrac{1}{2}(-\beta \pm \sqrt{\beta^2 - 4{\omega_0}^2}\,). \tag{38}$$

The general solution is therefore

$$x = Ae^{r_- t} + Be^{r_+ t}, \tag{39}$$

where A and B are constants. The initial values are

$$x(0) = A + B \tag{40}$$

$$x'(0) = Ar_- + Br_+, \tag{41}$$

so

$$A = x(0) + \frac{r_+ x(0) - x'(0)}{r_- - r_+} \tag{42}$$

$$B = -\frac{r_+ x(0) - x'(0)}{r_- - r_+}. \tag{43}$$

For a cosinusoidally forced overdamped oscillator with forcing function $g(t) = C\cos(\omega t)$, the particular solutions are

$$y_1(t) = e^{r_1 t} \tag{44}$$

$$y_2(t) = e^{r_2 t}, \tag{45}$$

where

$$r_1 \equiv \tfrac{1}{2}(-\beta + \sqrt{\beta^2 - 4{\omega_0}^2}\,) \tag{46}$$

$$r_2 \equiv \tfrac{1}{2}(-\beta - \sqrt{\beta^2 - 4{\omega_0}^2}\,). \tag{47}$$

These give the identities

$$r_1 + r_2 = -\beta \tag{48}$$
$$r_1 - r_2 = \sqrt{\beta^2 - 4\omega_0{}^2} \tag{49}$$

and

$$\begin{aligned}\omega_0{}^2 &= \tfrac{1}{4}[\beta - (r_1 - r_2)^2] = \tfrac{1}{4}[(r_1 + r_2)^2 - (r_1 - r_2)^2] \\ &= \tfrac{1}{4}[2r_1r_2 + 2r_1r_2] = r_1r_2.\end{aligned} \tag{50}$$

The WRONSKIAN is

$$\begin{aligned}W(t) &\equiv y_1y_2' - y_1'y_2 = e^{r_1t}r_2e^{r_2t} - r_1e^{r_1t}e^{r_2t} \\ &= (r_2 - r_1)e^{(r_1+r_2)t}.\end{aligned} \tag{51}$$

The particular solution is

$$y_p = -y_1v_1 + y_2v_2, \tag{52}$$

where

$$v_1 \equiv \int \frac{y_2 g(t)}{W(t)} = \frac{C}{r_2 - r_1}\frac{\omega\sin(\omega t) - r_2\cos(\omega t)}{e^{r_2t}(r_2{}^2 + \omega^2)} \tag{53}$$

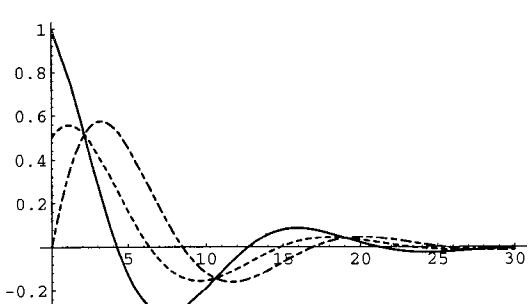
$$v_2 \equiv \int \frac{y_2 g(t)}{W(t)} = \frac{C}{r_2 - r_1}\frac{\omega\sin(\omega t) - r_1\cos(\omega t)}{e^{r_1t}(r_2{}^2 + \omega^2)}. \tag{54}$$

Therefore,

$$\begin{aligned}y_p &= C\frac{\cos(\omega t)(r_1r_2 - \omega^2) - \sin(\omega t)\omega(r_1 + r_2)}{(r_1{}^2 + \omega^2)(r_2{}^2 + \omega^2)} \\ &= C\frac{(\omega_0{}^2 - \omega^2)\cos(\omega t) + \beta\omega\sin(\omega t)}{\omega^2\beta^2 + (\omega^2 - \omega_0{}^2)} \\ &= \frac{C}{\omega^2\beta^2 + (\omega^2 - \omega_0{}^2)^2}\sqrt{(\omega^2 - \omega_0{}^2)^2 + \beta^2\omega^2} \\ &\quad\times\cos(\omega t + \delta) \\ &= \frac{C}{\sqrt{\beta^2\omega^2 + (\omega^2 - \omega_0{}^2)^2}}\cos(\omega t + \delta),\end{aligned} \tag{55}$$

where

$$\delta = \tan^{-1}\left(\frac{\beta\omega}{\omega^2 - \omega_0{}^2}\right). \tag{56}$$



Underdamped simple harmonic motion occurs when

$$\beta^2 - 4\omega_0{}^2 < 0, \tag{57}$$

so

$$\alpha \equiv \beta^2 - 4\omega_0{}^2 < 0. \tag{58}$$

The above plot shows an underdamped simple harmonic oscillator with $\omega = 0.3$, $\beta = 0.4$. The solid curve is for $(A, B) = (1, 0)$, the dot-dashed for $(0, 1)$, and the dotted for $(1/2,\ 1/2)$. Define

$$\gamma \equiv \sqrt{-\alpha} = \tfrac{1}{2}\sqrt{4\omega_0{}^2 - \beta^2}, \tag{59}$$

then solutions satisfy

$$r_\pm = -\tfrac{1}{2}\beta \pm i\gamma, \tag{60}$$

where

$$r_\pm \equiv \tfrac{1}{2}(-\beta \pm \sqrt{\beta^2 - 4\omega_0{}^2}\,), \tag{61}$$

and are of the form

$$x = e^{-(\beta/2 \pm i\gamma)t}. \tag{62}$$

Using the EULER FORMULA

$$e^{ix} = \cos x + i\sin x, \tag{63}$$

this can be rewritten

$$x = e^{-(\beta/2)t}\left[\cos(\gamma t) \pm i\sin(\gamma t)\right]. \tag{64}$$

We are interested in the *real* solutions. Since we are dealing here with a *linear homogeneous* ODE, linear sums of LINEARLY INDEPENDENT solutions are also solutions. Since we have a sum of such solutions in (64), it follows that the IMAGINARY and REAL PARTS separately satisfy the ODE and are therefore the solutions we seek. The constant in front of the sine term is arbitrary, so we can identify the solutions as

$$x_1 = e^{-(\beta/2)t}\cos(\gamma t) \tag{65}$$
$$x_2 = e^{-(\beta/2)t}\sin(\gamma t), \tag{66}$$

so the general solution is

$$x = e^{-(\beta/2)t}[A\cos(\gamma t) + B\sin(\gamma t)]. \tag{67}$$

The initial values are

$$x(0) = A \tag{68}$$
$$x'(0) = -\tfrac{1}{2}\beta A + B,\gamma \tag{69}$$

so A and B can be expressed in terms of the initial conditions by

$$A = x(0) \tag{70}$$
$$B = \frac{\beta x(0)}{2\gamma} + \frac{x'(0)}{\gamma}. \tag{71}$$

For a cosinusoidally forced underdamped oscillator with forcing function $g(t) = C\cos(\omega t)$, use

$$\gamma \equiv \tfrac{1}{2}\sqrt{4\omega_0{}^2 - \beta^2} \quad (72)$$
$$\alpha \equiv \tfrac{1}{2}\beta \quad (73)$$

to obtain

$$4\omega_0{}^2 - \beta^2 = 4\gamma^2 \quad (74)$$
$$\omega_0{}^2 = \gamma^2 + \tfrac{1}{4}\beta^2 = \gamma^2 + \alpha^2 \quad (75)$$
$$\beta = 2\alpha. \quad (76)$$

The particular solutions are

$$y_1(t) = e^{-\alpha t}\cos(\gamma t) \quad (77)$$
$$y_2(t) = e^{-\alpha t}\sin(\gamma t). \quad (78)$$

The WRONSKIAN is

$$\begin{aligned} W(t) &\equiv y_1 y_2' - y_1' y_2 \\ &= e^{-\alpha t}\cos(\gamma t)[-\alpha e^{-\alpha t}\sin(\gamma t) + e^{-\alpha t}\gamma\cos(\gamma t)] \\ &\quad - e^{-\alpha t}\sin(\gamma t)[-\alpha e^{-\alpha t}\cos(\gamma t) - e^{-\alpha t}\gamma\sin(\gamma t)] \\ &= e^{-2\alpha t}\{\alpha[-\sin(\gamma t)\cos(\gamma t) + \sin(\gamma t)\cos(\gamma t)] \\ &\quad + \gamma[\cos^2(\gamma t) + \sin^2(\gamma t)]\} \\ &= \gamma e^{-2\alpha t}. \end{aligned} \quad (79)$$

The particular solution is given by

$$y_p = -y_1 v_1 + y_2 v_2, \quad (80)$$

where

$$v_1 \equiv \int \frac{y_2 g(t)}{W(t)} = \frac{C}{\gamma}\int e^{\alpha t}\cos(\gamma t)\cos(\omega t)\,dt \quad (81)$$
$$v_2 \equiv \int \frac{y_2 g(t)}{W(t)} = \frac{C}{\gamma}\int e^{\alpha t}\cos(\gamma t)\cos(\omega t)\,dt. \quad (82)$$

Using computer algebra to perform the algebra, the particular solution is

$$\begin{aligned} y_p(t) &= C\frac{(\alpha^2 + \gamma^2 - \omega^2)\cos(\omega t) + 2\alpha\omega\sin(\omega t)}{[\alpha^2 + (\gamma - \omega)^2][\alpha^2 + (\gamma + \omega)^2]} \\ &= C\frac{(\omega_0{}^2 - \omega^2)\cos(\omega t) + \beta\omega\sin(\omega t)}{(\alpha^2 + \gamma^2 + \omega^2)^2 - 4\gamma^2\omega^2} \\ &= C\frac{(\omega_0{}^2 - \omega^2)\cos(\omega t) + \beta\omega\sin(\omega t)}{(\omega_0{}^2 + \omega^2)^2 - 4\frac{1}{4}(4\omega_0{}^2 - \beta^2)\omega^2} \\ &= C\frac{(\omega_0{}^2 - \omega^2)\cos(\omega t) + \beta\omega\sin(\omega t)}{(\omega_0{}^2 - \omega^2)^2 - \omega^2(4\omega_0{}^2 - \beta^2)} \\ &= \frac{C\sqrt{(\omega_0{}^2 - \omega^2)^2 + \beta^2\omega^2}}{(\omega_0{}^2 - \omega^2)^2 - \omega^2(4\omega_0{}^2 - \beta^2)}\cos(\omega t + \delta) \\ &= C\frac{\sqrt{(\omega_0{}^2 - \omega^2)^2 + \beta^2\omega^2}}{(\omega_0{}^2 - \omega^2)^2 - \omega^2(4\omega_0{}^2 - \beta^2)}\cos(\omega t + \delta), \end{aligned} \quad (83)$$

where

$$\delta = \tan^{-1}\left(\frac{\beta\omega}{\omega^2 - \omega_0{}^2}\right). \quad (84)$$

If the forcing function is sinusoidal instead of cosinusoidal, then

$$\delta' = \delta - \tfrac{1}{2}\pi = \tan^{-1}x - \tfrac{1}{2}\pi = \tan^{-1}\left(-\frac{1}{x}\right), \quad (85)$$

so

$$\delta' = \tan^{-1}\left(\frac{\omega_0{}^2 - \omega^2}{\beta\omega}\right). \quad (86)$$

Simple Harmonic Motion Quadratic Perturbation

Given a simple harmonic oscillator with a quadratic perturbation ϵx^2,

$$\ddot{x} + \omega_0{}^2 x - \alpha\epsilon x^2 = 0, \quad (1)$$

find the first-order solution using a perturbation method. Write

$$x \equiv x_0 + \epsilon x_1 + \ldots, \quad (2)$$

so

$$\ddot{x} = \ddot{x}_0 + \epsilon\ddot{x}_1 + \ldots. \quad (3)$$

Plugging (2) and (3) back into (1) gives

$$(\ddot{x}_0 + \epsilon\ddot{x}_1) + (\omega_0{}^2 x_0 + \omega_0{}^2\epsilon x_1) - \alpha\epsilon(x_0 + 2x_0 x_1\epsilon + \ldots) = 0. \quad (4)$$

Keeping only terms of order ϵ and lower and grouping, we obtain

$$(\ddot{x}_0 + \omega_0{}^2 x_0) + (\ddot{x}_1 + \omega_0{}^2 x_1 - \alpha x_0{}^2)\epsilon = 0. \quad (5)$$

Since this equation must hold for all POWERS of ϵ, we can separate it into the two differential equations

$$\ddot{x}_0 + \omega_0{}^2 x_0 = 0 \quad (6)$$

$$\ddot{x}_1 + \omega_0{}^2 x_1 = \alpha x_0{}^2. \quad (7)$$

The solution to (6) is just

$$x_0 = A\cos(\omega_0 t + \phi). \quad (8)$$

Setting our clock so that $\phi = 0$ gives

$$x_0 = A\cos(\omega_0 t). \quad (9)$$

Plugging this into (7) then gives

$$\ddot{x}_1 + \omega_0{}^2 x_1 = \alpha A^2\cos^2(\omega_0 t). \quad (10)$$

The two homogeneous solutions to (10) are

$$x_1 = \cos(\omega_0 t) \quad (11)$$
$$x_2 = \sin(\omega_0 t). \quad (12)$$

The particular solution to (10) is therefore given by

$$x_p(t) = -x_1(t)\int \frac{x_2(t)g(t)}{W(t)}\,dt + x_2(t)\int \frac{x_1(t)g(t)}{W(t)}\,dt, \tag{13}$$

where

$$g(t) = \alpha A^2 \cos^2(\omega_0 t), \tag{14}$$

and the WRONSKIAN is

$$\begin{aligned} W &\equiv x_1\dot{x}_2 - \dot{x}_1 x_2 \\ &= \cos(\omega_0 t)\omega_0\cos(\omega_0 t) - [-\omega_0\sin(\omega_0 t)]\sin(\omega_0 t) \\ &= \omega_0. \end{aligned} \tag{15}$$

Plugging everything into (13),

$$\begin{aligned} x_p &= \alpha A^2\left[-\cos(\omega_0 t)\int \frac{\sin(\omega_0 t)\cos^2(\omega_0 t)}{\omega_0}\,dt \right. \\ &\quad \left. +\sin(\omega_0 t)\int \frac{\cos^3(\omega_0 t)}{\omega_0}\,dt\right] \\ &= \frac{\alpha A^2}{\omega_0}\left\{\sin(\omega_0 t)\int [1-\sin^2(\omega_0 t)]\cos(\omega_0 t)\,dt \right. \\ &\quad \left. -\cos(\omega_0 t)\int \sin(\omega_0 t)\cos^2(\omega_0 t)\,dt\right\}. \end{aligned} \tag{16}$$

Now let

$$u \equiv \sin(\omega_0 t) \tag{17}$$

$$du = \omega_0\cos(\omega_0 t)\,dt \tag{18}$$

$$v \equiv \cos(\omega_0 t) \tag{19}$$

$$dv = -\omega_0\sin(\omega_0 t)\,dt. \tag{20}$$

Then

$$\begin{aligned} x_p &= \frac{\alpha A^2}{{\omega_0}^2}\left[\sin(\omega_0 t)\int (1-u^2)\,du + \cos(\omega_0 t)\int v^2\,dv\right] \\ &= \frac{\alpha A^2}{{\omega_0}^2}\left[\sin(\omega_0 t)(1-\tfrac{1}{3}u^3) + \cos(\omega_0 t)\tfrac{1}{3}v^3\right] \\ &= \frac{\alpha A^2}{{\omega_0}^2}\{\sin(\omega_0 t)[1-\tfrac{1}{3}\sin^3(\omega_0 t)] \\ &\quad +\tfrac{1}{3}\cos(\omega_0 t)\cos^3(\omega_0 t)\} \\ &= \frac{\alpha A^2}{{\omega_0}^2}\left\{\tfrac{1}{3}[\cos^4(\omega_0 t) - \sin^4(\omega_0 t)] + \sin^2(\omega_0 t)\right\} \\ &= \frac{\alpha A^2}{{\omega_0}^2}\left\{\tfrac{1}{3}[\cos^2(\omega_0 t) - \sin^2(\omega_0 t)] + \sin^2(\omega_0 t)\right\} \\ &= \frac{\alpha A^2}{{\omega_0}^2}\tfrac{1}{3}[\cos^2(\omega_0 t) + 2\sin^2(\omega_0 t)] \\ &= \frac{\alpha A^2}{3{\omega_0}^2}[2-\cos^2(\omega_0 t)] = \frac{\alpha A^2}{3{\omega_0}^2}\{2-\tfrac{1}{2}[1+\cos(2\omega_0 t)]\} \\ &= \frac{\alpha A^2}{6{\omega_0}^2}[3-\cos(2\omega_0 t)]. \end{aligned} \tag{21}$$

Plugging $x_0(t)$ and (21) into (2), we obtain the solution

$$x(t) = A\cos(\omega_0 t) - \frac{\alpha A^2}{6{\omega_0}^2}\epsilon[\cos(2\omega_0 t) - 3]. \tag{22}$$

Simple Harmonic Oscillator

see SIMPLE HARMONIC MOTION

Simple Interest

INTEREST which is paid only on the PRINCIPAL and not on the additional amount generated by previous INTEREST payments. A formula for computing simple interest is

$$a(t) = a(0)(1+rt),$$

where $a(t)$ is the sum of PRINCIPAL and INTEREST at time t for a constant interest rate r.

see also COMPOUND INTEREST, INTEREST

References

Kellison, S. G. *Theory of Interest, 2nd ed.* Burr Ridge, IL: Richard D. Irwin, 1991.

Simple Polygon

A POLYGON P is said to be simple (or JORDAN) if the only points of the plane belonging to two EDGES of P are the VERTICES of P. Such a polygon has a well-defined interior and exterior.

see also POLYGON, REGULAR POLYGON, TWO-EARS THEOREM

References

Toussaint, G. "Anthropomorphic Polygons." *Amer. Math. Monthly* **122**, 31–35, 1991.

Simple Ring

A NONZERO RING S whose only (two-sided) IDEALS are S itself and zero. Every commutative simple ring is a FIELD. Every simple ring is a PRIME RING.

see also FIELD, IDEAL, PRIME RING, RING

Simplex

The generalization of a tetrahedral region of space to n-D. The boundary of a k-simplex has $k+1$ 0-faces (VERTICES), $k(k+1)/2$ 1-faces (EDGES), and $\binom{k+1}{i+1}$ i-faces, where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT.

The simplex in 4-D is a regular TETRAHEDRON $ABCD$ in which a point E along the fourth dimension through the center of $ABCD$ is chosen so that $EA = EB = EC = ED = AB$. The 4-D simplex has SCHLÄFLI SYMBOL $\{3,3,3\}$.

n	Simplex
0	point
1	line segment
2	equilateral triangular plane region
3	tetrahedral region
4	4-simplex

The only irreducible spherical simplexes generated by reflection are A_n ($n \geq 1$), B_n ($n \geq 4$), C_n ($n \geq 2$), D_2^p ($p \geq 5$), E_6, E_7, E_8, F_4, G_3, and G_4. The only irreducible Euclidean simplexes generated by reflection are W_2, P_m $(m \geq 3)$, Q_m $(m \geq 5)$, R_m $(m \geq 3)$, S_m $(m \geq 4)$, V_3, T_7, T_8, T_9, and U_5.

The regular simplex in n-D with $n \geq 5$ is denoted α_n and has SCHLÄFLI SYMBOL $\{\underbrace{3,\ldots,3}_{3^{n-1}}\}$.

see also COMPLEX, CROSS POLYTOPE, EQUILATERAL TRIANGLE, LINE SEGMENT, MEASURE POLYTOPE, NERVE, POINT, SIMPLEX METHOD, TETRAHEDRON

References
Eppstein, D. "Triangles and Simplices." http://www.ics.uci.edu/~eppstein/junkyard/triangulation.html.

Simplex Method

A method for solving problems in LINEAR PROGRAMMING. This method, invented by G. B. Dantzig in 1947, runs along EDGES of the visualization SOLID to find the best answer. In 1970, Klee and Minty constructed examples in which the simplex method required an exponential number of steps, but such cases seem never to be encountered in practical applications.

A much more efficient (POLYNOMIAL-time) ALGORITHM was found in 1984 by N. Karmarkar. This method goes through the middle of the SOLID and then transforms and warps. It offers many advantages over the simplex method (Nemirovsky and Yudin 1994).

see also LINEAR PROGRAMMING

References
Nemirovsky, A. and Yudin, N. *Interior-Point Polynomial Methods in Convex Programming.* Philadelphia, PA: SIAM, 1994.
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Downhill Simplex Method in Multidimensions" and "Linear Programming and the Simplex Method." §10.4 and 10.8 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 402–406 and 423–436, 1992.
Tokhomirov, V. M. "The Evolution of Methods of Convex Optimization." *Amer. Math. Monthly* **103**, 65–71, 1996.

Simplicial Complex

A simplicial complex is a SPACE with a TRIANGULATION. Objects in the space made up of only the simplices in the triangulation of the space are called simplicial subcomplexes. When only simplicial complexes and subcomplexes are considered, defining HOMOLOGY is particularly easy (and, in fact, combinatorial because of its finite/counting nature). This kind of homology is called SIMPLICIAL HOMOLOGY.

see also HOMOLOGY (TOPOLOGY), NERVE, SIMPLICIAL HOMOLOGY, SPACE, TRIANGULATION

Simplicial Homology

The type of HOMOLOGY which results when the spaces being studied are restricted to SIMPLICIAL COMPLEXES and subcomplexes.

see also SIMPLICIAL COMPLEX

Simplicity

The number of operations needed to effect a GEOMETRIC CONSTRUCTION as determined in GEOMETROGRAPHY. If the number of operations of the five GEOMETROGRAPHIC types are denoted m_1, m_2, n_1, n_2, and n_3, respectively, then the simplicity is $m_1+m_2+n_1+n_2+n_3$ and the symbol $m_1S_1 + m_2S_2 + n_1C_1 + n_2C_2 + n_3C_3$. It is apparently an unsolved problem to determine if a given GEOMETRIC CONSTRUCTION is of smallest possible simplicity.

see also GEOMETRIC CONSTRUCTION, GEOMETROGRAPHY

References
De Temple, D. W. "Carlyle Circles and the Lemoine Simplicity of Polygonal Constructions." *Amer. Math. Monthly* **98**, 97–108, 1991.
Eves, H. *An Introduction to the History of Mathematics, 6th ed.* New York: Holt, Rinehart, and Winston, 1976.

Simply Connected

A CONNECTED DOMAIN is said to be simply connected (also called 1-connected) if any simple closed curve can be shrunk to a point continuously in the set. If the domain is CONNECTED but not simply, it is said to be MULTIPLY CONNECTED.

A SPACE S is simply connected if it is 0-connected and if every MAP from the 1-SPHERE to S extends continuously to a MAP from the 2-DISK. In other words, every loop in the SPACE is contractible.

see also CONNECTED SPACE, MULTIPLY CONNECTED

Simpson's Paradox

It is not necessarily true that averaging the averages of different populations gives the average of the combined population.

References
Paulos, J. A. *A Mathematician Reads the Newspaper.* New York: BasicBooks, p. 135, 1995.

Simpson's Rule

Let $h \equiv (b-a)/n$, and assume a function $f(x)$ is defined at points $f(a+kh) = y_k$ for $k = 0, \ldots, n$. Then

$$\int_a^b f(x)\,dx = \tfrac{1}{3}h(y_1 + 4y_2 + 2y_3 + 4y_4 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n) - R_n,$$

where the remainder is

$$R_n = \tfrac{1}{90}(b-a)^4 f^{(4)}(x^*)$$

for some $x^* \in [a,b]$.

see also BODE'S RULE, NEWTON-COTES FORMULAS, SIMPSON'S 3/8 RULE, TRAPEZOIDAL RULE

References
Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 886, 1972.

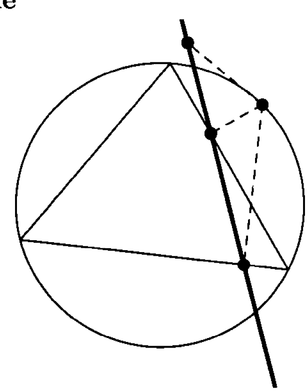
Simpson's 3/8 Rule

$$\int_{x_1}^{x_4} f(x)\,dx = \tfrac{3}{8}h(f_1+3f_2+3f_3+f_4) - \tfrac{3}{80}h^5 f^{(4)}(\xi).$$

see also BODE'S RULE, NEWTON-COTES FORMULAS, SIMPSON'S RULE

References
Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 886, 1972.

Simson Line



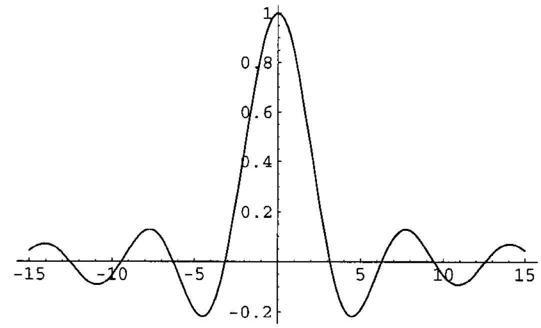
The Simson line is the LINE containing the feet of the perpendiculars from a point on the CIRCUMCIRCLE of a TRIANGLE to the sides (or their extensions) of the TRIANGLE. The Simson line is sometimes known as the WALLACE-SIMSON LINE, since it does not appear in any work of Simson (Johnson 1929, p. 137).

The ANGLE between the Simson lines of two points P and P' is half the ANGLE of the arc PP'. The Simson line of any VERTEX is the ALTITUDE through that VERTEX. The Simson line of a point opposite a VERTEX is the corresponding side. If $T_1T_2T_3$ is the Simson line of a point T of the CIRCUMCIRCLE, then the triangles TT_1T_2 and TA_2A_1 are directly similar.

see also CIRCUMCIRCLE

References
Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 40–41 and 43–45, 1967.
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 137–139, 1929.

Sinc Function



A function also called the SAMPLING FUNCTION and defined by

$$\operatorname{sinc}(x) \equiv \begin{cases} 1 & \text{for } x=0 \\ \frac{\sin x}{x} & \text{otherwise,} \end{cases} \tag{1}$$

where $\sin x$ is the SINE function. Let $\Pi(x)$ be the RECTANGLE FUNCTION, then the FOURIER TRANSFORM of $\Pi(x)$ is the sinc function

$$\mathcal{F}[\Pi(x)] = \operatorname{sinc}(\pi k). \tag{2}$$

The sinc function therefore frequently arises in physical applications such as Fourier transform spectroscopy as the so-called INSTRUMENT FUNCTION, which gives the instrumental response to a DELTA FUNCTION input. Removing the instrument functions from the final spectrum requires use of some sort of DECONVOLUTION algorithm.

The sinc function can be written as a complex INTEGRAL by noting that

$$\begin{aligned}\operatorname{sinc}(nx) &\equiv \frac{\sin(nx)}{nx} = \frac{1}{nx}\frac{e^{inx}-e^{-inx}}{2i} \\ &= \frac{1}{2inx}[e^{itx}]_{-n}^{n} = \frac{1}{2n}\int_{-n}^{n} e^{ixt}\,dt. \end{aligned} \tag{3}$$

The sinc function can also be written as the INFINITE PRODUCT

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right). \tag{4}$$

Definite integrals involving the sinc function include

$$\int_0^\infty \operatorname{sinc}(x)\,dx = \tfrac{1}{2}\pi \tag{5}$$

$$\int_0^\infty \operatorname{sinc}^2(x)\,dx = \tfrac{1}{2}\pi \tag{6}$$

$$\int_0^\infty \operatorname{sinc}^3(x)\,dx = \tfrac{3}{8}\pi \tag{7}$$

$$\int_0^\infty \operatorname{sinc}^4(x)\,dx = \tfrac{1}{3}\pi \tag{8}$$

$$\int_0^\infty \operatorname{sinc}^5(x)\,dx = \tfrac{115}{384}\pi. \tag{9}$$

These are all special cases of the amazing general result

$$\int_0^\infty \frac{\sin^a x}{x^b}\,dx = \frac{\pi^{1-c}(-1)^{\lfloor (a-b)/2\rfloor}}{2^{a-c}(b-1)!} \times \sum_{k=0}^{\lfloor a/2\rfloor - c} (-1)^k \binom{a}{k}(a-2k)^{b-1}[\ln(a-2k)]^c, \quad (10)$$

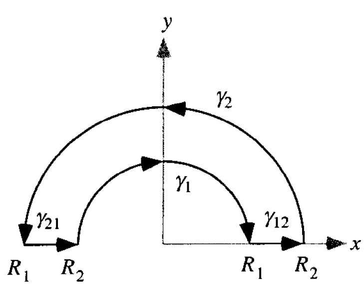
where a and b are POSITIVE integers such that $a \geq b > c$, $c \equiv a - b \pmod 2$, $\lfloor x \rfloor$ is the FLOOR FUNCTION, and 0^0 is taken to be equal to 1 (Kogan). This spectacular formula simplifies in the special case when n is a POSITIVE EVEN integer to

$$\int_0^\infty \frac{\sin^{2n} x}{x^{2n}}\,dx = \frac{\pi}{2(2n-1)!}\left\langle {2n-1 \atop n-1} \right\rangle, \quad (11)$$

where $\left\langle {n \atop k} \right\rangle$ is an EULERIAN NUMBER (Kogan). The solution of the integral can also be written in terms of the RECURRENCE RELATION for the coefficients

$$c(a,b) = \begin{cases} \frac{\pi}{2^{a+1-b}}\binom{a-1}{\frac{1}{2}(a-1)} & \text{for } b=1 \text{ or } b=2 \\ \frac{a}{(b-1)(b-2)}[(a-1)c(a-2,b-2) \\ \quad -a\cdot c(a,b-2)] & \text{otherwise} \end{cases} \quad (12)$$

(Zimmerman).



The half-infinite integral of sinc(x) can be derived using CONTOUR INTEGRATION. In the above figure, consider the path $\gamma \equiv \gamma_1 + \gamma_{12} + \gamma_2 + \gamma_{21}$. Now write $z = Re^{i\theta}$. On an arc, $dz = iRe^{i\theta}\,d\theta$ and on the x-AXIS, $dz = e^{i\theta}\,dR$. Write

$$\int_{-\infty}^\infty \frac{\sin x}{x}\,dx = \Im \int_\gamma \frac{e^{iz}}{z}\,dx, \quad (13)$$

where $\Im$ denotes the IMAGINARY POINT. Now define

$$\begin{aligned} I &\equiv \int_\gamma \frac{e^{iz}}{z}\,dz \\ &= \lim_{R_1\to 0}\int_\pi^0 \frac{\exp(iR_1e^{i\theta})}{R_1e^{i\theta}}\,i\theta R_1 e^{i\theta}\,d\theta \\ &\quad + \lim_{R_1\to 0}\lim_{R_2\to\infty}\int_{R_1}^{R_2} \frac{e^{iR}}{R}\,dR \\ &\quad + \lim_{R_2\to\infty}\int_0^\pi \frac{\exp(iz)}{z}\,dx + \lim_{R_1\to 0}\int_{R_2}^{R_1}\frac{e^{-iR}}{-R}\,(-dR), \end{aligned} \quad (14)$$

where the second and fourth terms use the identities $e^{i0} = 1$ and $e^{i\pi} = -1$. Simplifying,

$$\begin{aligned} I &= \lim_{R_1\to 0}\int_\pi^0 \exp(iR_1e^{i\theta})i\theta\,d\theta + \int_{0+}^\infty \frac{e^{iR}}{R}\,dR \\ &\quad + \lim_{R_2\to\infty}\int_0^\pi \frac{\exp(iz)}{z}\,dz + \int_\infty^{0^+}\frac{e^{-iR}}{-R}\,(-dR) \\ &= -\int_0^\pi i\theta\,d\theta + \int_{0+}^\infty \frac{e^{iR}}{R}\,dR + 0 + \int_{-\infty}^{0^-}\frac{e^{iR}}{R}\,dR, \end{aligned} \quad (15)$$

where the third term vanishes by JORDAN'S LEMMA. Performing the integration of the first term and combining the others yield

$$I = -i\pi + \int_{-\infty}^\infty \frac{e^{iz}}{z}\,dz = 0. \quad (16)$$

Rearranging gives

$$\int_{-\infty}^\infty \frac{e^{iz}}{z}\,dz = i\pi, \quad (17)$$

so

$$\int_{-\infty}^\infty \frac{\sin z}{z}\,dz = \pi. \quad (18)$$

The same result is arrived at using the method of RESIDUES by noting

$$\begin{aligned} I &= 0 + \tfrac{1}{2}2\pi i\,\mathrm{Res}[f(z)]_{z=0} \\ &= i\pi\left[(z-0)\frac{e^{iz}}{z}\right]_{z=0} = i\pi[e^{iz}]_{z=0} \\ &= i\pi, \end{aligned} \quad (19)$$

so

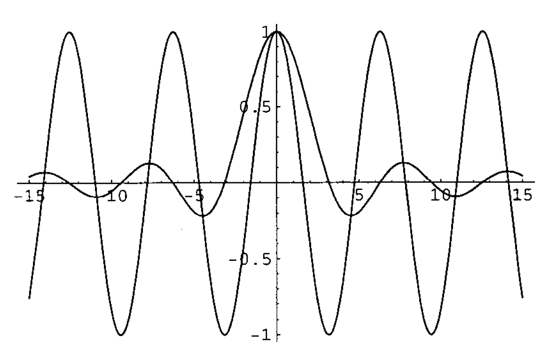
$$\Im(I) = \pi. \quad (20)$$

Since the integrand is symmetric, we therefore have

$$\int_0^\infty \frac{\sin x}{x}\,dx = \tfrac{1}{2}\pi, \quad (21)$$

giving the SINE INTEGRAL evaluated at 0 as

$$\mathrm{si}(0) = -\int_0^\infty \frac{\sin x}{x}\,dx = -\tfrac{1}{2}\pi. \quad (22)$$



An interesting property of $\operatorname{sinc}(x)$ is that the set of LOCAL EXTREMA of $\operatorname{sinc}(x)$ corresponds to its intersections with the COSINE function $\cos(x)$, as illustrated above.

see also FOURIER TRANSFORM, FOURIER TRANSFORM—RECTANGLE FUNCTION, INSTRUMENT FUNCTION, JINC FUNCTION, SINE, SINE INTEGRAL

References
Kogan, S. "A Note on Definite Integrals Involving Trigonometric Functions." `http://www.mathsoft.com/asolve/constant/pi/sin/sin.html`.
Morrison, K. E. "Cosine Products, Fourier Transforms, and Random Sums." *Amer. Math. Monthly* **102**, 716–724, 1995.

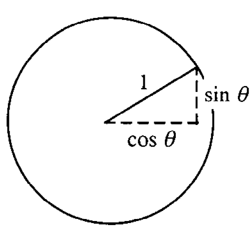
Sinclair's Soap Film Problem

Find the shape of a soap film (i.e., MINIMAL SURFACE) which will fill two inverted conical FUNNELS facing each other is known as Sinclair's soap film problem (Bliss 1925, p. 121). The soap film will assume the shape of a CATENOID.

see also CATENOID, FUNNEL, MINIMAL SURFACE

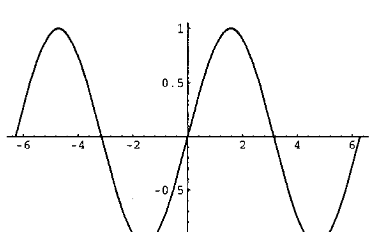
References
Bliss, G. A. *Calculus of Variations.* Chicago, IL: Open Court, pp. 121–122, 1925.
Isenberg, C. *The Science of Soap Films and Soap Bubbles.* New York: Dover, p. 81, 1992.
Sinclair, M. E. "On the Minimum Surface of Revolution in the Case of One Variable End Point." *Ann. Math.* **8**, 177–188, 1907.

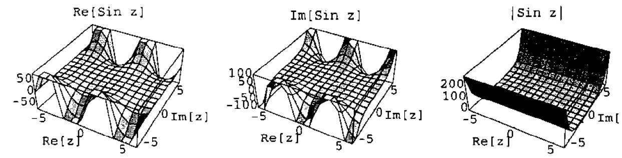
Sine



Let θ be an ANGLE measured counterclockwise from the x-AXIS along the arc of the UNIT CIRCLE. Then $\sin\theta$ is the vertical coordinate of the arc endpoint. As a result of this definition, the sine function is periodic with period 2π. By the PYTHAGOREAN THEOREM, $\sin\theta$ also obeys the identity

$$\sin^2\theta + \cos^2\theta = 1. \qquad (1)$$





The sine function can be defined algebraically by the infinite sum

$$\sin x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} x^{2n-1} \qquad (2)$$

and INFINITE PRODUCT

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right). \qquad (3)$$

It is also given by the IMAGINARY PART of the complex exponential

$$\sin x = \Im[e^{ix}]. \qquad (4)$$

The multiplicative inverse of the sine function is the COSECANT, defined as

$$\csc x \equiv \frac{1}{\sin x}. \qquad (5)$$

Using the results from the EXPONENTIAL SUM FORMULAS

$$\begin{aligned}\sum_{n=0}^{\infty} \sin(nx) &= \Im\left[\sum_{n=0}^{\infty} e^{inx}\right] \\ &= \Im\left[\frac{\sin(\frac{1}{2}Nx)}{\sin(\frac{1}{2}x)} e^{i(N-1)x/2}\right] \\ &= \frac{\sin(\frac{1}{2}Nx)}{\sin(\frac{1}{2}x)} \sin[\tfrac{1}{2}x(N-1)]. \end{aligned} \qquad (6)$$

Similarly,

$$\begin{aligned}\sum_{n=0}^{\infty} p^n \sin(nx) &= \Im\left[\sum_{n=0}^{\infty} p^n e^{inx}\right] \\ &= \Im\left[\frac{1-pe^{-ix}}{1-2p\cos x + p^2}\right] = \frac{p\sin x}{1-2p\cos x+p^2}. \end{aligned} \qquad (7)$$

Other identities include

$$\sin(n\theta) = 2\cos\theta \sin[(n-1)\theta] - \sin[(n-2)\theta] \qquad (8)$$

$$\begin{aligned}\sin(nx) = \binom{n}{1} \cos^{n-1} x \sin x - \binom{n}{3} \cos^{n-3} x \sin^3 x \\ + \binom{n}{5} \cos^{n-5} x \sin^5 x - \ldots, \end{aligned} \qquad (9)$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT.

Cvijović and Klinowski (1995) show that the sum

$$S_\nu(\alpha) = \sum_{k=0}^{\infty} \frac{\sin(2k+1)\alpha}{(2k+1)^\nu} \tag{10}$$

has closed form for $\nu = 2n+1$,

$$S_{2n+1}(\alpha) = \frac{(-1)^n}{4(2n)!}\pi^{2n+1} E_{2n}\left(\frac{\alpha}{\pi}\right), \tag{11}$$

where $E_n(x)$ is an EULER POLYNOMIAL.

A CONTINUED FRACTION representation of $\sin x$ is

$$\sin x = \cfrac{x}{1 + \cfrac{x^2}{(2\cdot 3 - x^2) + \cfrac{2\cdot 3x^2}{(4\cdot 5 - x^2) + \cfrac{4\cdot 5x^2}{(6\cdot 7 - x^2) + \ldots}}}}. \tag{12}$$

The value of $\sin(2\pi/n)$ is IRRATIONAL for all n except 4 and 12, for which $\sin(\pi/2) = 1$ and $\sin(\pi/6) = 1/2$.

The FOURIER TRANSFORM of $\sin(2\pi k_0 x)$ is given by

$$\begin{aligned} \mathcal{F}[\sin(2\pi k_0 x)] &= \int_{-\infty}^{\infty} e^{-2\pi i k_0 x} \sin(2\pi k_0 x)\,dx \\ &= \tfrac{1}{2} i[\delta(k+k_0) - \delta(k-k_0)]. \end{aligned} \tag{13}$$

Definite integrals involving $\sin x$ include

$$\int_0^\infty \sin(x^2)\,dx = \tfrac{1}{4}\sqrt{2\pi} \tag{14}$$

$$\int_0^\infty \sin(x^3)\,dx = \tfrac{1}{6}\Gamma(\tfrac{1}{3}) \tag{15}$$

$$\int_0^\infty \sin(x^4)\,dx = -\cos(\tfrac{5}{8}\pi)\Gamma(\tfrac{5}{4}) \tag{16}$$

$$\int_0^\infty \sin(x^5)\,dx = \tfrac{1}{4}(\sqrt{5}-1)\Gamma(\tfrac{6}{5}), \tag{17}$$

where $\Gamma(x)$ is the GAMMA FUNCTION.

see also ANDREW'S SINE, COSECANT, COSINE, FOURIER TRANSFORM—SINE, HYPERBOLIC SINE, SINC FUNCTION, TANGENT, TRIGONOMETRY

References

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Sine-Gordon Equation

A PARTIAL DIFFERENTIAL EQUATION which appears in differential geometry and relativistic field theory. Its name is a pun on its similar form to the KLEIN-GORDON EQUATION. The sine-Gordon equation is

$$v_{tt} - v_{xx} + \sin v = 0, \tag{1}$$

where v_{tt} and v_{xx} are PARTIAL DERIVATIVES. The equation can be transformed by defining

$$\xi \equiv \tfrac{1}{2}(x-t) \tag{2}$$

$$\eta \equiv \tfrac{1}{2}(x+t), \tag{3}$$

giving

$$v_{\xi\eta} = \sin v. \tag{4}$$

Traveling wave analysis gives

$$z - z_0 = \sqrt{c^2-1}\int \frac{df}{\sqrt{2[d - 2\sin^2(\frac{1}{2}f)]}}. \tag{5}$$

For $d = 0$,

$$z - z_0 = \pm\sqrt{1-c^2}\,\ln[\pm\tan(\tfrac{1}{4}f)] \tag{6}$$

$$f(z) = \pm 4\tan^{-1}[e^{\pm(z-z_0)/(1-c^2)^{1/2}}]. \tag{7}$$

Letting $z \equiv \xi\eta$ then gives

$$zf'' + f' = \sin f. \tag{8}$$

Letting $g \equiv e^{if}$ gives

$$g'' - \frac{g'^2}{f} + \frac{2g' - g^2 + 1}{2z} = 0, \tag{9}$$

which is the third PAINLEVÉ TRANSCENDENT. Look for a solution of the form

$$v(x,t) = 4\tan^{-1}\left[\frac{\phi(x)}{\psi(t)}\right]. \tag{10}$$

Taking the partial derivatives gives

$$\phi_{xx} = -k^2\phi^4 + m^2\phi^2 + n^2 \tag{11}$$

$$\psi_{tt} = k^2\psi^4 + (m^2-1)\psi^2 - n^2, \tag{12}$$

which can be solved in terms of ELLIPTIC FUNCTIONS. A single SOLITON solution exists with $k = n = 0$, $m > 1$:

$$v = 4\tan^{-1}\left[\exp\left(\frac{\pm x - \beta t}{\sqrt{1-\beta^2}}\right)\right], \tag{13}$$

where

$$\beta \equiv \frac{\sqrt{m^2-1}}{m}. \tag{14}$$

A two-SOLITON solution exists with $k = 0$, $m > 1$:

$$v = 4\tan^{-1}\left[\frac{\sinh(\beta m x)}{\beta\cosh(\beta m t)}\right]. \tag{15}$$

A SOLITON-antisoliton solution exists with $k \neq 0$, $n = 0$, $m^2 > 1$:

$$v = -4\tan^{-1}\left[\frac{\sinh(\beta m x)}{\beta\cosh(m t)}\right]. \tag{16}$$

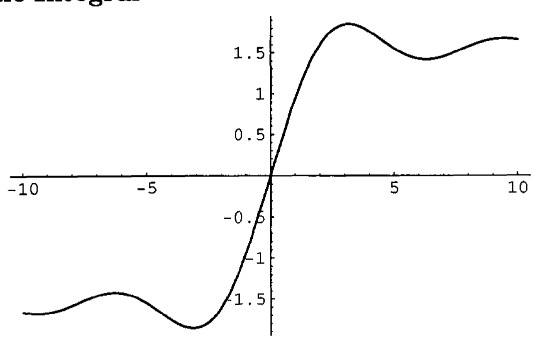
A "breather" solution is

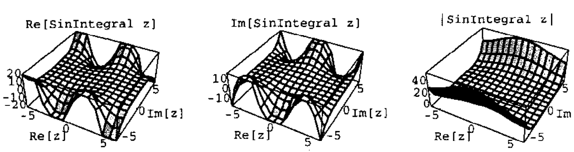
$$v = -4\tan^{-1}\left[\frac{m}{\sqrt{1-m^2}}\frac{\sin(\sqrt{1-m^2}\,t)}{\cosh(m x)}\right]. \tag{17}$$

References

Infeld, E. and Rowlands, G. *Nonlinear Waves, Solitons, and Chaos.* Cambridge, England: Cambridge University Press, pp. 199–200, 1990.

Sine Integral





There are two types of "sine integrals" commonly defined,

$$\mathrm{Si}(x) \equiv \int_0^z \frac{\sin t}{t}\,dt \tag{1}$$

and

$$\begin{aligned}\mathrm{si}(x) &\equiv -\int_x^\infty \frac{\sin t}{t}\,dt \\ &= \frac{1}{2i}[\mathrm{ei}(ix) - \mathrm{ei}(-ix)] \\ &= \frac{1}{2i}[\mathrm{e}_1(ix) - \mathrm{e}_1(-ix)] \qquad (3)\\ &= \mathrm{Si}(z) - \tfrac{1}{2}\pi, \qquad (4)\end{aligned} \tag{2}$$

where $\mathrm{ei}(x)$ is the EXPONENTIAL INTEGRAL and

$$\mathrm{e}_1(x) \equiv -\mathrm{ei}(-x). \tag{5}$$

$\mathrm{Si}(x)$ is the function returned by the *Mathematica*® (Wolfram Research, Champaign, IL) command `SinIntegral[x]` and displayed above. The half-infinite integral of the SINC FUNCTION is given by

$$\mathrm{si}(0) = -\int_0^\infty \frac{\sin x}{x}\,dx = -\tfrac{1}{2}\pi. \tag{6}$$

To compute the integral of a sine function times a power

$$I \equiv \int x^{2n}\sin(mx)\,dx, \tag{7}$$

use INTEGRATION BY PARTS. Let

$$u = x^{2n} \qquad dv = \sin(mx)\,dx \tag{8}$$

$$du = 2nx^{2n-1}\,dx \qquad v = -\frac{1}{m}\cos(mx), \tag{9}$$

so

$$I = -\frac{1}{m}x^{2n}\cos(mx) + \frac{2n}{m}\int x^{2n-1}\cos(mx)\,dx. \tag{10}$$

Using INTEGRATION BY PARTS again,

$$u = x^{2n-1} \qquad dv = \cos(mx)\,dx \tag{11}$$

$$du = (2n-1)x^{2n-2}\,dx \qquad v = \frac{1}{m}\sin(mx) \tag{12}$$

$$\begin{aligned}\int x^{2n}\sin(mx)\,dx &= -\frac{1}{m}x^{2n}\cos(mx) \\ &\quad + \frac{2n}{m}\left[\frac{1}{m}x^{2n-1}\cos(mx) \right. \\ &\quad \left. - \frac{2n-1}{m}\int x^{2n-2}\sin(mx)\,dx\right] \\ &= -\frac{1}{m}x^{2n}\sin(mx) + \frac{2n}{m^2}x^{2n-1}\sin(mx) \\ &\quad - \frac{(2n)(2n-1)}{m^2}\int x^{2n-2}\sin(mx)\,dx \\ &= -\frac{1}{m}x^{2n}\cos(mx) + \frac{2n}{m^2}x^{2n-1}\sin(mx) + \ldots \\ &\quad + \frac{(2n)!}{m^{2n}}\int x^0\sin(mx)\,dx \\ &= -\frac{1}{m}x^{2n}\cos(mx) + \frac{2n}{m^2}x^{2n-1}\sin(mx) + \ldots \\ &\quad - \frac{(2n)!}{m^{2n+1}}\cos(mx) \\ &= \cos(mx)\sum_{k=0}^{n}(-1)^{k+1}\frac{(2n)!}{(2n-2k)!\,m^{2k+1}}x^{2n-2k} \\ &\quad + \sin(mx)\sum_{k=1}^{n}(-1)^{k+1}\frac{(2n)!}{(2k-2n-1)!\,m^{2k}}x^{2n-2k+1}.\end{aligned} \tag{13}$$

Letting $k' \equiv n-k$, so

$$\begin{aligned}\int x^{2n}\sin(mx)\,dx &= \cos(mx)\sum_{k=0}^{n}(-1)^{n-k+1}\frac{(2n)!}{(2k)!m^{2n-2k+1}}x^{2k}\\ &\quad+\sin(mx)\sum_{k=0}^{n-1}(-1)^{n-k+1}\frac{(2n)!}{(2k-1)!m^{2n-2k}}x^{2k+1}\\ &= (-1)^{n+1}(2n)!\left[\cos(mx)\sum_{k=0}^{n}\frac{(-1)^k}{(2k)!m^{2n-2k+1}}x^{2k}\right.\\ &\quad\left.+\sin(mx)\sum_{k=1}^{n}\frac{(-1)^{k+1}}{(2k-3)!m^{2n-2k+2}}x^{2k-1}\right].\end{aligned} \tag{14}$$

General integrals of the form

$$I(k,l)=\int_0^\infty \frac{\sin^k x}{x^l}\,dx \tag{15}$$

are related to the SINC FUNCTION and can be computed analytically.

see also CHI, COSINE INTEGRAL, EXPONENTIAL INTEGRAL, NIELSEN'S SPIRAL, SHI, SICI SPIRAL, SINC FUNCTION

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Sine-Tangent Theorem

If

$$\frac{\sin\alpha}{\sin\beta}=\frac{m}{n},$$

then

$$\frac{\tan[\frac{1}{2}(\alpha-\beta)]}{\tan[\frac{1}{2}(\alpha+\beta)]}=\frac{m-n}{m+n}.$$

Sines Law

see LAW OF SINES

Singly Even Number

An EVEN NUMBER of the form $4n+2$ (i.e., an INTEGER which is DIVISIBLE by 2 but not by 4). The first few for $n=0, 1, 2, \ldots$ are 2, 6, 10, 14, 18, ... (Sloane's A016825)

see also DOUBLY EVEN NUMBER, EVEN NUMBER, ODD NUMBER

References

Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, p. 30, 1996.

Sloane, N. J. A. Sequence A016825 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Singular Homology

The general type of HOMOLOGY which is what mathematicians generally mean when they say "homology." Singular homology is a more general version than Poincaré's original SIMPLICIAL HOMOLOGY.

see also HOMOLOGY (TOPOLOGY), SIMPLICIAL HOMOLOGY

Singular Point (Algebraic Curve)

A singular point of an ALGEBRAIC CURVE is a point where the curve has "nasty" behavior such as a CUSP or a point of self-intersection (when the underlying field K is taken as the REALS). More formally, a point (a,b) on a curve $f(x,y)=0$ is singular if the x and y PARTIAL DERIVATIVES of f are both zero at the point (a,b). (If the field K is not the REALS or COMPLEX NUMBERS, then the PARTIAL DERIVATIVE is computed formally using the usual rules of CALCULUS.)

Consider the following two examples. For the curve

$$x^3-y^2=0,$$

the CUSP at (0, 0) is a singular point. For the curve

$$x^2+y^2=-1,$$

$(0,i)$ is a nonsingular point and this curve is nonsingular.

see also ALGEBRAIC CURVE, CUSP

Singular Point (Differential Equation)

Consider a second-order ORDINARY DIFFERENTIAL EQUATION

$$y''+P(x)y'+Q(x)y=0.$$

If $P(x)$ and $Q(x)$ remain FINITE at $x=x_0$, then x_0 is called an ORDINARY POINT. If either $P(x)$ or $Q(x)$ diverges as $x\to x_0$, then x_0 is called a singular point. Singular points are further classified as follows:

1. If either $P(x)$ or $Q(x)$ diverges as $x\to x_0$ but $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ remain FINITE as $x\to x_0$, then $x=x_0$ is called a REGULAR SINGULAR POINT (or NONESSENTIAL SINGULARITY).

2. If $P(x)$ diverges more quickly than $1/(x - x_0)$, so $(x - x_0)P(x)$ approaches INFINITY as $x \to x_0$, or $Q(x)$ diverges more quickly than $1/(x - x_0)^2 Q$ so that $(x - x_0)^2 Q(x)$ goes to INFINITY as $x \to x_0$, then x_0 is called an IRREGULAR SINGULARITY (or ESSENTIAL SINGULARITY).

see also IRREGULAR SINGULARITY, REGULAR SINGULAR POINT, SINGULARITY

References

Arfken, G. "Singular Points." §8.4 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 451–454, 1985.

Singular Point (Function)

Singular points (also simply called "singularities") are points z_0 in the DOMAIN of a FUNCTION f where f fails to be ANALYTIC. ISOLATED SINGULARITIES may be classified as ESSENTIAL SINGULARITIES, POLES, or REMOVABLE SINGULARITIES.

ESSENTIAL SINGULARITIES are POLES of INFINITE order.

A POLE of order n is a singularity z_0 of $f(z)$ for which the function $(z - z_0)^n f(z)$ is nonsingular and for which $(z - z_0)^k f(z)$ is singular for $k = 0, 1, \ldots, n - 1$.

REMOVABLE SINGULARITIES are singularities for which it is possible to assign a COMPLEX NUMBER in such a way that $f(z)$ becomes ANALYTIC. For example, the function $f(z) = z^2/z$ has a REMOVABLE SINGULARITY at 0, since $f(z) = z$ everywhere but 0, and $f(z)$ can be set equal to 0 at $z = 0$. REMOVABLE SINGULARITIES are not POLES.

The function $f(z) = \csc(1/z)$ has POLES at $z = 1/(2\pi n)$, and a nonisolated singularity at 0.

see also ESSENTIAL SINGULARITY, IRREGULAR SINGULARITY, ORDINARY POINT, POLE, REGULAR SINGULAR POINT, REMOVABLE SINGULARITY, SINGULAR POINT (DIFFERENTIAL EQUATION)

References

Arfken, G. "Singularities." §7.1 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 396–400, 1985.

Singular Series

$$\rho_{2s}(n) = \frac{\pi^s}{\Gamma(s)} n^{s-1} \sum_{p,q} \left(\frac{S_{p,q}}{q} \right)^{2s} e^{2np\pi i/q},$$

where $S_{p,q}$ is a GAUSSIAN SUM, and $\Gamma(s)$ is the GAMMA FUNCTION.

Singular System

A system is singular if the CONDITION NUMBER is INFINITE and ILL-CONDITIONED if it is too large.

see also CONDITION NUMBER, ILL-CONDITIONED

Singular Value

A MODULUS k_r such that

$$\frac{K'(k_r)}{K(k_r)} = \sqrt{r},$$

where $K(k)$ is a complete ELLIPTIC INTEGRAL OF THE FIRST KIND, and $K'(k_r) \equiv K(\sqrt{1 - k_r^2})$. The ELLIPTIC LAMBDA FUNCTION $\lambda^*(r)$ gives the value of k_r.

Abel (quoted in Whittaker and Watson 1990, p. 525) proved that if r is an INTEGER, or more generally whenever

$$\frac{K'(k)}{K(k)} = \frac{a + b\sqrt{n}}{c + d\sqrt{n}},$$

where a, b, c, d, and n are INTEGERS, then the MODULUS k is the ROOT of an algebraic equation with INTEGER COEFFICIENTS.

see also ELLIPTIC INTEGRAL SINGULAR VALUE, ELLIPTIC INTEGRAL OF THE FIRST KIND, ELLIPTIC LAMBDA FUNCTION, MODULUS (ELLIPTIC INTEGRAL)

References

Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis, 4th ed.* Cambridge, England: Cambridge University Press, pp. 524–528, 1990.

Singular Value Decomposition

An expansion of a REAL $M \times N$ MATRIX by ORTHOGONAL OUTER PRODUCTS according to

$$A = \sum_{k=1}^{K} s_k \mathbf{u}_k \mathbf{v}_k^{\mathrm{T}}, \tag{1}$$

where $s_1 \geq s_2 \geq \ldots \geq 0$,

$$K \equiv \min\{M, N\} \tag{2}$$

and

$$\mathbf{u}_k^{\mathrm{T}} \mathbf{u}_{k'} = \mathbf{v}_l^{\mathrm{T}} \mathbf{v}_{k'} = \delta_{kk'}. \tag{3}$$

Here δ_{ij} is the KRONECKER DELTA and A^{T} is the MATRIX TRANSPOSE.

see also CHOLESKY DECOMPOSITION, LU DECOMPOSITION, QR DECOMPOSITION

References

Nash, J. C. "The Singular-Value Decomposition and Its Use to Solve Least-Squares Problems." Ch. 3 in *Compact Numerical Methods for Computers: Linear Algebra and Function Minimisation, 2nd ed.* Bristol, England: Adam Hilger, pp. 30–48, 1990.
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Singular Value Decomposition." §2.6 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 51–63, 1992.

Singularity

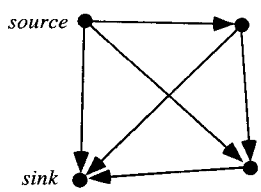
In general, a point at which an equation, surface, etc., blows up or becomes DEGENERATE.

see also ESSENTIAL SINGULARITY, ISOLATED SINGULARITY, SINGULAR POINT (ALGEBRAIC CURVE), SINGULAR POINT (DIFFERENTIAL EQUATION), SINGULAR POINT (FUNCTION), WHITNEY SINGULARITY

Sinh

see HYPERBOLIC SINE

Sink (Directed Graph)



A vertex of a DIRECTED GRAPH with no exiting edges, also called a TERMINAL.

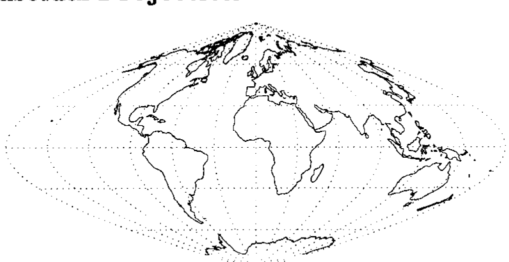
see also DIRECTED GRAPH, NETWORK, SOURCE

Sink (Map)

A stable fixed point of a MAP which, in a dissipative DYNAMICAL SYSTEM, is an ATTRACTOR.

see also ATTRACTOR, DYNAMICAL SYSTEM

Sinusoidal Projection



An equal AREA MAP PROJECTION.

$$x = (\lambda - \lambda_0)\cos\phi \quad (1)$$
$$y = \phi. \quad (2)$$

The inverse FORMULAS are

$$\phi = y \quad (3)$$
$$\lambda = \lambda_0 + \frac{x}{\cos\phi}. \quad (4)$$

References

Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, pp. 243–248, 1987.

Sinusoidal Spiral

A curve of the form

$$r^n = a^n \cos(n\theta)$$

with n RATIONAL, which is not a true SPIRAL. Sinusoidal spirals were first studied by Maclaurin. Special cases are given in the following table.

n	Curve
-2	hyperbola
-1	line
$-\frac{1}{2}$	parabola
$-\frac{1}{3}$	Tschirnhausen cubic
0	logarithmic spiral
$\frac{1}{3}$	Cayley sextic
$\frac{1}{2}$	cardioid
1	circle
2	Bernoulli lemniscate

References

Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, p. 184, 1972.

Lee, X. "Sinusoid." http://www.best.com/~xah/SpecialPlaneCurves_dir/Sinusoid_dir/sinusoid.html.

Lockwood, E. H. *A Book of Curves.* Cambridge, England: Cambridge University Press, p. 175, 1967.

MacTutor History of Mathematics Archive. "Sinusoidal Spirals." http://www-groups.dcs.st-and.ac.uk/~history/Curves/Sinusoidal.html.

Sinusoidal Spiral Inverse Curve

The INVERSE CURVE of a SINUSOIDAL SPIRAL

$$r = a^{(1/n)}[\cos(nt)]^{1/n}$$

with INVERSION CENTER at the origin and inversion radius k is another SINUSOIDAL SPIRAL

$$r = ka^{(1/n)}[\cos(nt)]^{1/n}.$$

Sinusoidal Spiral Pedal Curve

The PEDAL CURVE of a SINUSOIDAL SPIRAL

$$r = a^{(1/n)}[\cos(nt)]^{1/n}$$

with PEDAL POINT at the center is another SINUSOIDAL SPIRAL

$$x = \cos^{1+1/n}(nt)\cos[(n+1)t]$$
$$y = \cos^{1+1/n}(nt)\sin[(n+1)t].$$

Sister Celine's Method

A method for finding RECURRENCE RELATIONS for hypergeometric polynomials directly from the series expansions of the polynomials. The method is effective and easily implemented, but usually slower than ZEILBERGER'S ALGORITHM. Given a sum $f(n) = \sum_k F(n,k)$, the method operates by finding a recurrence of the form

$$\sum_{i=0}^{I}\sum_{j=0}^{J} a_{ij}(n)F(n-j,k-i) = 0$$

by proceeding as follows (Petkovšek *et al.* 1996, p. 59):

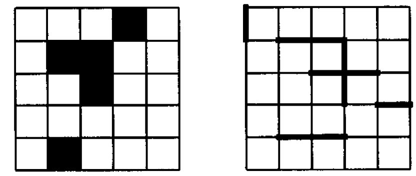
1. Fix trial values of I and J.
2. Assume a recurrence formula of the above form where $a_{ij}(n)$ are to be solved for.
3. Divide each term of the assumed recurrence by $F(n,k)$ and reduce every ratio $F(n-j,k-i)/F(n,k)$ by simplifying the ratios of its constituent factorials so that only RATIONAL FUNCTIONS in n and k remain.
4. Put the resulting expression over a common DENOMINATOR, then collect the numerator as a POLYNOMIAL in k.
5. Solve the system of linear equations that results after setting the coefficients of each power of k in the NUMERATOR to 0 for the unknown coefficients a_{ij}.
6. If no solution results, start again with larger I or J.

Under suitable hypotheses, a "fundamental theorem" (Verbaten 1974, Wilf and Zeilberger 1992, Petkovšek *et al.* 1996) guarantees that this algorithm always succeeds for large enough I and J (which can be estimated in advance). The theorem also generalizes to multivariate sums and to q- and multi-q-sums (Wilf and Zeilberger 1992, Petkovšek *et al.* 1996).

see also GENERALIZED HYPERGEOMETRIC FUNCTION, GOSPER'S ALGORITHM, HYPERGEOMETRIC IDENTITY, HYPERGEOMETRIC SERIES, ZEILBERGER'S ALGORITHM

References
Fasenmyer, Sister M. C. *Some Generalized Hypergeometric Polynomials.* Ph.D. thesis. University of Michigan, Nov. 1945.
Fasenmyer, Sister M. C. "Some Generalized Hypergeometric Polynomials." *Bull. Amer. Math. Soc.* **53**, 806–812, 1947.
Fasenmyer, Sister M. C. "A Note on Pure Recurrence Relations." *Amer. Math. Monthly* **56**, 14–17, 1949.
Petkovšek, M.; Wilf, H. S.; and Zeilberger, D. "Sister Celine's Method." Ch. 4 in *A=B.* Wellesley, MA: A. K. Peters, pp. 55–72, 1996.
Rainville, E. D. Chs. 14 and 18 in *Special Functions.* New York: Chelsea, 1971.
Verbaten, P. "The Automatic Construction of Pure Recurrence Relations." *Proc. EUROSAM '74, ACM-SIGSAM Bull.* **8**, 96–98, 1974.
Wilf, H. S. and Zeilberger, D. "An Algorithmic Proof Theory for Hypergeometric (Ordinary and "q") Multisum/Integral Identities." *Invent. Math.* **108**, 575–633, 1992.

Site Percolation



site percolation *bond percolation*

A PERCOLATION which considers the lattice vertices as the relevant entities (left figure).

see also BOND PERCOLATION, PERCOLATION THEORY

Siteswap

A siteswap is a sequence encountered in JUGGLING in which each term is a POSITIVE integer, encoded in BINARY. The transition rule from one term to the next consists of changing some 0 to 1, subtracting 1, and then dividing by 2, with the constraint that the DIVISION by two must be exact. Therefore, if a term is EVEN, the bit to be changed must be the units bit. In siteswaps, the number of 1-bits is a constant.

Each transition is characterized by the bit position of the toggled bit (denoted here by the numeral on top of the arrow). For example,

$$\begin{aligned}111 &\xrightarrow{5} 10011 \xrightarrow{2} 1011 \xrightarrow{5} 10101 \xrightarrow{1} 1011 \xrightarrow{2} 111 \\ &\xrightarrow{6} 100011 \xrightarrow{3} 10101 \xrightarrow{3} 1110 \xrightarrow{0} 111 \xrightarrow{4} 1011 \ldots\end{aligned}$$

The second term is given from the first as follows: 000111 with bit 5 flipped becomes 100111, or 39. Subtract 1 to obtain 38 and divide by two to obtain 19, which is 10011.

see also JUGGLING

References
Juggling Information Service. "Siteswaps." `http://www.juggling.org/help/siteswap`.

Six-Color Theorem

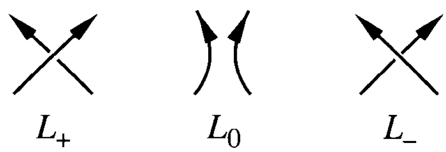
To color any map on the SPHERE or the PLANE requires at most six-colors. This number can be easily be reduced to five, and the FOUR-COLOR THEOREM demonstrates that the NECESSARY number is, in fact, four.

see also FOUR-COLOR THEOREM, HEAWOOD CONJECTURE, MAP COLORING

References
Franklin, P. "A Six Colour Problem." *J. Math. Phys.* **13**, 363–369, 1934.
Hoffman, I. and Soifer, A. "Another Six-Coloring of the Plane." *Disc. Math.* **150**, 427–429, 1996.
Saaty, T. L. and Kainen, P. C. *The Four-Color Problem: Assaults and Conquest.* New York: Dover, 1986.

Skein Relationship

A relationship between KNOT POLYNOMIALS for links in different orientations (denoted below as L_+, L_0, and L_-). J. H. Conway was the first to realize that the ALEXANDER POLYNOMIAL could be defined by a relationship of this type.



see also ALEXANDER POLYNOMIAL, HOMFLY POLYNOMIAL, SIGNATURE (KNOT)

Skeleton

The GRAPH obtained by collapsing a POLYHEDRON into the PLANE. The number of topologically distinct skeletons $N(n)$ with n VERTICES is given in the following table.

n	$N(n)$
4	1
5	2
6	7

References

Gardner, M. *Martin Gardner's New Mathematical Diversions from Scientific American.* New York: Simon and Schuster, p. 233, 1966.

Skeleton Division

A LONG DIVISION in which most or all of the digits are replaced by a symbol (usually asterisks) to form a CRYPTARITHM.

see also CRYPTARITHM

Skew Conic

Also known as a GAUCHE CONIC, SPACE CONIC, TWISTED CONIC, or CUBICAL CONIC SECTION. A third-order SPACE CURVE having up to three points in common with a plane and having three points in common with the plane at infinity. A skew cubic is determined by six points, with no four of them COPLANAR. A line is met by up to four tangents to a skew cubic.

A line joining two points of a skew cubic (REAL or conjugate imaginary) is called a SECANT of the curve, and a line having one point in common with the curve is called a SEMISECANT or TRANSVERSAL. Depending on the nature of the roots, the skew conic is classified as follows:

1. The three ROOTS are REAL and distinct (CUBICAL HYPERBOLA).
2. One root is REAL and the other two are COMPLEX CONJUGATES (CUBICAL ELLIPSE).
3. Two of the ROOTS coincide (CUBICAL PARABOLIC HYPERBOLA).
4. All three ROOTS coincide (CUBICAL PARABOLA).

see also CONIC SECTION, CUBICAL ELLIPSE, CUBICAL HYPERBOLA, CUBICAL PARABOLA, CUBICAL PARABOLIC HYPERBOLA

Skew Field

A FIELD in which the commutativity of multiplication is not required, more commonly called a DIVISION ALGEBRA.

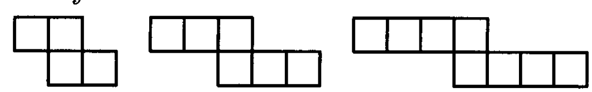
see also DIVISION ALGEBRA, FIELD

Skew Lines

Two or more LINES which have no intersections but are not PARALLEL, also called AGONIC LINES. Since two LINES in the PLANE must intersect or be PARALLEL, skew lines can exist only in three or more DIMENSIONS.

see also GALLUCCI'S THEOREM, REGULUS

Skew Polyomino



see also L-POLYOMINO, SQUARE POLYOMINO, STRAIGHT POLYOMINO, T-POLYOMINO

Skew Quadrilateral

A four-sided QUADRILATERAL not contained in a plane. The problem of finding the minimum bounding surface of a skew quadrilateral was solved by Schwarz (1890) in terms of ABELIAN INTEGRALS and has the shape of a SADDLE. It is given by solving

$$(1 + {f_y}^2)f_{xx} - 2f_x f_y f_{xy} + (1 + {f_x}^2)f_{yy} = 0.$$

see also QUADRILATERAL

References

Isenberg, C. *The Science of Soap Films and Soap Bubbles.* New York: Dover, p. 81, 1992.

Forsyth, A. R. *Calculus of Variations.* New York: Dover, p. 503, 1960.

Schwarz, H. A. *Gesammelte Mathematische Abhandlungen, 2nd ed.* New York: Chelsea.

Skew Symmetric Matrix

A MATRIX A where

$$\mathsf{A}^{\mathrm{T}} = -\mathsf{A},$$

with A^{T} denoting the MATRIX TRANSPOSE.

see also MATRIX TRANSPOSE, SYMMETRIC MATRIX

Skewes Number

The Skewes number (or first Skewes number) is the number Sk_1 above which $\pi(n) < \mathrm{Li}(n)$ must fail (assuming that the RIEMANN HYPOTHESIS is true), where $\pi(n)$ is the PRIME COUNTING FUNCTION and $\mathrm{Li}(n)$ is the LOGARITHMIC INTEGRAL.

$$\mathrm{Sk}_1 = e^{e^{e^{79}}} \approx 10^{10^{10^{34}}}.$$

The Skewes number has since been reduced to $e^{e^{27/4}} \approx 8.185 \times 10^{370}$ by te Riele (1987), although Conway and Guy (1996) claim that the best current limit is 10^{1167}. In 1914, Littlewood proved that the inequality must, in fact, fail infinitely often.

The second Skewes number Sk_2 is the number above which $\pi(n) < \mathrm{Li}(n)$ must fail (assuming that the RIEMANN HYPOTHESIS is false). It is much larger than the Skewes number Sk_1,

$$\mathrm{Sk}_2 = 10^{10^{10^{10^{3}}}}.$$

see also GRAHAM'S NUMBER, RIEMANN HYPOTHESIS

References

Asimov, I. "Skewered!" *Of Matters Great and Small.* New York: Ace Books, 1976. Originally published in *Magazine of Fantasy and Science Fiction,* Nov. 1974.

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 63, 1987.

Boas, R. P. "The Skewes Number." In *Mathematical Plums* (Ed. R. Honsberger). Washington, DC: Math. Assoc. Amer., 1979.

Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, p. 61, 1996.

Lehman, R. S. "On the Difference $\pi(x) - \mathrm{li}(x)$." *Acta Arith.* **11**, 397–410, 1966.

te Riele, H. J. J. "On the Sign of the Difference $\pi(x) - \mathrm{li}(x)$." *Math. Comput.* **48**, 323–328, 1987.

Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, p. 30, 1991.

Skewness

The degree of asymmetry of a distribution. If the distribution has a longer tail less than the maximum, the function has NEGATIVE skewness. Otherwise, it has POSITIVE skewness. Several types of skewness are defined. The FISHER SKEWNESS is defined by

$$\gamma_1 = \frac{\mu_3}{\mu_2{}^{3/2}} = \frac{\mu_3}{\sigma^3}, \tag{1}$$

where μ_3 is the third MOMENT, and $\mu_2{}^{1/2} \equiv \sigma$ is the STANDARD DEVIATION. The PEARSON SKEWNESS is defined by

$$\beta_1 = \left(\frac{\mu_3}{\sigma^3}\right)^2 = \gamma_1{}^2. \tag{2}$$

The MOMENTAL SKEWNESS is defined by

$$\alpha^{(m)} \equiv \tfrac{1}{2}\gamma_1. \tag{3}$$

The PEARSON MODE SKEWNESS is defined by

$$\frac{[\text{mean}] - [\text{mode}]}{\sigma}. \tag{4}$$

PEARSON'S SKEWNESS COEFFICIENTS are defined by

$$\frac{3[\text{mean}] - [\text{mode}]}{s} \tag{5}$$

and

$$\frac{3[\text{mean}] - [\text{median}]}{s}. \tag{6}$$

The BOWLEY SKEWNESS (also known as QUARTILE SKEWNESS COEFFICIENT) is defined by

$$\frac{(Q_3 - Q_2) - (Q_2 - Q_1)}{Q_3 - Q_1} = \frac{Q_1 - 2Q_2 + Q_3}{Q_3 - Q_1}, \tag{7}$$

where the Qs denote the INTERQUARTILE RANGES. The MOMENTAL SKEWNESS is

$$\alpha^{(m)} \equiv \tfrac{1}{2}\gamma = \frac{\mu_3}{2\sigma^3}. \tag{8}$$

An ESTIMATOR for the FISHER SKEWNESS γ_1 is

$$g_1 = \frac{k_3}{k_2{}^{3/2}}, \tag{9}$$

where the ks are k-STATISTICS. The STANDARD DEVIATION of g_1 is

$$\sigma_{g_1}{}^2 \approx \frac{6}{N}. \tag{10}$$

see also BOWLEY SKEWNESS, FISHER SKEWNESS, GAMMA STATISTIC, KURTOSIS, MEAN, MOMENTAL SKEWNESS, PEARSON SKEWNESS, STANDARD DEVIATION

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 928, 1972.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Moments of a Distribution: Mean, Variance, Skewness, and So Forth." §14.1 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 604–609, 1992.

Sklar's Theorem

Let H be a 2-D distribution function with marginal distribution functions F and G. Then there exists a COPULA C such that

$$H(x,y) = C(F(x), G(y)).$$

Conversely, for any univariate distribution functions F and G and any COPULA C, the function H is a two-dimensional distribution function with marginals F and G. Furthermore, if F and G are continuous, then C is unique.

Skolem-Mahler-Lerch Theorem

If $\{a_0, a_1, \ldots\}$ is a Recurrence Sequence, then the set of all k such that $a_k = 0$ is the union of a finite (possibly Empty) set and a finite number (possibly zero) of full arithmetical progressions, where a full arithmetic progression is a set of the form $\{r, r+d, r+2d, \ldots\}$ with $r \in [0, d)$.

References

Myerson, G. and van der Poorten, A. J. "Some Problems Concerning Recurrence Sequences." *Amer. Math. Monthly* **102**, 698–705, 1995.

Skolem Paradox

Even though Arithmetic is uncountable, it possesses a countable "model."

Skolem Sequence

A Skolem sequence of order n is a sequence $S = \{s_1, s_2, \ldots, s_{2n}\}$ of $2n$ integers such that

1. For every $k \in \{1, 2, \ldots, n\}$, there exist exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$, and
2. If $s_i = s_j = k$ with $i < j$, then $j - i = k$.

References

Colbourn, C. J. and Dinitz, J. H. (Eds.) "Skolem Sequences." Ch. 43 in *CRC Handbook of Combinatorial Designs.* Boca Raton, FL: CRC Press, pp. 457–461, 1996.

Slant Height

The height of an object (such as a Cone) measured along a side from the edge of the base to the apex.

Slice Knot

A Knot K in $\mathbb{S}^3 = \partial\mathbb{D}^4$ is a slice knot if it bounds a Disk Δ^2 in $\mathbb{D}^4$ which has a Tubular Neighborhood $\Delta^2 \times \mathbb{D}^2$ whose intersection with $\mathbb{S}^3$ is a Tubular Neighborhood $K \times \mathbb{D}^2$ for K.

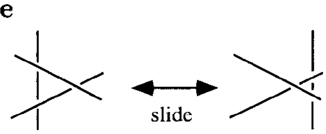
Every Ribbon Knot is a slice knot, and it is conjectured that every slice knot is a Ribbon Knot.

see also Ribbon Knot, Tubular Neighborhood

References

Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, p. 218, 1976.

Slide Move



The Reidemeister Move of type III.

see also Reidemeister Moves

Slide Rule

A mechanical device consisting of a sliding portion and a fixed case, each marked with logarithmic axes. By lining up the ticks, it is possible to do Multiplication by taking advantage of the additive property of Logarithms. More complicated slide rules also allow the extraction of roots and computation of trigonometric functions. The development of the desk calculator (and subsequently pocket calculator) rendered slide rules largely obsolete beginning in the 1960s.

see also Abacus, Ruler, Straightedge

References

Electronic Teaching Laboratories. *Simplify Math: Learn to Use the Slide Rule.* New Augusta, IN: Editors and Engineers, 1966.

Saffold, R. *The Slide Rule.* Garden City, NY: Doubleday, 1962.

Slightly Defective Number

see Almost Perfect Number

Slightly Excessive Number

see Quasiperfect Number

Slip Knot

see Running Knot

Slope

A quantity which gives the inclination of a curve or line with respect to another curve or line. For a Line in the Plane making an Angle θ with the x-Axis, the Slope m is a constant given by

$$m \equiv \frac{\Delta y}{\Delta x} = \tan\theta,$$

where Δx and Δy are changes in the two coordinates over some distance. It is meaningless to talk about the slope in 3-D unless the slope *with respect to what* is specified.

Slothouber-Graatsma Puzzle

Assemble six $1 \times 2 \times 2$ blocks and three $1 \times 1 \times 1$ blocks into a $3 \times 3 \times 3$ Cube.

see also Box-Packing Theorem, Conway Puzzle, Cube Dissection, de Bruijn's Theorem, Klarner's Theorem, Polycube

References

Honsberger, R. *Mathematical Gems II.* Washington, DC: Math. Assoc. Amer., pp. 75–77, 1976.

Slutzky-Yule Effect

A Moving Average may generate an irregular oscillation even if none exists in the original data.

see also Moving Average

Sluze Pearls

see PEARLS OF SLUZE

Smale-Hirsch Theorem

The SPACE of IMMERSIONS of a MANIFOLD in another MANIFOLD is HOMOTOPICALLY equivalent to the space of bundle injections from the TANGENT SPACE of the first to the TANGENT BUNDLE of the second.

see also HOMOTOPY, IMMERSION, MANIFOLD, TANGENT BUNDLE, TANGENT SPACE

Smale Horseshoe Map

The basic topological operations for constructing an ATTRACTOR consist of stretching (which gives sensitivity to initial conditions) and folding (which gives the attraction). Since trajectories in PHASE SPACE cannot cross, the repeated stretching and folding operations result in an object of great topological complexity.

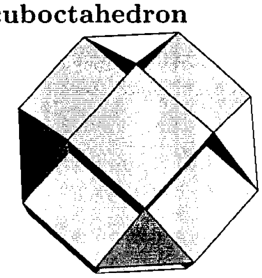
The Smale horseshoe map consists of a sequence of operations on the unit square. First, stretch by a factor of 2 in the x direction, then compress by $2a$ in the y direction. Then, fold the rectangle and fit it back into the square. Repeating this generates the horseshoe attractor. If one looks at a cross-section of the final structure, it is seen to correspond to a CANTOR SET.

see also ATTRACTOR, CANTOR SET

References

Gleick, J. *Chaos: Making a New Science.* New York: Penguin, pp. 50–51, 1988.
Rasband, S. N. *Chaotic Dynamics of Nonlinear Systems.* New York: Wiley, p. 77, 1990.
Tabor, M. *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, 1989.

Small Cubicuboctahedron



UNIFORM POLYHEDRON U_{13} whose DUAL POLYHEDRON is the SMALL HEXACRONIC ICOSITETRAHEDRON. It has WYTHOFF SYMBOL $\frac{3}{2}\,4\,|\,4$. Its faces are $8\{3\}+6\{4\}+6\{8\}$. The CIRCUMRADIUS for the solid with unit edge length is

$$R=\tfrac{1}{2}\sqrt{5+2\sqrt{2}}.$$

FACETED versions include the GREAT RHOMBICUBOCTAHEDRON (UNIFORM) and SMALL RHOMBIHEXAHEDRON.

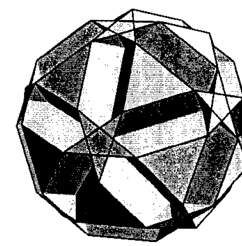
References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 104–105, 1971.

Small Ditrigonal Dodecacronic Hexecontahedron

The DUAL POLYHEDRON of the SMALL DITRIGONAL DODECICOSIDODECAHEDRON.

Small Ditrigonal Dodecicosidodecahedron



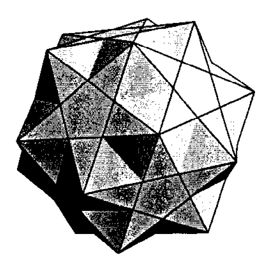
The UNIFORM POLYHEDRON U_{43} whose DUAL POLYHEDRON is the SMALL DITRIGONAL DODECACRONIC HEXECONTAHEDRON. It has WYTHOFF SYMBOL $3\,\frac{5}{3}\,|\,5$. Its faces are $20\{3\}+12\{\frac{5}{2}\}+12\{10\}$. Its CIRCUMRADIUS with $a=1$ is

$$R=\tfrac{1}{4}\sqrt{34+6\sqrt{5}}.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 126–127, 1971.

Small Ditrigonal Icosidodecahedron



The UNIFORM POLYHEDRON U_{30} whose DUAL POLYHEDRON is the SMALL TRIAMBIC ICOSAHEDRON. It has WYTHOFF SYMBOL $3\,|\,3\,\frac{5}{2}$. Its faces are $20\{3\}+12\{\frac{5}{2}\}$. A FACETED version is the DITRIGONAL DODECADODECAHEDRON. Its CIRCUMRADIUS with $a=1$ is

$$R=\tfrac{1}{2}\sqrt{3}.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 106–107, 1971.

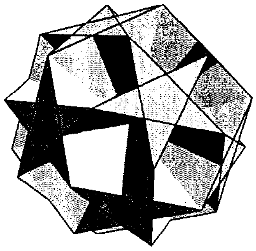
Small Dodecacronic Hexecontahedron

The DUAL POLYHEDRON of the SMALL DODECICOSIDODECAHEDRON.

Small Dodecahemicosacron

The DUAL POLYHEDRON of the SMALL DODECAHEMICOSAHEDRON.

Small Dodecahemicosahedron



The UNIFORM POLYHEDRON U_{62} whose DUAL POLYHEDRON is the SMALL DODECAHEMICOSACRON. It has WYTHOFF SYMBOL $\frac{5}{3}\,\frac{5}{2}\,|\,3$. Its faces are $10\{6\}+12\{\frac{5}{2}\}$. It is a FACETED version of the ICOSIDODECAHEDRON. Its CIRCUMRADIUS with unit edge length is

$$R=1.$$

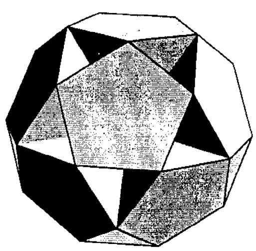
References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, p. 155, 1971.

Small Dodecahemidodecacron

The DUAL POLYHEDRON of the SMALL DODECAHEMIDODECAHEDRON.

Small Dodecahemidodecahedron



The UNIFORM POLYHEDRON U_{51} whose DUAL POLYHEDRON is the SMALL DODECAHEMIDODECACRON. It has WYTHOFF SYMBOL $2\,5\,\frac{\frac{3}{2}}{\frac{5}{2}}$. Its faces are $30\{4\}+12\{10\}$. Its CIRCUMRADIUS with $a=1$ is

$$R=\tfrac{1}{2}\sqrt{11+4\sqrt{5}}\,.$$

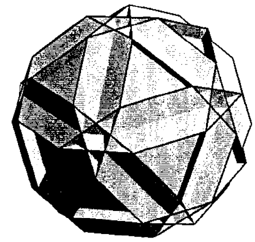
References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 113–114, 1971.

Small Dodecicosacron

The DUAL POLYHEDRON of the SMALL DODECICOSAHEDRON.

Small Dodecicosahedron



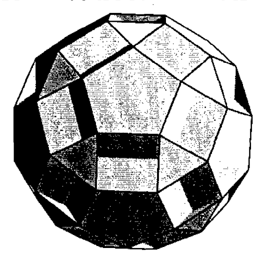
The UNIFORM POLYHEDRON U_{50} whose DUAL POLYHEDRON is the SMALL DODECICOSACRON. It has WYTHOFF SYMBOL $3\,5\,\frac{\frac{3}{2}}{\frac{5}{4}}\,\Big|$. Its faces are $20\{6\}+12\{10\}$. Its CIRCUMRADIUS with $a=1$ is

$$R=\tfrac{1}{4}\sqrt{34+6\sqrt{5}}\,.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 141–142, 1971.

Small Dodecicosidodecahedron



The UNIFORM POLYHEDRON U_{33} whose DUAL POLYHEDRON is the SMALL DODECACRONIC HEXECONTAHEDRON. It has WYTHOFF SYMBOL $\frac{3}{2}\,5\,|\,5$. Its faces are $20\{3\}+12\{5\}+12\{10\}$. It is a FACETED version of the SMALL RHOMBICOSIDODECAHEDRON. Its CIRCUMRADIUS with $a=1$ is

$$R=\tfrac{1}{2}\sqrt{11+4\sqrt{5}}\,.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 110–111, 1971.

Small Hexacronic Icositetrahedron

The DUAL POLYHEDRON of the SMALL CUBICUBOCTAHEDRON.

Small Hexagonal Hexecontahedron

The DUAL POLYHEDRON of the SMALL SNUB ICOSICOSIDODECAHEDRON.

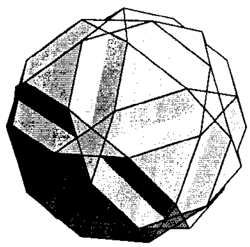
Small Hexagrammic Hexecontahedron

The DUAL POLYHEDRON of the SMALL RETROSNUB ICOSICOSIDODECAHEDRON.

Small Icosacronic Hexecontahedron

The DUAL POLYHEDRON of the SMALL ICOSICOSIDODECAHEDRON.

Small Icosicosidodecahedron



The UNIFORM POLYHEDRON U_{31} whose DUAL POLYHEDRON is the SMALL ICOSACRONIC HEXECONTAHEDRON. It has WYTHOFF SYMBOL $\frac{5}{4}\,5\,|\,5$. Its faces are $12\{5\}+6\{10\}$. Its CIRCUMRADIUS with $a=1$ is

$$R=\phi=\tfrac{1}{2}(1+\sqrt{5}),$$

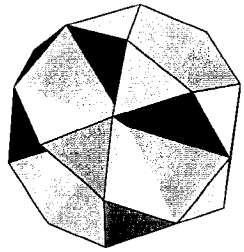
where ϕ is the GOLDEN RATIO.

References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, p. 143, 1971.

Small Icosihemidodecacron

The DUAL POLYHEDRON of the SMALL ICOSIHEMIDODECAHEDRON.

Small Icosihemidodecahedron



The UNIFORM POLYHEDRON U_{49} whose DUAL POLYHEDRON is the SMALL ICOSIHEMIDODECACRON. It has WYTHOFF SYMBOL $\frac{3}{2}\,3\,|\,5$. Its faces are $20\{3\}+6\{10\}$. It is a FACETED version of the ICOSIDODECAHEDRON. Its CIRCUMRADIUS with $a=1$ is

$$R=\phi=\tfrac{1}{2}(1+\sqrt{5}).$$

References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, p. 140, 1971.

Small Inverted Retrosnub Icosicosidodecahedron

see SMALL RETROSNUB ICOSICOSIDODECAHEDRON

Small Multiple Method

An algorithm for computing a UNIT FRACTION.

Small Number

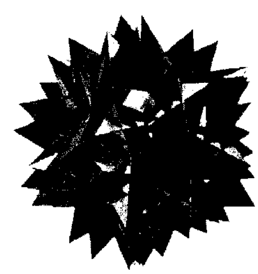
Guy's "STRONG LAW OF SMALL NUMBERS" states that there aren't enough small numbers to meet the many demands made of them. Guy (1988) also gives several interesting and misleading facts about small numbers:

1. 10% of the first 100 numbers are SQUARE NUMBERS.
2. A QUARTER of the numbers <100 are PRIMES.
3. All numbers less than 10, except for 6, are PRIME POWERS.
4. Half the numbers less than 10 are FIBONACCI NUMBERS.

see also LARGE NUMBER, STRONG LAW OF SMALL NUMBERS

References
Guy, R. K. "The Strong Law of Small Numbers." *Amer. Math. Monthly* **95**, 697–712, 1988.

Small Retrosnub Icosicosidodecahedron

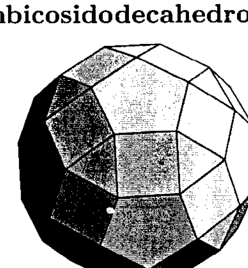


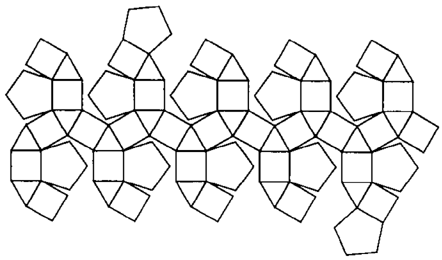
The UNIFORM POLYHEDRON U_{72} also called the SMALL INVERTED RETROSNUB ICOSICOSIDODECAHEDRON whose DUAL POLYHEDRON is the SMALL HEXAGRAMMIC HEXECONTAHEDRON. It has WYTHOFF SYMBOL $|\,\frac{3}{2}\,\frac{3}{2}\,\frac{5}{2}$. Its faces are $100\{3\}+12\{\frac{5}{2}\}$. It has CIRCUMRADIUS with $a=1$

$$R=\tfrac{1}{4}\sqrt{13+3\sqrt{5}-\sqrt{102+46\sqrt{5}}}$$
$$\approx 0.580694800133921.$$

References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 194–199, 1971.

Small Rhombicosidodecahedron



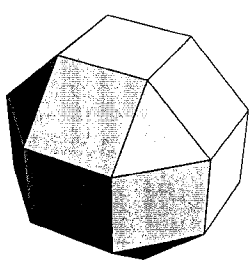


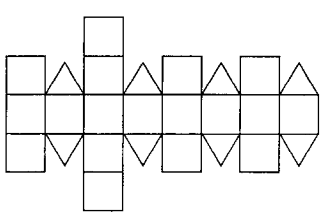
An ARCHIMEDEAN SOLID whose DUAL POLYHEDRON is the DELTOIDAL HEXECONTAHEDRON. It has SCHLÄFLI SYMBOL $\mathrm{r}\left\{{3 \atop 5}\right\}$. It is also UNIFORM POLYHEDRON U_{27} with WYTHOFF SYMBOL $3\,5\,|\,2$. Its faces are $20\{3\} + 30\{4\} + 12\{5\}$. The SMALL DODECICOSIDODECAHEDRON and SMALL RHOMBIDODECAHEDRON are FACETED versions. The INRADIUS, MIDRADIUS, and CIRCUMRADIUS for $a = 1$ are

$$r = \tfrac{1}{41}(15 + 2\sqrt{5})\sqrt{11 + 4\sqrt{5}} = 2.12099\ldots$$
$$\rho = \tfrac{1}{2}\sqrt{10 + 4\sqrt{5}} = 2.17625\ldots$$
$$R = \tfrac{1}{2}\sqrt{11 + 4\sqrt{5}} = 2.23295\ldots.$$

see also GREAT RHOMBICOSIDODECAHEDRON (ARCHIMEDEAN), GREAT RHOMBICOSIDODECAHEDRON (UNIFORM)

Small Rhombicuboctahedron





An ARCHIMEDEAN SOLID also (inappropriately) called the TRUNCATED ICOSIDODECAHEDRON. This name is inappropriate since truncation would yield rectangular instead of square faces. Its DUAL POLYHEDRON is the DELTOIDAL ICOSITETRAHEDRON, also called the TRAPEZOIDAL ICOSITETRAHEDRON. It has SCHLÄFLI SYMBOL $\mathrm{r}\left\{{3 \atop 4}\right\}$. It is also UNIFORM POLYHEDRON U_{10} and has WYTHOFF SYMBOL $3\,4\,|\,2$. Its INRADIUS, MIDRADIUS, and CIRCUMRADIUS for $a = 1$ are

$$r = \tfrac{1}{17}(6 + \sqrt{2})\sqrt{5 + 2\sqrt{2}} = 1.22026\ldots$$
$$\rho = \tfrac{1}{2}\sqrt{4 + 2\sqrt{2}} = 1.30656\ldots$$
$$R = \tfrac{1}{2}\sqrt{5 + 2\sqrt{2}} = 1.39897\ldots.$$

A version in which the top and bottom halves are rotated with respect to each other is known as the ELONGATED SQUARE GYROBICUPOLA.

see also ELONGATED SQUARE GYROBICUPOLA, GREAT RHOMBICUBOCTAHEDRON (ARCHIMEDEAN), GREAT RHOMBICUBOCTAHEDRON (UNIFORM)

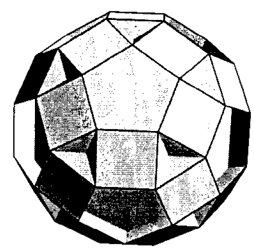
References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 137–138, 1987.

Small Rhombidodecacron

The DUAL POLYHEDRON of the SMALL RHOMBIDODECAHEDRON.

Small Rhombidodecahedron



The UNIFORM POLYHEDRON U_{39} whose DUAL POLYHEDRON is the SMALL RHOMBIDODECACRON. It has WYTHOFF SYMBOL $2\,5\,\frac{3}{2}\,\frac{5}{2}$. Its faces are $30\{4\} + 12\{10\}$. It is a FACETED version of the SMALL RHOMBICOSIDODECAHEDRON. Its CIRCUMRADIUS with $a = 1$ is

$$R = \tfrac{1}{2}\sqrt{11 + 4\sqrt{5}}.$$

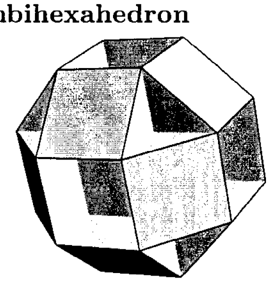
References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 113–114, 1971.

Small Rhombihexacron

The DUAL POLYHEDRON of the SMALL RHOMBIHEXAHEDRON.

Small Rhombihexahedron

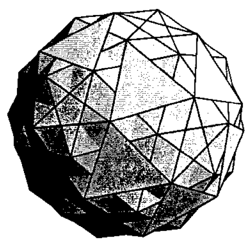


The UNIFORM POLYHEDRON U_{18} whose DUAL POLYHEDRON is the SMALL RHOMBIHEXACRON. It has WYTHOFF SYMBOL $2\,4\,\frac{3}{2}\,\frac{4}{2}\,|$. Its faces are $12\{4\} + 6\{8\}$. It is a FACETED version of the SMALL RHOMBICUBOCTAHEDRON. Its CIRCUMRADIUS with $a = 1$ is

$$R = \tfrac{1}{2}\sqrt{5+2\sqrt{2}}.$$

References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, p. 134, 1971.

Small Snub Icosicosidodecahedron



The UNIFORM POLYHEDRON U_{32} whose DUAL POLYHEDRON is the SMALL HEXAGONAL HEXECONTAHEDRON. It has WYTHOFF SYMBOL $|\,3\,3\,\frac{5}{2}$ (Har'El 1993 gives the symbol as $|\,\frac{5}{2}\,3\,3$.) Its faces are $100\{3\} + 12\{\frac{5}{2}\}$. Its CIRCUMRADIUS for $a = 1$ is

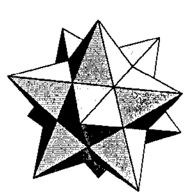
$$\begin{aligned} R &= \tfrac{1}{4}\sqrt{13+3\sqrt{5}+\sqrt{102+46\sqrt{5}}} \\ &= 1.4581903307387\ldots. \end{aligned}$$

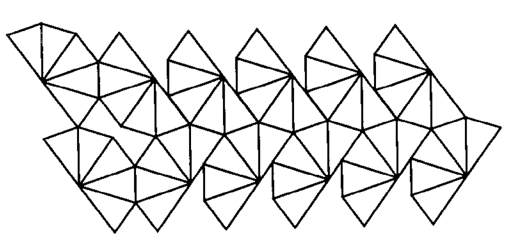
References
Har'El, Z. "Uniform Solution for Uniform Polyhedra." *Geometriae Dedicata* **47**, 57–110, 1993.
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 172–173, 1971.

Small Stellapentakis Dodecahedron

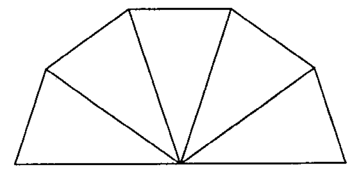
The DUAL POLYHEDRON of the TRUNCATED GREAT DODECAHEDRON.

Small Stellated Dodecahedron





One of the KEPLER-POINSOT SOLIDS whose DUAL POLYHEDRON is the GREAT DODECAHEDRON. Its SCHLÄFLI SYMBOL is $\{\frac{5}{2}, 5\}$. It is also UNIFORM POLYHEDRON U_{34} and has WYTHOFF SYMBOL $5\,|\,2\,\frac{5}{2}$. It was originally called the URCHIN by Kepler. It is composed of 12 PENTAGRAMMIC faces. Its faces are $12\{\frac{5}{2}\}$. The easiest way to construct it is to build twelve pentagonal PYRAMIDS



and attach them to the faces of a DODECAHEDRON. The CIRCUMRADIUS of the small stellated dodecahedron with $a = 1$ is

$$R = \tfrac{1}{2}5^{1/4}\phi^{-1/2} = \tfrac{1}{4}5^{1/4}\sqrt{2(\sqrt{5}-1)}.$$

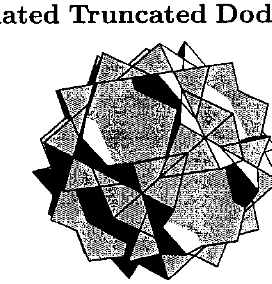
see also GREAT DODECAHEDRON, GREAT ICOSAHEDRON, GREAT STELLATED DODECAHEDRON, KEPLER-POINSOT SOLID

References
Fischer, G. (Ed.). Plate 103 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, p. 102, 1986.
Rawles, B. *Sacred Geometry Design Sourcebook: Universal Dimensional Patterns.* Nevada City, CA: Elysian Pub., p. 219, 1997.

Small Stellated Triacontahedron

see MEDIAL RHOMBIC TRIACONTAHEDRON

Small Stellated Truncated Dodecahedron

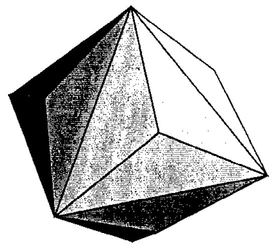


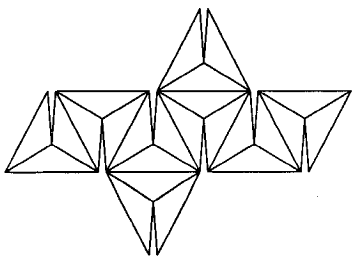
The UNIFORM POLYHEDRON U_{58} also called the QUASITRUNCATED SMALL STELLATED DODECAHEDRON whose DUAL POLYHEDRON is the GREAT PENTAKIS DODECAHEDRON. It has SCHLÄFLI SYMBOL $t'\{\frac{5}{2}, 5\}$ and WYTHOFF SYMBOL $2\,5\,|\,\frac{5}{3}$. Its faces are $12\{5\} + 12\{\frac{10}{3}\}$. Its CIRCUMRADIUS with $a = 1$ is

$$R = \tfrac{1}{4}\sqrt{34-10\sqrt{5}}.$$

References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, p. 151, 1971.

Small Triakis Octahedron





The DUAL POLYHEDRON of the TRUNCATED CUBE.

see also GREAT TRIAKIS OCTAHEDRON

Small Triambic Icosahedron

The DUAL POLYHEDRON of the SMALL DITRIGONAL ICOSIDODECAHEDRON.

Small World Problem

The small world problem asks for the probability that two people picked at random have at least one acquaintance in common.

see also BIRTHDAY PROBLEM

Smarandache Ceil Function

A SMARANDACHE-like function which is defined where $S_k(n)$ is defined as the smallest integer for which $n|S_k(n)^k$. The Smarandache $S_k(n)$ function can therefore be obtained by replacing any factors which are kth powers in n by their k roots. The functions $S_k(n)$ for $k = 2, 3, \ldots, 6$ for values such that $S_k(n) \neq n$ are tabulated by Begay (1997).

$S_1(n) = n$, so the first few values of $S_1(n)$ are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, ... (Sloane's A000027). The first few values of $S_2(n)$ are 1, 2, 3, 2, 5, 6, 7, 4, 3, 10, 11, 6, 13, 14, 15, 4, 17, 6, 19, 10, ... (Sloane's A019554) The first few values of $S_3(n)$ are 1, 2, 3, 2, 5, 6, 7, 2, 3, 10, 11, 6, 13, 14, 15, 4, 17, 6, 19, 10, ... (Sloane's A019555) The first few values of $S_4(n)$ are 1, 2, 3, 2, 5, 6, 7, 2, 3, 10, 11, 6, 13, 14, 15, 2, 17, 6, 19, 10, ... (Sloane's A007947).

see also PSEUDOSMARANDACHE FUNCTION, SMARANDACHE FUNCTION, SMARANDACHE-KUREPA FUNCTION, SMARANDACHE NEAR-TO-PRIMORIAL FUNCTION, SMARANDACHE SEQUENCES, SMARANDACHE-WAGSTAFF FUNCTION, SMARANDACHE FUNCTION

References

Begay, A. "Smarandache Ceil Functions." *Bull. Pure Appl. Sci.* **16E**, 227–229, 1997.

"Functions in Number Theory." `http://www.gallup.unm.edu/~smarandache/FUNCT1.TXT`.

Sloane, N. J. A. Sequences A007947, A019554, A019555, and A0472/M000027 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Smarandache, F. *Collected Papers, Vol. 2.* Kishinev, Moldova: Kishinev University Press, 1997.

Smarandache, F. *Only Problems, Not Solutions!, 4th ed.* Phoenix, AZ: Xiquan, 1993.

Smarandache Constants

The first Smarandache constant is defined as

$$S_1 \equiv \sum_{n=2}^{\infty} \frac{1}{[S(n)]!} > 1.093111,$$

where $S(n)$ is the SMARANDACHE FUNCTION. Cojocaru and Cojocaru (1996a) prove that S_1 exists and is bounded by $0.717 < S_1 < 1.253$. The lower limit given above is obtained by taking 40,000 terms of the sum.

Cojocaru and Cojocaru (1996b) prove that the second Smarandache constant

$$S_2 \equiv \sum_{n=2}^{\infty} \frac{S(n)}{n!} \approx 1.71400629359162$$

is an IRRATIONAL NUMBER.

Cojocaru and Cojocaru (1996c) prove that the series

$$S_3 \equiv \sum_{n=2}^{\infty} \frac{1}{\prod_{i=2}^{n} S(i)} \approx 0.719960700043708$$

converges to a number $0.71 < S_3 < 1.01$, and that

$$S_4(a) \equiv \sum_{n=2}^{\infty} \frac{n^a}{\prod_{i=2}^{n} S(i)}$$

converges for a fixed REAL NUMBER $a \geq 1$. The values for small a are

$$\begin{aligned} S_4(1) &\approx 1.72875760530223 \\ S_4(2) &\approx 4.50251200619297 \\ S_4(3) &\approx 13.0111441949445 \\ S_4(4) &\approx 42.4818449849626 \\ S_4(5) &\approx 158.105463729329. \end{aligned}$$

Sandor (1997) shows that the series

$$S_5 \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n-1} S(n)}{n!}$$

converges to an IRRATIONAL. Burton (1995) and Dumitrescu and Seleacu (1996) show that the series

$$S_6 \equiv \sum_{n=2}^{\infty} \frac{S(n)}{(n+1)!}$$

converges. Dumitrescu and Seleacu (1996) show that the series

$$S_7 \equiv \sum_{n=r}^{\infty} \frac{S(n)}{(n+r)!}$$

and

$$S_8 \equiv \sum_{n=r}^{\infty} \frac{S(n)}{(n-r)!}$$

converge for r a natural number (which must be nonzero in the latter case). Dumitrescu and Seleacu (1996) show that

$$S_9 \equiv \sum_{n=2}^{\infty} \frac{1}{\sum_{i=2}^{n} \frac{S(i)}{i!}}$$

converges. Burton (1995) and Dumitrescu and Seleacu (1996) show that the series

$$S_{10} \sum_{n=2}^{\infty} \frac{1}{[S(n)]^{\alpha} \sqrt{S(n)!}}$$

and

$$S_{11} \sum_{n=2}^{\infty} \frac{1}{[S(n)]^{\alpha} \sqrt{[S(n)+1]!}}$$

converge for $\alpha > 1$.

see also SMARANDACHE FUNCTION

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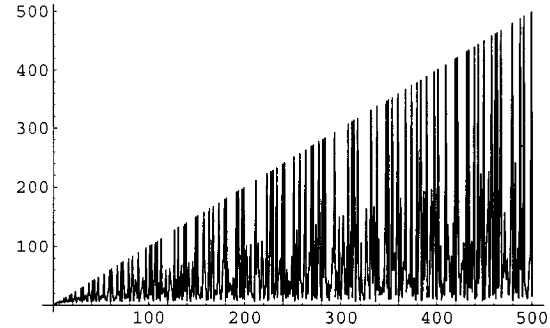
Smarandache Function



The smallest value $S(n)$ for a given n for which $n|S(n)!$ (n divides $S(n)$ FACTORIAL). For example, the number 8 does not divide $1!$, $2!$, $3!$, but does divide $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 8 \cdot 3$, so $S(8) = 4$. For a PRIME p, $S(p) = p$, and for an EVEN PERFECT NUMBER r, $S(r)$ is PRIME (Ashbacher 1997).

The Smarandache numbers for $n = 1, 2, \ldots$ are 1, 2, 3, 4, 5, 3, 7, 4, 6, 5, 11, ... (Sloane's A002034). Letting $a(n)$ denote the smallest value of n for which $S(n) = 1, 2, \ldots$, then $a(n)$ is given by 1, 2, 3, 4, 5, 9, 7, 32, 27, 25, 11, 243, ... (Sloane's A046021). Some values of $S(n)$ first occur only for very large n, for example, $S(59,049) = 24$, $S(177,147) = 27$, $S(134,217,728) = 30$, $S(43,046,721) = 36$, and $S(9,765,625) = 45$. D. Wilson points out that if we let

$$I(n,p) = \frac{n - \Sigma(n,p)}{p-1},$$

be the power of the PRIME p in $n!$, where $\Sigma(n,p)$ is the sum of the base-p digits of n, then it follows that

$$a(n) = \min p^{I(n-1,p)+1},$$

where the minimum is taken over the PRIMES p dividing n. This minimum appears to always be achieved when p is the GREATEST PRIME FACTOR of n.

The incrementally largest values of $S(n)$ are 1, 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 29, ... (Sloane's A046022), which occur for $n = 1, 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 29, \ldots$, i.e., the values where $S(n) = n$.

Tutescu (1996) conjectures that the DIOPHANTINE EQUATION $S(n) = S(n+1)$ has no solution.

see also FACTORIAL, GREATEST PRIME FACTOR, PSEUDOSMARANDACHE FUNCTION, SMARANDACHE CEIL FUNCTION, SMARANDACHE CONSTANTS, SMARANDACHE-KUREPA FUNCTION, SMARANDACHE NEAR-TO-PRIMORIAL FUNCTION, SMARANDACHE-WAGSTAFF FUNCTION

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Smarandache-Kurepa Function

Given the sum-of-factorials function

$$\Sigma(n) = \sum_{k=1}^{n} k!,$$

SK(p) is the smallest integer for p PRIME such that $1 + \Sigma[\mathrm{SK}(p-1)]$ is divisible by p. The first few known values of SK(p) are 2, 4, 6, 6, 5, 7, 7, 12, 22, 16, 55, 54, 42, 24, ... for $p = 2$, 5, 7, 11, 17, 19, 23, 31, 37, 41, 61, 71, 73, 89, ... The values for $p = 3$, 13, 29, 43, 47, 53, 67, 79, 83, ..., if they are finite, must be very large (e.g., $\mathrm{SK}(3) > 100,000$).

see also PSEUDOSMARANDACHE FUNCTION, SMARANDACHE CEIL FUNCTION, SMARANDACHE FUNCTION, SMARANDACHE-WAGSTAFF FUNCTION, SMARANDACHE FUNCTION

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Smarandache Near-to-Primorial Function

$SNTP(n)$ is the smallest PRIME such that $p\# - 1$, $p\#$, or $p\# + 1$ is divisible by n, where $p\#$ is the PRIMORIAL of p. Ashbacher (1996) shows that $SNTP(n)$ only exists

1. If there are no square or higher powers in the factorization of n, or
2. If there exists a PRIME $q < p$ such that $n | (q\# \pm 1)$, where p is the smallest power contained in the factorization of n.

Therefore, $SNTP(n)$ does not exist for the SQUAREFUL numbers $n = 4$, 8, 9, 12, 16, 18, 20, 24, 25, 27, 28, ... (Sloane's A002997) The first few values of $SNTP(n)$, where defined, are 2, 2, 2, 3, 3, 3, 5, 7, ... (Sloane's A046026).

see also PRIMORIAL, SMARANDACHE FUNCTION

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Smarandache Paradox

Let A be some attribute (e.g., possible, present, perfect, etc.). If all is A, then the non-A must also be A. For example, "All is possible, the impossible too," and "Nothing is perfect, not even the perfect."

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Smarandache Sequences

Smarandache sequences are any of a number of simply generated INTEGER SEQUENCES resembling those considered in published works by Smarandache such as the CONSECUTIVE NUMBER SEQUENCES and EUCLID NUMBERS (Iacobescu 1997). Other Smarandache-type sequences are given below.

1. The concatenation of n copies of the INTEGER n: 1, 22, 333, 4444, 55555, ... (Sloane's A000461; Marimutha 1997),
2. The concatenation of the first n FIBONACCI NUMBERS: 1, 11, 112, 1123, 11235, ... (Sloane's A019523; Marimutha 1997),
3. The smallest number that is the sum of squares of *two* distinct earlier terms: 1, 2, 5, 26, 29, 677, ... (Sloane's A008318, Bencze 1997),
4. The smallest number that is the sum of squares of any number of distinct earlier terms: 1, 1, 2, 4, 5, 6, 16, 17, ... (Sloane's A008319, Bencze 1997),
5. The smallest number that is *not* the sum of squares of *two* distinct earlier terms: 1, 2, 3, 4, 6, 7, 8, 9, 11, ... (Sloane's A008320, Bencze 1997),
6. The smallest number that is *not* the sum of squares of any number of distinct earlier terms: 1, 2, 3, 6, 7, 8, 11, ... (Sloane's A008321, Bencze 1997),

7. The smallest number that is a sum of cubes of *two* distinct earlier terms: 1, 2, 9, 730, 737, ... (Sloane's A008322, Bencze 1997),
8. The smallest number that is a sum of cubes of any number of distinct earlier terms: 1, 1, 2, 8, 9, 512, 513, 514, ... (Sloane's A008323, Bencze 1997),
9. The smallest number that is *not* a sum of cubes of *two* of distinct earlier terms: 1, 2, 3, 4, 5, 6, 7, 8, 10, ... (Sloane's A008380, Bencze 1997),
10. The smallest number that is *not* a sum of cubes of any number of distinct earlier terms: 1, 2, 3, 4, 5, 6, 7, 10, 11, ... (Sloane's A008381, Bencze 1997),
11. The number of PARTITIONS of a number $n = 1, 2, \ldots$ into SQUARE NUMBERS: 1, 1, 1, 1, 2, 2, 2, 2, 3, 4, 4, 4, 5, 6, 6, 6, 8, 9, 10, 10, 12, 13, ... (Sloane's A001156, Iacobescu 1997),
12. The number of PARTITIONS of a number $n = 1, 2, \ldots$ into CUBIC NUMBERS: 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, ... (Sloane's A003108, Iacobescu 1997),
13. Two copies of the first n POSITIVE integers: 11, 1212, 123123, 12341234, ... (Sloane's A019524, Iacobescu 1997),
14. Numbers written in base of triangular numbers: 1, 2, 10, 11, 12, 100, 101, 102, 110, 1000, 1001, 1002, ... (Sloane's A000462, Iacobescu 1997),
15. Numbers written in base of double factorial numbers: 1, 10, 100, 101, 110, 200, 201, 1000, 1001, 1010, ... (Sloane's A019513, Iacobescu 1997),
16. Sequences starting with terms $\{a_1, a_2\}$ which contain no three-term arithmetic progressions starting with $\{1, 2\}$: 1, 2, 4, 5, 10, 11, 13, 14, 28, ... (Sloane's A033155, Iacobescu 1997, Mudge 1997, Weisstein),
17. Numbers of the form $(n!)^2 + 1$: 2, 5, 37, 577, 14401, 518401, 25401601, 1625702401, 131681894401, ... (Sloane's A020549, Iacobescu 1997),
18. Numbers of the form $(n!)^3 + 1$: 2, 9, 217, 13825, 1728001, 373248001, 128024064001, ... (Sloane's A019514, Iacobescu 1997),
19. Numbers of the form $1 + 1!2!3!\cdots n!$: 2, 3, 13, 289, 34561, 24883201, 125411328001, 5056584744960001, ... (Sloane's A019515, Iacobescu 1997),
20. Sequences starting with terms $\{a_1, a_2\}$ which contain no three-term geometric progressions starting with $\{1, 2\}$: 1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, ... (Sloane's A000452, Iacobescu 1997),
21. Numbers repeating the digit 1 p_n times, where p_n is the nth prime: 11, 111, 11111, 1111111, ... (Sloane's A031974, Iacobescu 1997). These are a subset of the REPUNITS,
22. Integers with all 2s, 3s, 5s, and 7s (prime digits) removed: 1, 4, 6, 8, 9, 10, 11, 1, 1, 14, 1, 16, 1, 18, 19, 0, ... (Sloane's A019516, Iacobescu 1997),
23. Integers with all 0s, 1s, 4s, and 9s (square digits) removed: 2, 3, 5, 6, 7, 8, 2, 3, 5, 6, 7, 8, 2, 2, 22, 23, ... (Sloane's A031976, Iacobescu 1997).
24. (Smarandache-Fibonacci triples) Integers n such that $S(n) = S(n-1) + S(n-2)$, where $S(k)$ is the SMARANDACHE FUNCTION: 3, 11, 121, 4902, 26245, ... (Sloane's A015047; Aschbacher and Mudge 1995; Ibstedt 1997, pp. 19–23; Begay 1997). The largest known is 19,448,047,080,036,
25. (Smarandache-Radu triplets) Integers n such that there are no primes between the smaller and larger of $S(n)$ and $S(n+1)$: 224, 2057, 265225, ... (Sloane's A015048; Radu 1994/1995, Begay 1997, Ibstedt 1997). The largest known is 270,329,975,921, 205,253,634,707,051,822,848,570,391,313,
26. (Smarandache crescendo sequence): Integers obtained by concatenating strings of the first $n+1$ integers for $n = 0, 1, 2, \ldots$: 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, ... (Sloane's A002260; Brown 1997, Brown and Castillo 1997). The nth term is given by $n - m(m+1)/2 + 1$, where $m = \lfloor (\sqrt{8n+1} - 1)/2 \rfloor$, with $\lfloor x \rfloor$ the FLOOR FUNCTION (Hamel 1997),
27. (Smarandache descrescendo sequence): Integers obtained by concatenating strings of the first n integers for $n = \ldots, 2, 1$: 1, 2, 1, 3, 2, 1, 4, 3, 2, 1, ... (Sloane's A004736; Smarandache 1997, Brown 1997),
28. (Smarandache crescendo pyramidal sequence): Integers obtained by concatenating strings of rising and falling integers: 1, 1, 2, 1, 1, 2, 3, 2, 1, 1, 2, 3, 4, 3, 2, 1, ... (Sloane's A004737; Brown 1997, Brown and Castillo 1997, Smarandache 1997),
29. (Smarandache descrescendo pyramidal sequence): Integers obtained by concatenating strings of falling and rising integers: 1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, 2, 1, 2, 3, 4, ... (Brown 1997),
30. (Smarandache crescendo symmetric sequence): 1, 1, 1, 2, 2, 1, 1, 2, 3, 3, 2, 1, ... (Sloane's A004739, Brown 1997, Smarandache 1997),
31. (Smarandache descrescendo symmetric sequence): 1, 1, 2, 1, 1, 2, 3, 2, 1, 1, 2, 3, ... (Sloane's A004740; Brown 1997, Smarandache 1997),
32. (Smarandache permutation sequence): Numbers obtained by concatenating sequences of increasing length of increasing ODD NUMBERS and decreasing EVEN NUMBERS: 1, 2, 1, 3, 4, 2, 1, 3, 5, 6, 4, 2, ... (Sloane's A004741; Brown 1997, Brown and Castillo 1997),
33. (Smarandache pierced chain sequence): Numbers of the form $c(n) = 101\underbrace{0101}_{n}$ for $n = 0, 1, \ldots$: 101, 1010101, 10101010101, ... (Sloane's A031982; Ashbacher 1997). In addition, $c(n)/101$ contains no PRIMES (Ashbacher 1997),

34. (Smarandache symmetric sequence): 1, 11, 121, 1221, 12321, 123321, ... (Sloane's A007907; Smarandache 1993, Dumitrescu and Seleacu 1994, sequence 3; Mudge 1995),
35. (Smarandache square-digital sequence): square numbers all of whose digits are also squares: 1, 4, 9, 49, 100, 144, ... (Sloane's A019544; Mudge 1997),
36. (Square-digits): numbers composed of digits which are squares: 1, 4, 9, 10, 14, 19, 40, 41, 44, 49, ... (Sloane's A066030),
37. (Smarandache square-digital sequence): square-digit numbers which are themselves squares: 1, 4, 9, 49, 100, 144, ... (Sloane's A019544; Mudge 1997),
38. (Cube-digits): numbers composed of digits which are cubes: 1, 4, 10, 11, 14, 40, 41, 44, 100, 101, ... (Sloane's A046031),
39. (Smarandache cube-digital sequence): cube-digit numbers which are themselves cubes: 1, 8, 1000, 8000, 1000000, ... (Sloane's A019545; Mudge 1997),
40. (Prime-digits): numbers composed of digits which are primes: 2, 3, 5, 7, 22, 23, 25, 27, 32, 33, 35, ... (Sloane's A046034),
41. (Smarandache prime-digital sequence): prime-digit numbers which are themselves prime: 2, 3, 5, 7, 23, 37, 53, ... (Smith 1996, Mudge 1997).

see also ADDITION CHAIN, CONSECUTIVE NUMBER SEQUENCES, CUBIC NUMBER, EUCLID NUMBER, EVEN NUMBER, FIBONACCI NUMBER, INTEGER SEQUENCE, ODD NUMBER, PARTITION, SMARANDACHE FUNCTION, SQUARE NUMBER

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Smarandache-Wagstaff Function

Given the sum-of-FACTORIALS function

$$\Sigma(n) = \sum_{k=1}^{n} k!,$$

$SW(p)$ is the smallest integer for p PRIME such that $\Sigma[SW(p)]$ is divisible by p. The first few known values are 2, 4, 5, 12, 19, 24, 32, 19, 20, 20, 20, 7, 57, 6, ... for $p = 3$, 11, 17, 23, 29, 37, 41, 43, 53, 67, 73, 79, 97, ... The values for 5, 7, 13, 31, ..., if they are finite, must be very large.

see also FACTORIAL, SMARANDACHE FUNCTION

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Smith Brothers

Consecutive SMITH NUMBERS. The first two brothers are (728, 729) and (2964, 2965).

see also SMITH NUMBER

Smith Conjecture

The set of fixed points which do not move as a knot is transformed into itself is not a KNOT. The conjecture was proved in 1978.

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Smith's Markov Process Theorem

Consider

$$P_2(y_1,t|y_3,t_3) = \int P_2(y_1,t_1|y_2,t_1)P_3(y_1,t_1;y_2,t_2|y_3,t_3)\,dy_2. \quad (1)$$

If the probability distribution is governed by a MARKOV PROCESS, then

$$\begin{aligned} P_3(y_1,t_1;y_2,t_2|y_3,t_3) &= P_2(y_2,t_2|y_3,t_3) \\ &= P_2(y_2|y_3,t_3-t_2). \end{aligned} \quad (2)$$

Assuming no time dependence, so $t_1 \equiv 0$,

$$P_2(y_1|y_3,t_3) = \int P_2(y_1|y_2,t_2)P_2(y_2|y_3,t_3-t_2)\,dy_2. \quad (3)$$

see also MARKOV PROCESS

Smith's Network Theorem

In a NETWORK with three EDGES at each VERTEX, the number of HAMILTONIAN CIRCUITS through a specified EDGE is 0 or EVEN.

see also EDGE (GRAPH), HAMILTONIAN CIRCUIT, NETWORK

Smith Normal Form

A form for INTEGER matrices.

Smith Number

A COMPOSITE NUMBER the SUM of whose DIGITS is the sum of the DIGITS of its PRIME factors (excluding 1). (The PRIMES are excluded since they trivially satisfy this condition). One example of a Smith number is the BEAST NUMBER

$$666 = 2 \cdot 3 \cdot 3 \cdot 37,$$

since

$$6+6+6 = 2+3+3+(3+7) = 18.$$

Another Smith number is

$$4937775 = 3 \cdot 5 \cdot 5 \cdot 65837,$$

since

$$4+9+3+7+7+7+5 = 3+5+5+(6+5+8+3+7) = 42.$$

The first few Smith numbers are 4, 22, 27, 58, 85, 94, 121, 166, 202, 265, 274, 319, 346, ... (Sloane's A006753). There are 360 Smith numbers less than 10^4 and 29,928 $\leq 10^6$. McDaniel (1987a) showed that an infinite number exist.

A generalized k-Smith number can also be defined as a number m satisfying $S_p(m) = kS(m)$, where S_p is the sum of prime factors and S is the sum of digits. There are 47 1-Smith numbers, 21 2-Smith numbers, three 3-Smith numbers, and one 7-Smith, 9-Smith, and 14-Smith number < 1000.

A Smith number can be constructed from every factored REPUNIT R_n. The largest known Smith number is

$$9 \times R_{1031}(10^{4594} + 3 \times 10^{2297} + 1)^{1476} \times 10^{3913210}.$$

see also MONICA SET, PERFECT NUMBER, REPUNIT, SMITH BROTHERS, SUZANNE SET

References

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Yates, S. "Smith Numbers Congruent to 4 (mod 9)." *J. Recr. Math.* **19**, 139–141, 1987.

Smooth Manifold

Another word for a C^∞ (infinitely differentiable) MANIFOLD. A smooth manifold is a TOPOLOGICAL MANIFOLD together with its "functional structure" (Bredon 1995) and so differs from a TOPOLOGICAL MANIFOLD because the notion of differentiability exists on it. Every smooth manifold is a TOPOLOGICAL MANIFOLD, but not necessarily vice versa. (The first nonsmooth TOPOLOGICAL MANIFOLD occurs in 4-D.) In 1959, Milnor showed that a 7-D HYPERSPHERE can be made into a smooth manifold in 28 ways.

see also DIFFERENTIABLE MANIFOLD, HYPERSPHERE, MANIFOLD, TOPOLOGICAL MANIFOLD

References

Bredon, G. E. *Topology & Geometry.* New York: Springer-Verlag, p. 69, 1995.

Smooth Number

An INTEGER is k-smooth if it has no PRIME FACTORS $> k$. The probability that a random POSITIVE INTEGER $\leq n$ is k-smooth is $\psi(n,k)/n$, where $\psi(n,k)$ is the number of k-smooth numbers $\leq n$. This fact is important in application of Kraitchik's extension of FERMAT'S FACTORIZATION METHOD because it is related to the number of random numbers which must be examined to find a suitable subset whose product is a square.

Since about $\pi(k)$ k-smooth numbers must be found (where $\pi(k)$ is the PRIME COUNTING FUNCTION), the number of random numbers which must be examined is about $\pi(k)n/\psi(n,k)$. But because it takes about $\pi(k)$ steps to determine if a number is k-smooth using TRIAL DIVISION, the expected number of steps needed to find a subset of numbers whose product is a square is $\sim [\pi(k)]^2 n/\psi(n,k)$ (Pomerance 1996). Canfield *et al.* (1983) showed that this function is minimized when

$$k \sim \exp(\tfrac{1}{2}\sqrt{\ln n \ln\ln n}\,)$$

and that the minimum value is about

$$\exp(2\sqrt{\ln n \ln\ln n}\,).$$

In the CONTINUED FRACTION FACTORIZATION ALGORITHM, n can be taken as $2\sqrt{n}$, but in FERMAT'S FACTORIZATION METHOD, it is $n^{1/2+\epsilon}$. k is an estimate for the largest PRIME in the FACTOR BASE (Pomerance 1996).

References

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Pomerance, C. "A Tale of Two Sieves." *Not. Amer. Math. Soc.* **43**, 1473–1485, 1996.

Smooth Surface

A surface PARAMETERIZED in variables u and v is called smooth if the TANGENT VECTORS in the u and v directions satisfy

$$\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0},$$

where $\mathbf{A} \times \mathbf{B}$ is a CROSS PRODUCT.

Snake

A simple circuit in the d-HYPERCUBE which has no chords (i.e., for which all snake edges are edges of the HYPERCUBE). Klee (1970) asked for the maximum length $s(d)$ of a d-snake. Klee (1970) gave the bounds

$$\frac{7}{4(d-1)} \leq \frac{s(d)}{2^d}\frac{1}{2} - \frac{1-12\cdots 2^{-d}}{7d(d-1)^2+2} \qquad (1)$$

for $d \geq 6$ (Danzer and Klee 1967, Douglas 1969), as well as numerous references. Abbott and Katchalski (1988) show

$$s(d) \geq 77 \cdot 2^{d-8}, \qquad (2)$$

and Snevily (1994) showed that

$$s(d) \leq 2^{n-1}\left(1 - \frac{1}{20n-41}\right) \qquad (3)$$

for $n \leq 12$, and conjectured

$$s(d) \leq 3 \cdot 2^{n-3} + 2 \qquad (4)$$

for $n \leq 5$. The first few values for $s(d)$ for $d = 1, 2, \ldots$, are 2, 4, 6, 8, 14, 26, ... (Sloane's A000937).

see also HYPERCUBE

References

Abbott, H. L. and Katchalski, M. "On the Snake in the Box Problem." *J. Combin. Th. Ser. B* **44**, 12–24, 1988.
Danzer, L. and Klee, V. "Length of Snakes in Boxes." *J. Combin. Th.* **2**, 258–265, 1967.
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Snake Eyes

A roll of two 1s (the lowest roll possible) on a pair of six-sided DICE. The probability of rolling snake eyes is 1/36, or 2.777...%.

see also BOXCARS

Snake Oil Method

The expansion of the two sides of a sum equality in terms of POLYNOMIALS in x^m and y^k, followed by closed form summation in terms of x and y. For an example of the technique, see Bloom (1995).

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Bloom, D. M. "A Semi-Unfriendly Identity." Problem 10206. Solution by R. J. Chapman. *Amer. Math. Monthly* **102**, 657–658, 1995.
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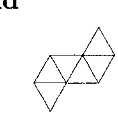
Snake Polyiamond



A 6-POLYIAMOND.

References

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Snedecor's F-Distribution

If a random variable X has a CHI-SQUARED DISTRIBUTION with m degrees of freedom ($\chi_m{}^2$) and a random variable Y has a CHI-SQUARED DISTRIBUTION with n degrees of freedom ($\chi_n{}^2$), and X and Y are independent, then

$$F \equiv \frac{X/m}{Y/n} \tag{1}$$

is distributed as Snedecor's F-distribution with m and n degrees of freedom

$$f(F(m,n)) = \frac{\Gamma\left(\frac{m+n}{2}\right)\left(\frac{m}{n}\right)^{m/2} F^{(m-2)/2}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)\left(1+\frac{m}{n}F\right)^{(m+n)/2}} \tag{2}$$

for $0 < F < \infty$. The MOMENTS about 0 are

$$\mu_1' = \frac{n}{n-2} \tag{3}$$

$$\mu_2' = \frac{n^2(m+2)}{m(n-2)(n-4)} \tag{4}$$

$$\mu_3' = \frac{n^3(m+2)(m+4)}{m^2(n-2)(n-4)(n-6)} \tag{5}$$

$$\mu_4' = \frac{n^4(m+2)(m+4)(m+6)}{m^3(n-2)(n-4)(n-6)(n-8)}, \tag{6}$$

so the MOMENTS about the MEAN are given by

$$\mu_2 = \frac{2n^2(m+n-2)}{m(n-2)^2(n-4)} \tag{7}$$

$$\mu_3 = \frac{8n^3(m+n-2)(2m+n-2)}{m^2(n-2)^3(n-4)(n-6)} \tag{8}$$

$$\mu_4 = \frac{12n^4(m+n-2)}{m^3(n-2)^4(n-4)(n-6)(n-8)} g(m,n), \tag{9}$$

where

$$g(m,n) = mn^2 + 4n^2 + m^2 n + 8mn - 16n + 10m^2 - 20m + 16, \tag{10}$$

and the MEAN, VARIANCE, SKEWNESS, and KURTOSIS are

$$\mu = \mu_1' = \frac{n}{n-2} \tag{11}$$

$$\sigma^2 = \frac{2n^2(m+n-2)}{m(n-2)^2(n-4)} \tag{12}$$

$$\gamma_1 = \frac{\mu_3}{\sigma^3} = 2\sqrt{\frac{2(n-4)}{m(m+n-2)}}\,\frac{2m+n-2}{n-6} \tag{13}$$

$$\gamma_2 = \frac{\mu_4}{\sigma^4} - 3 = \frac{12h(m,n)}{m(m+n-2)(n-6)(n-8)}, \tag{14}$$

where

$$h(m,n) \equiv n^3 + 5mn^2 - 8n^2 + 5m^2 n - 32mn + 20n - 22m^2 + 44m - 16. \tag{15}$$

Letting

$$w \equiv \frac{\frac{mF}{n}}{1+\frac{mF}{n}} \tag{16}$$

gives a BETA DISTRIBUTION.

see also BETA DISTRIBUTION, CHI-SQUARED DISTRIBUTION, STUDENT'S t-DISTRIBUTION

References
Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 536, 1987.

Snellius-Pothenot Problem

A SURVEYING PROBLEM which asks: Determine the position of an unknown accessible point P by its bearings from three inaccessible known points A, B, and C.

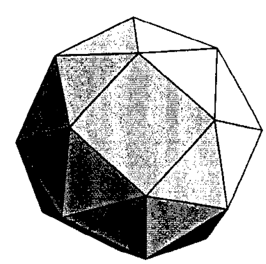
see also SURVEYING PROBLEMS

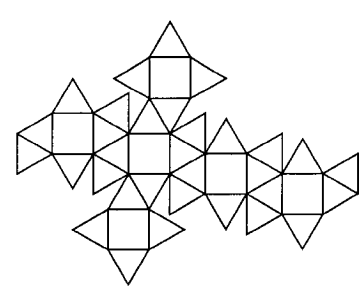
References
Dörrie, H. "Annex to a Survey." §40 in *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, pp. 193–197, 1965.

Snowflake

see EXTERIOR SNOWFLAKE, KOCH ANTISNOWFLAKE, KOCH SNOWFLAKE, PENTAFLAKE

Snub Cube





An ARCHIMEDEAN SOLID also called the SNUB CUBOCTAHEDRON whose VERTICES are the 24 points on the surface of a SPHERE for which the smallest distance between any two is as great as possible. It has two ENANTIOMERS, and its DUAL POLYHEDRON is the PENTAGONAL ICOSITETRAHEDRON. It has SCHLÄFLI SYMBOL $s\left\{\begin{smallmatrix}3\\4\end{smallmatrix}\right\}$. It is also UNIFORM POLYHEDRON U_{12} and has WYTHOFF SYMBOL $|\,2\,3\,4$. Its faces are $32\{3\} + 6\{4\}$.

The INRADIUS, MIDRADIUS, and CIRCUMRADIUS for $a = 1$ are

$$r = 1.157661791\ldots$$
$$\rho = 1.247223168\ldots$$
$$R = \tfrac{1}{2}\sqrt{\frac{x^2 - 8x + 4}{x^2 - 5x + 4}} = 1.3437133737446\ldots,$$

where

$$x \equiv (19 + 3\sqrt{33}\,)^{1/3},$$

and the exact expressions for r and ρ can be computed using

$$r = \frac{R^2 - \frac{1}{4}a^2}{R}$$
$$\rho = \sqrt{R^2 - \tfrac{1}{4}a^2}.$$

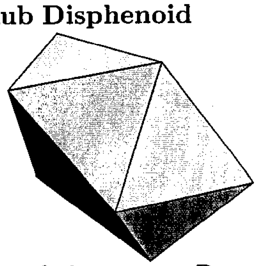
see also SNUB DODECAHEDRON

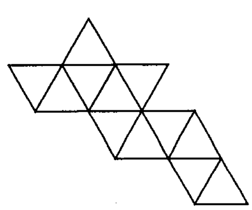
References
Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 139, 1987.
Coxeter, H. S. M.; Longuet-Higgins, M. S.; and Miller, J. C. P. "Uniform Polyhedra." *Phil. Trans. Roy. Soc. London Ser. A* **246**, 401–450, 1954.

Snub Cuboctahedron

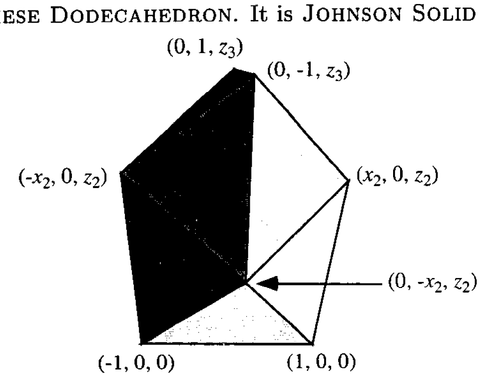
see SNUB CUBE

Snub Disphenoid





One of the convex DELTAHEDRA also known as the SIAMESE DODECAHEDRON. It is JOHNSON SOLID J_{84}.



The coordinates of the VERTICES may be found by solving the set of four equations

$$1 + {x_2}^2 + {z_1}^2 = 4$$
$$(x_2 - 1)^2 + (z_3 - z_1)^2 = 4$$
$${x_2}^2 + (z_3 - z_2)^2 = 4$$
$${x_2}^2 + {x_2}^2 + (z_2 - z_1)^2 = 4$$

for the four unknowns x_2, z_1, z_2, and z_3. Numerically,

$$x_2 = 1.28917$$
$$z_1 = 1.15674$$
$$z_2 = 1.97898$$
$$z_3 = 3.13572.$$

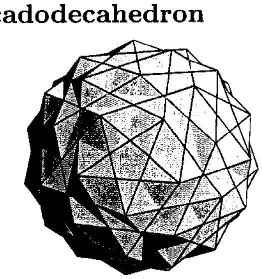
The analytic solution requires solving the CUBIC EQUATION and gives

$$x_2 = 1 - 7\cdot 2^{-2/3}(1 - i\sqrt{3})\alpha^{-1} - \tfrac{1}{6}\cdot 2^{-1/3}(1 + i\sqrt{3})\alpha$$
$$z_1 = \tfrac{1}{3}\cdot 2^{-1/2}[-48 + 6\beta(1 + i\sqrt{3}) + \beta^2(1 - i\sqrt{3}) + 147\beta\gamma(\sqrt{3} - i) + 42\beta^2\gamma(\sqrt{3} + i)]^{1/2},$$

where

$$\alpha \equiv (12i\sqrt{237} - 54)^{1/3}$$
$$\beta \equiv 3^{1/3}(2i\sqrt{237} - 9)^{1/3}$$
$$\gamma \equiv (9i + 2\sqrt{237}\,)^{-1}.$$

Snub Dodecadodecahedron



The UNIFORM POLYHEDRON U_{40} whose DUAL POLYHEDRON is the MEDIAL PENTAGONAL HEXECONTAHEDRON. It has WYTHOFF SYMBOL $|\,2\,\frac{5}{2}\,5$. Its faces are $12\{\frac{5}{2}\} + 60\{3\} + 12\{5\}$. It has CIRCUMRADIUS for $a = 1$ of

$$R = 1.27443994.$$

see also SNUB CUBE

References
Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 139, 1987.
Coxeter, H. S. M.; Longuet-Higgins, M. S.; and Miller, J. C. P. "Uniform Polyhedra." *Phil. Trans. Roy. Soc. London Ser. A* **246**, 401–450, 1954.
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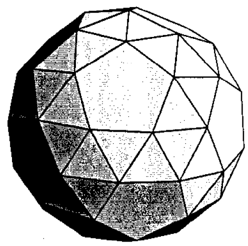
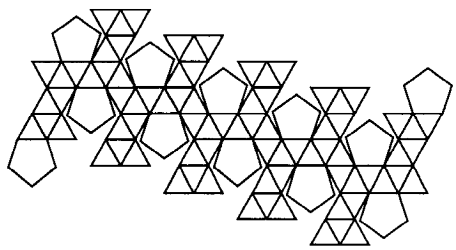
Snub Dodecahedron





An ARCHIMEDEAN SOLID, also called the SNUB ICOSIDODECAHEDRON, whose DUAL POLYHEDRON is the PENTAGONAL HEXECONTAHEDRON. It has SCHLÄFLI SYMBOL $s\left\{ {3 \atop 5} \right\}$. It is also UNIFORM POLYHEDRON U_{29} and has WYTHOFF SYMBOL $|\,2\,3\,5$. Its faces are $80\{3\}+12\{5\}$. For $a=1$, it has INRADIUS, MIDRADIUS, and CIRCUMRADIUS

$$\begin{aligned} r &= 2.039873155\ldots \\ \rho &= 2.097053835\ldots \\ R &= \frac{1}{2}\sqrt{\frac{8\cdot 2^{2/3}-16x+2^{1/3}x^2}{8\cdot 2^{2/3}-10x+2^{1/3}x^2}} \\ &= 2.15583737511564\ldots, \end{aligned}$$

where

$$x \equiv \left(49+27\sqrt{5}+3\sqrt{6}\sqrt{93+49\sqrt{5}}\right)^{1/3},$$

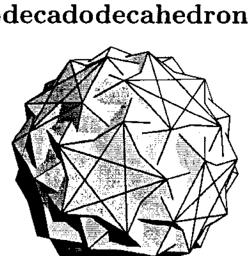
and the exact expressions for r and ρ can be computed using

$$\begin{aligned} r &= \frac{R^2-\frac{1}{4}a^2}{R} \\ \rho &= \sqrt{R^2-\tfrac{1}{4}a^2}. \end{aligned}$$

References

Coxeter, H. S. M.; Longuet-Higgins, M. S.; and Miller, J. C. P. "Uniform Polyhedra." *Phil. Trans. Roy. Soc. London Ser. A* **246**, 401–450, 1954.

Snub Icosidodecadodecahedron



The UNIFORM POLYHEDRON U_{46} whose DUAL POLYHEDRON is the MEDIAL HEXAGONAL HEXECONTAHEDRON. It has WYTHOFF SYMBOL $|\,3\,\frac{5}{3}\,5$. Its faces are $12\{\frac{5}{2}\}+80\{3\}+12\{5\}$. It has CIRCUMRADIUS for $a=1$ of

$$\begin{aligned} R &= \frac{1}{2}\sqrt{\frac{2^{4/3}-14x+2^{2/3}x^2}{2^{4/3}-8x+2^{2/3}x^2}} \\ &= 1.12689791279994\ldots, \end{aligned}$$

where

$$x=(25+3\sqrt{69}\,)^{1/3}.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 177–178, 1971.

Snub Icosidodecahedron

see SNUB DODECAHEDRON

Snub Polyhedron

A polyhedron with extra triangular faces, given by the SCHLÄFLI SYMBOL $s\left\{ {p \atop q} \right\}$.

see also RHOMBIC POLYHEDRON, TRUNCATED POLYHEDRON

Snub Square Antiprism

see JOHNSON SOLID

Soap Bubble

see BUBBLE

Soccer Ball

see TRUNCATED ICOSAHEDRON

Sociable Numbers

Numbers which result in a periodic ALIQUOT SEQUENCE. If the period is 1, the number is called a PERFECT NUMBER. If the period is 2, the two numbers are called an AMICABLE PAIR. If the period is $t \geq 3$, the number is called sociable of order t. Only two sociable numbers were known prior to 1970, the sets of orders 5 and 28 discovered by Poulet (1918). In 1970, Cohen discovered nine groups of order 4.

The table below summarizes the number of sociable cycles known as given in the compilation by Moews (1995).

order	known
3	0
4	38
5	1
6	2
8	2
9	1
28	1

see also ALIQUOT SEQUENCE, PERFECT NUMBER, UNITARY SOCIABLE NUMBERS

References

Beeler, M.; Gosper, R. W.; and Schroeppel, R. *HAKMEM.* Cambridge, MA: MIT Artificial Intelligence Laboratory, Memo AIM-239, Item 61, Feb. 1972.

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Moews, D. and Moews, P. C. "A Search for Aliquot Cycles and Amicable Pairs." *Math. Comput.* **61**, 935–938, 1993.

Moews, D. "A List of Aliquot Cycles of Length Greater than 2." Rev. Dec. 18, 1995. `http://xraysgi.ims.uconn.edu:8080/sociable.txt`.

Poulet, P. Question 4865. *L'interméd. des Math.* **25**, 100–101, 1918.

te Riele, H. J. J. "Perfect Numbers and Aliquot Sequences." In *Computational Methods in Number Theory, Part I.* (Eds. H. W. Lenstra Jr. and R. Tijdeman). Amsterdam, Netherlands: Mathematisch Centrum, pp. 141–157, 1982.

Weisstein, E. W. "Sociable and Amicable Numbers." `http://www.astro.virginia.edu/~eww6n/math/notebooks/Sociable.m`.

Social Choice Theory

The theory of analyzing a decision between a collection of alternatives made by a collection of n voters with separate opinions. Any choice for the entire group should reflect the desires of the individual voters to the extent possible.

Fair choice procedures usually satisfy ANONYMITY (invariance under permutation of voters), DUALITY (each alternative receives equal weight for a single vote), and MONOTONICITY (a change favorable for X does not hurt X). Simple majority vote is anonymous, dual, and monotone. MAY'S THEOREM states a stronger result.

see also ANONYMOUS, DUAL VOTING, MAY'S THEOREM, MONOTONIC VOTING, VOTING

References

Taylor, A. *Mathematics and Politics: Strategy, Voting, Power, and Proof.* New York: Springer-Verlag, 1995.

Young, S. C.; Taylor, A. D.; and Zwicker, W. S. "Counting Quota Systems: A Combinatorial Question from Social Choice Theory." *Math. Mag.* **68**, 331–342, 1995.

Socrates' Paradox

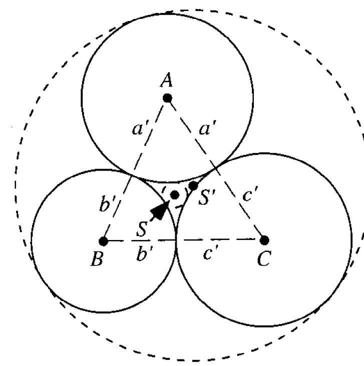
Socrates is reported to have stated: "One thing I know is that I know nothing."

see also LIAR'S PARADOX

References

Pickover, C. A. *Keys to Infinity.* New York: W. H. Freeman, p. 134, 1995.

Soddy Circles



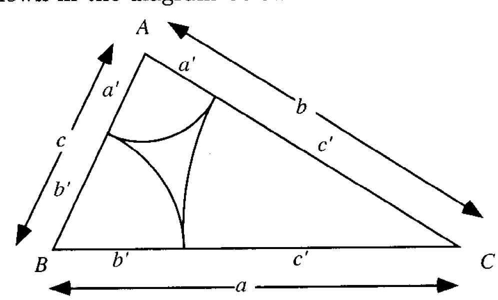
Given three distinct points A, B, and C, let three CIRCLES be drawn, one centered about each point and each one tangent to the other two. Call the RADII r_i ($r_3 = a'$, $r_1 = b'$, $r_2 = c'$). Then the CIRCLES satisfy

$$a' + b' = c \tag{1}$$

$$a' + c' = b \tag{2}$$

$$b' + c' = a, \tag{3}$$

as shown in the diagram below.



Solving for the RADII then gives

$$a' = \tfrac{1}{2}(b + c - a) \tag{4}$$

$$b' = \tfrac{1}{2}(a + c - b) \tag{5}$$

$$c' = \tfrac{1}{2}(a + b - c). \tag{6}$$

The above TRIANGLE has sides a, b, and c, and SEMIPERIMETER

$$s \equiv \tfrac{1}{2}(a + b + c). \tag{7}$$

Plugging in,

$$2s = (a' + b') + (a' + c') + (b' + c') = 2(a' + b' + c'), \tag{8}$$

giving

$$a' + b' + c' = s. \tag{9}$$

In addition,

$$a = b' + c' = a' + b' + c' - a' = s - a'. \tag{10}$$

Switching a and a' to opposite sides of the equation and noting that the above argument applies equally well to b' and c' then gives

$$a' = s - a \tag{11}$$

$$b' = s - b \tag{12}$$

$$c' = s - c. \tag{13}$$

As can be seen from the first figure, there exist exactly two nonintersecting CIRCLES which are TANGENT to all three CIRCLES. These are called the inner and outer Soddy circles (S and S', respectively), and their centers are called the inner and outer SODDY POINTS.

The inner Soddy circle is the solution to the FOUR COINS PROBLEM. The center S of the inner Soddy circle is the EQUAL DETOUR POINT, and the center of the outer Soddy circle S' is the ISOPERIMETRIC POINT (Kimberling 1994).

Frederick Soddy (1936) gave the FORMULA for finding the RADII of the Soddy circles (r_4) given the RADII r_i ($i = 1$, 2, 3) of the other three. The relationship is

$$2({\epsilon_1}^2 + {\epsilon_2}^2 + {\epsilon_3}^2 + {\epsilon_4}^2) = (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)^2, \tag{14}$$

where $\epsilon_i \equiv \pm\kappa_i = \pm 1/r_i$ are the so-called BENDS, defined as the signed CURVATURES of the CIRCLES. If the contacts are all external, the signs are all taken as POSITIVE, whereas if one circle surrounds the other three, the sign of this circle is taken as NEGATIVE (Coxeter 1969). Using the QUADRATIC FORMULA to solve for ϵ_4, expressing in terms of radii instead of curvatures, and simplifying gives

$$r_4^{\pm} = \frac{r_1 r_2 r_3}{r_2 r_3 + r_1(r_2 + r_3) \pm 2\sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)}}. \tag{15}$$

Here, the NEGATIVE solution corresponds to the outer Soddy circle and the POSITIVE one to the inner Soddy circle.

This FORMULA is called the DESCARTES CIRCLE THEOREM since it was known to Descartes. However, Soddy also extended it to SPHERES. Gosper has further extended the result to $n + 2$ mutually tangent n-D HYPERSPHERES, whose CURVATURES satisfy

$$\left(\sum_{i=0}^{n+1} \kappa_i\right)^2 - n \sum_{i=0}^{n+1} {\kappa_i}^2 = 0. \tag{16}$$

Solving for κ_{n+1} gives

$$\kappa_{n+1} = \frac{\sqrt{n}\sqrt{\left(\sum_{i=0}^{n} \kappa_i\right)^2 - (n-1)\sum_{i=0}^{n} {\kappa_i}^2} + \sum_{i=0}^{n} \kappa_i}{n-1}. \tag{17}$$

For (at least) $n = 2$ and 3, the RADICAL equals

$$f(n) V \kappa_0 \kappa_1 \cdots \kappa_n, \tag{18}$$

where V is the CONTENT of the SIMPLEX whose vertices are the centers of the $n+1$ independent HYPERSPHERES. The RADICAND can also become NEGATIVE, yielding an IMAGINARY κ_{n+1}. For $n = 3$, this corresponds to a sphere touching three large bowling balls and a small BB, all mutually tangent, which is an impossibility.

Bellew has derived a generalization applicable to a CIRCLE surrounded by n CIRCLES which are, in turn, circumscribed by another CIRCLE. The relationship is

$$\begin{aligned}&[n(c_n-1)^2+1]\sum_{i=1}^{n} {\kappa_i}^2 + n(3n{c_n}^2 - 2n - 6){c_n}^2(c_n-1)^2 = \\ &\frac{1}{[n(c_n-1)+1]^2} \times \{n(c_n-1)^2+1]\sum_{i=1}^{n}\kappa_i \\ &+ nc_n(c_n-1)(n{c_n}^2 + (3-n)c_n - 4])\},\end{aligned} \tag{19}$$

where

$$c_n \equiv \csc\left(\frac{\pi}{n}\right). \tag{20}$$

For $n = 3$, this simplifies to the Soddy formula.

see also APOLLONIUS CIRCLES, APOLLONIUS' PROBLEM, ARBELOS, BEND (CURVATURE), CIRCUMCIRCLE, DESCARTES CIRCLE THEOREM, FOUR COINS PROBLEM, HART'S THEOREM, PAPPUS CHAIN, SPHERE PACKING, STEINER CHAIN

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Soddy's Hexlet

see HEXLET

Soddy Line

A LINE on which the INCENTER I, GERGONNE POINT Ge, and inner and outer SODDY POINTS S and S' lie (the latter two of which are the EQUAL DETOUR POINT and the ISOPERIMETRIC POINT). The Soddy line can be given parametrically by

$$I + \lambda Ge,$$

where λ is a parameter. It is also given by

$$\sum (f - e)\alpha = 0,$$

where cyclic permutations of d, e, and f are taken and the sum is over TRILINEAR COORDINATES α, β, and γ.

λ	Center
-4	outer Griffiths point Gr'
-2	outer Oldknow point Ol'
$-\frac{4}{3}$	outer Rigby point Ri'
-1	outer Soddy center S'
0	incenter I
1	inner Soddy center S
$\frac{4}{3}$	inner Rigby point Ri
2	inner Oldknow point Ol
4	inner Griffiths point Gr
∞	Gergonne point

S', I, S, and Ge are COLLINEAR and form a HARMONIC RANGE (Vandeghen 1964, Oldknow 1996). There are a total of 22 HARMONIC RANGES for sets of four points out of these 10 (Oldknow 1996).

The Soddy line intersects the EULER LINE in the DE LONGCHAMPS POINT, and the GERGONNE LINE in the FLETCHER POINT.

see also DE LONGCHAMPS POINT, EULER LINE, FLETCHER POINT, GERGONNE POINT, GRIFFITHS POINTS, HARMONIC RANGE, INCENTER, OLDKNOW POINTS, RIGBY POINTS, SODDY POINTS

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Vandeghen, A. "Soddy's Circles and the De Longchamps Point of a Triangle." *Amer. Math. Monthly* **71**, 176–179, 1964.

Soddy Points

Given three mutually tangent CIRCLES, there exist exactly two nonintersecting CIRCLES TANGENT to all three CIRCLES. These are called the inner and outer SODDY CIRCLES, and their centers are called the inner and outer Soddy points. The outer Soddy circle is the solution to the FOUR COINS PROBLEM. The center S of the inner Soddy circle is the EQUAL DETOUR POINT, and the center of the outer Soddy circle S' is the ISOPERIMETRIC POINT (Kimberling 1994).

see also EQUAL DETOUR POINT, ISOPERIMETRIC POINT, SODDY CIRCLES

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Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, p. 181, 1994.

Sofa Constant

see MOVING SOFA CONSTANT

Sol Geometry

The GEOMETRY of the LIE GROUP R SEMIDIRECT PRODUCT with R^2, where R acts on R^2 by $(t, (x, y)) \to (e^t x, e^{-t} y)$.

see also THURSTON'S GEOMETRIZATION CONJECTURE

Soldner's Constant

Consider the following formulation of the PRIME NUMBER THEOREM,

$$\pi(x) = \sum \frac{\mu(m)}{m} \int_c^x \frac{dt}{\ln t},$$

where $\mu(m)$ is the MÖBIUS FUNCTION and c (sometimes also denoted μ) is Soldner's constant. Ramanujan found $c = 1.45136380\ldots$ (Hardy 1969, Le Lionnais 1983, Berndt 1994). Soldner (cited in Nielsen 1965) derived the correct value of c as $1.4513692346\ldots$, where c is the root of

$$L(x) = \lim_{\epsilon \to 0} \int_0^{1-\epsilon} \frac{dt}{\ln t} + \int_{1+\epsilon}^{\infty} \frac{dt}{\ln t}$$

(Le Lionnais 1983).

References

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Solenoidal Field

A solenoidal VECTOR FIELD satisfies

$$\nabla \cdot \mathbf{B} = 0 \tag{1}$$

for every VECTOR $\mathbf{B}$, where $\nabla \cdot \mathbf{B}$ is the DIVERGENCE. If this condition is satisfied, there exists a vector $\mathbf{A}$, known as the VECTOR POTENTIAL, such that

$$\mathbf{B} \equiv \nabla \times \mathbf{A}, \tag{2}$$

where $\nabla \times \mathbf{A}$ is the CURL. This follows from the vector identity

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0. \tag{3}$$

If $\mathbf{A}$ is an IRROTATIONAL FIELD, then

$$\mathbf{A} \times \mathbf{r} \tag{4}$$

is solenoidal. If $\mathbf{u}$ and $\mathbf{v}$ are irrotational, then

$$\mathbf{u} \times \mathbf{v} \tag{5}$$

is solenoidal. The quantity

$$(\nabla u) \times (\nabla v), \tag{6}$$

where ∇u is the GRADIENT, is always solenoidal. For a function ϕ satisfying LAPLACE'S EQUATION

$$\nabla^2 \phi = 0, \tag{7}$$

it follows that $\nabla \phi$ is solenoidal (and also IRROTATIONAL).

see also BELTRAMI FIELD, CURL, DIVERGENCE, DIVERGENCELESS FIELD, GRADIENT, IRROTATIONAL FIELD, LAPLACE'S EQUATION, VECTOR FIELD

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Solid

A closed 3-D figure (which may, according to some terminology conventions, be self-intersecting). Among the simplest solids are the SPHERE, CUBE, CONE, CYLINDER, and more generally, the POLYHEDRA.

see also APPLE, ARCHIMEDEAN SOLID, CATALAN SOLID, CONE, CORK PLUG, CUBE, CUBOCTAHEDRON, CYLINDER, CYLINDRICAL HOOF, CYLINDRICAL WEDGE, DODECAHEDRON, GEODESIC DOME, GREAT DODECAHEDRON, GREAT ICOSAHEDRON, GREAT RHOMBICOSIDODECAHEDRON (ARCHIMEDEAN), GREAT RHOMBICUBOCTAHEDRON (ARCHIMEDEAN), GREAT STELLATED DODECAHEDRON, ICOSAHEDRON, ICOSIDODECAHEDRON, JOHNSON SOLID, KEPLER-POINSOT SOLID, LEMON, MÖBIUS STRIP, OCTAHEDRON, PLATONIC SOLID, POLYHEDRON, PSEUDOSPHERE, RHOMBICOSIDODECAHEDRON, RHOMBICUBOCTAHEDRON, SMALL STELLATED DODECAHEDRON, SNUB CUBE, SNUB DODECAHEDRON, SOLID OF REVOLUTION, SPHERE, STEINMETZ SOLID, STELLA OCTANGULA, SURFACE, TETRAHEDRON, TORUS, TRUNCATED CUBE, TRUNCATED DODECAHEDRON, TRUNCATED ICOSAHEDRON, TRUNCATED OCTAHEDRON, TRUNCATED TETRAHEDRON, UNIFORM POLYHEDRON, WULFF SHAPE

Solid Angle

Defined as the SURFACE AREA Ω of a UNIT SPHERE which is subtended by a given object S. Writing the SPHERICAL COORDINATES as ϕ for the COLATITUDE (angle from the pole) and θ for the LONGITUDE (azimuth),

$$\Omega \equiv A_{\text{projected}} = \iint_S \sin\phi \, d\theta \, d\phi.$$

Solid angle is measured in STERADIANS, and the solid angle corresponding to all of space being subtended is 4π STERADIANS.

see also SPHERE, STERADIAN

Solid Geometry

That portion of GEOMETRY dealing with SOLIDS, as opposed to PLANE GEOMETRY. Solid geometry is concerned with POLYHEDRA, SPHERES, 3-D SOLIDS, lines in 3-space, PLANES, and so on.

see also GEOMETRY, PLANE GEOMETRY, SPHERICAL GEOMETRY

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Solid Partition

Solid partitions are generalizations of PLANE PARTITIONS. MacMohan (1960) conjectured the GENERATING FUNCTION for the number of solid partitions was

$$f(z) = \frac{1}{(1-z)(1-z^2)^3(1-z^3)^6(1-z^4)^{10}\cdots},$$

but this was subsequently shown to disagree at $n = 6$ (Atkin *et al.* 1967). Knuth (1970) extended the tabulation of values, but was unable to find a correct generating function. The first few values are 1, 4, 10, 26, 59, 140, ... (Sloane's A000293).

References
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Solid of Revolution

To find the VOLUME of a solid of rotation by adding up a sequence of thin cylindrical shells, consider a region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the LINE $x = a$, and on the right by the LINE $x = b$. When the region is rotated about the y-AXIS, the resulting VOLUME is given by

$$V = 2\pi \int_b^a x[f(x) - g(x)]\, dx.$$

To find the volume of a solid of rotation by adding up a sequence of thin flat disks, consider a region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the LINE $x = a$, and on the right by the LINE $x = b$. When the region is rotated about the x-AXIS, the resulting VOLUME is

$$V = \pi \int_b^a \{[f(x)]^2 - [g(x)]^2\}\,dx.$$

see also SURFACE OF REVOLUTION, VOLUME

Solidus

The diagonal slash "/" used to denote DIVISION for in-line equations such as a/b, $1/(x-1)^2$, etc. The solidus is also called a DIAGONAL.

see also DIVISION, OBELUS

Solitary Number

A number which does not have any FRIENDS. Solitary numbers include all PRIMES and POWERS of PRIMES. More generally, numbers for which $(n, \sigma(n)) = 1$ are solitary, where (a, b) is the GREATEST COMMON DIVISOR of a and b and $\sigma(n)$ is the DIVISOR FUNCTION. The first few solitary numbers are 1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 21, ... (Sloane's A014567).

see also FRIEND

References

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Soliton

A stable isolated (i.e., solitary) traveling wave solution to a set of equations.

see also LAX PAIR, SINE-GORDON EQUATION

References

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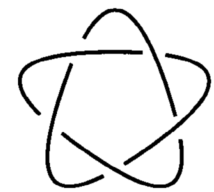
Solomon's Seal Knot



The (5,2) TORUS KNOT 05_{001} with BRAID WORD $\sigma_1{}^5$.

Solomon's Seal Lines

The 27 REAL or IMAGINARY straight LINES which lie on the general CUBIC SURFACE and the 45 triple tangent PLANES to the surface. All are related to the 28 BITANGENTS of the general QUARTIC CURVE.

Schoutte (1910) showed that the 27 lines can be put into a ONE-TO-ONE correspondence with the vertices of a particular POLYTOPE in 6-D space in such a manner that all incidence relations between the lines are mirrored in the connectivity of the POLYTOPE and conversely (Du Val 1931). A similar correspondence can be made between the 28 bitangents and a 7-D POLYTOPE (Coxeter 1928) and between the tritangent planes of the canonical curve of genus four and an 8-D POLYTOPE (Du Val 1933).

see also BRIANCHON'S THEOREM, CUBIC SURFACE, DOUBLE SIXES, PASCAL'S THEOREM, QUARTIC SURFACE, STEINER SET

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Solomon's Seal Polygon

see HEXAGRAM

Solvable Congruence

A CONGRUENCE that has a solution.

Solvable Group

A solvable group is a group whose composition indices are all PRIME NUMBERS. Equivalently, a solvable is a GROUP having a "normal series" such that each "normal factor" is ABELIAN. The term solvable derives from this type of group's relationship to GALOIS'S THEOREM, namely that the SYMMETRIC GROUP S_n is insoluble for $n \geq 5$ while it is solvable for $n = 1$, 2, 3, and 4. As a result, the POLYNOMIAL equations of degree ≥ 5 are not solvable using finite additions, multiplications, divisions, and root extractions.

Every FINITE GROUP of order < 60, every ABELIAN GROUP, and every SUBGROUP of a solvable group is solvable.

see also ABELIAN GROUP, COMPOSITION SERIES, GALOIS'S THEOREM, SYMMETRIC GROUP

References

Lomont, J. S. *Applications of Finite Groups.* New York: Dover, p. 26, 1993.

Solvable Lie Group

The connected closed SUBGROUPS (up to an ISOMORPHISM) of COMPLEX MATRICES that are closed under conjugate transpose and have a discrete finite center. Examples include SPECIAL LINEAR GROUPS, SYMPLECTIC GROUPS, and certain isometry groups of QUADRATIC FORMS.

see also LIE GROUP

References

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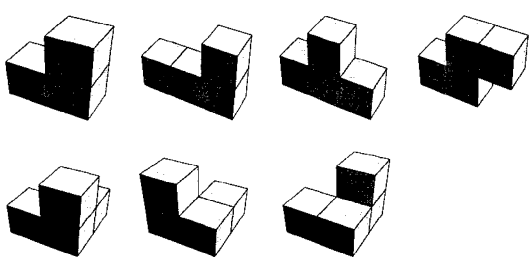
Soma Cube



A solid DISSECTION puzzle invented by Piet Hein during a lecture on Quantum Mechanics by Werner Heisenberg. There are seven soma pieces composed of all the *irregular* face-joined cubes (POLYCUBES) with ≤ 4 cubes. The object is to assemble the pieces into a CUBE. There are 240 essentially distinct ways of doing so (Beeler *et al.* 1972, Berlekamp *et al.* 1982), as first enumerated one rainy afternoon in 1961 by J. H. Conway and Mike Guy.

A commercial version of the cube colors the pieces black, green, orange, white, red, and blue. When the 48 symmetries of the cube, three ways of assembling the black piece, and 2^5 ways of assembling the green, orange, white, red, and blue pieces are counted, the total number of solutions rises to 1,105,920.

see also CUBE DISSECTION, POLYCUBE

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Steinhaus, H. *Mathematical Snapshots, 3rd American ed.* New York: Oxford University Press, pp. 168–169, 1983.

Somer-Lucas Pseudoprime

An ODD COMPOSITE NUMBER N is called a Somer-Lucas d-pseudoprime (with $d \geq 1$) if there EXISTS a nondegenerate LUCAS SEQUENCE $U(P,Q)$ with $U_0 = 0$, $U_1 = 1$, $D = P^2 - 4Q$, such that $(N,D) = 1$ and the rank appearance of N in the sequence $U(P,Q)$ is $(1/a)(N - (D/N))$, where (D/N) denotes the JACOBI SYMBOL.

see also LUCAS SEQUENCE, PSEUDOPRIME

References

Ribenboim, P. "Somer-Lucas Pseudoprimes." §2.X.D in *The New Book of Prime Number Records, 3rd ed.* New York: Springer-Verlag, pp. 131–132, 1996.

Sommerfeld's Formula

There are (at least) two equations known as Sommerfeld's formula. The first is

$$J_\nu(z) = \frac{1}{2\pi} \int_{-\eta+i\infty}^{2\pi-\eta+i\infty} e^{iz\cos t} e^{i\nu(t-\pi/2)}\,dt,$$

where $J_\nu(z)$ is a BESSEL FUNCTION OF THE FIRST KIND. The second states that under appropriate restrictions,

$$\int_0^\infty J_0(\tau r) e^{-|x|\sqrt{\tau^2-k^2}} \frac{\tau\,d\tau}{\sqrt{\tau^2-k^2}} = \frac{e^{ik\sqrt{\tau^2+k^2}}}{\sqrt{r^2+x^2}}.$$

see also WEYRICH'S FORMULA

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Somos Sequence

The Somos sequences are a set of related symmetrical RECURRENCE RELATIONS which, surprisingly, always give integers. The Somos sequence of order k is defined by

$$a_n = \frac{\sum_{j=1}^{\lfloor k/2 \rfloor} a_{n-j} a_{n-(k-j)}}{a_{n-k}},$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION and $a_j = 1$ for $j = 0, \ldots, k-1$. The 2- and 3-Somos sequences consist entirely of 1s. The k-Somos sequences for $k = 4$, 5, 6, and 7 are

$$a_n = \frac{a_{n-1}a_{n-3} + {a_{n-2}}^2}{a_{n-4}}$$
$$a_n = \frac{a_{n-1}a_{n-4} + a_{n-2}a_{n-3}}{a_{n-5}}$$
$$a_n = \frac{1}{a_{n-6}}[a_{n-1}a_{n-5} + a_{n-2}a_{n-4} + {a_{n-3}}^2]$$
$$a_n = \frac{1}{a_{n-7}}[a_{n-1}a_{n-6} + a_{n-2}a_{n-5} + a_{n-3}a_{n-4}],$$

giving 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, ... (Sloane's A006720), 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, 274, 1217, ... (Sloane's A006721), 1, 1, 1, 1, 1, 3, 5, 9, 23, 75, 421, 1103, ... (Sloane's A006722), 1, 1, 1, 1, 1, 1, 3, 5, 9, 17, 41, 137, 769, ... (Sloane's A006723). Gale (1991) gives simple proofs of the integer-only property of the 4-Somos and 5-Somos sequences. Hickerson proved 6-Somos generates only integers using computer algebra, and empirical evidence suggests 7-Somos is also integer-only.

However, the k-Somos sequences for $k \geq 8$ do not give integers. The values of n for which a_n first becomes nonintegral for the k-Somos sequence for $k = 8$, 9, ... are 17, 19, 20, 22, 24, 27, 28, 30, 33, 34, 36, 39, 41, 42, 44, 46, 48, 51, 52, 55, 56, 58, 60, ... (Sloane's A030127).

see also GÖBEL'S SEQUENCE, HERONIAN TRIANGLE

References

Buchholz, R. H. and Rathbun, R. L. "An Infinite Set of Heron Triangles with Two Rational Medians." *Amer. Math. Monthly* **104**, 107–115, 1997.

Gale, D. "Mathematical Entertainments: The Strange and Surprising Saga of the Somos Sequences." *Math. Intel.* **13**, 40–42, 1991.

Sloane, N. J. A. Sequences A030127, A006720/M0857, A006721/M0735, A006722/M2457, and A006723/M2456 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Sondat's Theorem

The PERSPECTIVE AXIS bisects the line joining the two ORTHOCENTERS.

see also ORTHOCENTER, PERSPECTIVE AXIS

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, p. 259, 1929.

Sonine's Integral

$$J_m(x) = \frac{2x^{m-n}}{2^{m-n}\Gamma(m-n)} \int_0^1 J_n(xt)t^{n+1}(1-t^2)^{m-n-1}\,dt,$$

where $J_m(x)$ is a BESSEL FUNCTION OF THE FIRST KIND and $\Gamma(x)$ is the GAMMA FUNCTION.

see also HANKEL'S INTEGRAL, POISSON INTEGRAL

Sonine Polynomial

A polynomial which differs from the associated LAGUERRE POLYNOMIAL by only a normalization constant,

$$\begin{aligned} S_r^s(x) &= \frac{1}{s!}e^x x^{-r} \frac{d^s}{dx^s}(e^{-x}x^{r+s}) = \frac{(-1)^r}{(r+s)!}L_{r+s}^r(x) \\ &= \frac{x^s}{s!(r+s)!0!} - \frac{x^{s-1}}{(s-1)!(r+s-1)!1!} \\ &\quad + \frac{x^{r-2}}{(r-2)!(r+s-2)!2!} - \ldots \\ &= \frac{1}{s!(r+s)!}x^{-(r+1)/2}e^{x/2}W_{s+r/2+1/2,r/2}(x), \end{aligned}$$

where $W_{k,m}(z)$ is a WHITTAKER FUNCTION.

see also LAGUERRE POLYNOMIAL, WHITTAKER FUNCTION

Sonine-Schafheitlin Formula

$$\begin{aligned} &\int_0^\infty J_\mu(at)J_\nu(bt)t^{-\lambda}\,dt \\ &= \frac{a^\mu\Gamma[(\mu+\nu-\lambda+1)/2]}{2^\lambda b^{\mu-\lambda+1}\Gamma[(-\mu+\nu+\lambda+1)/2]\Gamma(\mu+1)} \\ &\times {}_2F_1((\mu+\nu-\lambda+1)/2, (\mu-\nu-\lambda+1)/2; \mu+1; a^2/b^2), \end{aligned}$$

where $\Re[\mu+\nu-\lambda+1] > 0$, $\Re[\lambda] > -1$, $0 < a < b$, $J_\nu(x)$ is a BESSEL FUNCTION OF THE FIRST KIND, $\Gamma(x)$ is the GAMMA FUNCTION, and ${}_2F_1(a, b; c; x)$ is a HYPERGEOMETRIC FUNCTION.

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1474, 1980.

Sophie Germain Prime

A PRIME p is said to be a Sophie Germain prime if both p and $2p+1$ are PRIME. The first few Sophie Germain primes are 2, 3, 5, 11, 23, 29, 41, 53, 83, 89, 113, 131, ... (Sloane's A005384).

Around 1825, Sophie Germain proved that the first case of FERMAT'S LAST THEOREM is true for such primes, i.e., if p is a Sophie Germain prime, there do not exist INTEGERS x, y, and z different from 0 and not multiples of p such that

$$x^p + y^p = z^p.$$

Sophie Germain primes p of the form $p = k \cdot 2^n - 1$ (which makes $2p+1$ a PRIME) are COMPOSITE MERSENNE NUMBERS. Since the largest known COMPOSITE MERSENNE NUMBER is M_p with $p = 39051 \times 2^{6001} - 1$, p is the largest known Sophie Germain prime.

see also CUNNINGHAM CHAIN, FERMAT'S LAST THEOREM, MERSENNE NUMBER, TWIN PRIMES

References
Dubner, H. "Large Sophie Germain Primes." *Math. Comput.* **65**, 393–396, 1996.
Ribenboim, P. "Sophie Germane Primes." §5.2 in *The New Book of Prime Number Records.* New York: Springer-Verlag, pp. 329–332, 1996.
Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, pp. 154–157, 1993.
Sloane, N. J. A. Sequence A005384 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Sorites Paradox

Sorites paradoxes are a class of paradoxical arguments also known as little-by-little arguments. The name "sorites" derives from the Greek word *soros*, meaning "pile" or "heap". Sorites paradoxes are exemplified by the problem that a single grain of wheat does not comprise a heap, nor do two grains of wheat, three grains of wheat, etc. However, at some point, the collection of grains becomes large enough to be called a heap, but there is apparently no definite point where this occurs.

see also UNEXPECTED HANGING PARADOX

Sort-Then-Add Sequence

A sequence produced by sorting the digits of a number and adding them to the previous number. The algorithm terminates when a sorted number is obtained. For $n = 1, 2, \ldots$, the algorithm terminates on 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 11, 12, 13, 14, 15, 16, 17, 18, 19, 22, 33, ... (Sloane's A033862). The first few numbers not known to terminate are 316, 452, 697, 1376, 2743, 5090, ... (Sloane's A033861). The least numbers of sort-then-add persistence $n = 1, 2, \ldots$, are 1, 10, 65, 64, 175, 98, 240, 325, 302, 387, 198, 180, 550, ... (Sloane's A033863).

see also 196-ALGORITHM, RATS SEQUENCE

References
Sloane, N. J. A. Sequences A033861, A033862, and A033863 in "An On-Line Version of the Encyclopedia of Integer Sequences."

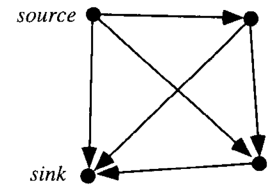
Sorting

Sorting is the rearrangement of numbers (or other orderable objects) in a list into their correct lexographic order. Alphabetization is therefore a form of sorting. Because of the extreme importance of sorting in almost all database applications, a great deal of effort has been expended in the creation and analysis of efficient sorting algorithms.

see also HEAPSORT, ORDERING, QUICKSORT

References
Knuth, D. E. *The Art of Computer Programming, Vol. 3: Sorting and Searching, 2nd ed.* Reading, MA: Addison-Wesley, 1973.
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Sorting." Ch. 8 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 320–339, 1992.

Source



A vertex of a DIRECTED GRAPH with no entering edges.

see also DIRECTED GRAPH, NETWORK, SINK (DIRECTED GRAPH)

Sous-Double

A MULTIPERFECT NUMBER P_3. Six sous-doubles are known, and these are believed to comprise all sous-doubles.

see also MULTIPERFECT NUMBER, SOUS-TRIPLE

Souslin's Hypothesis

Every dense linear order complete set without endpoints having at most ω disjoint intervals is order isomorphic to the CONTINUUM of REAL NUMBERS, where ω is the set of NATURAL NUMBERS.

References
Iyanaga, S. and Kawada, Y. (Eds.). "Souslin's Hypothesis." §35E.4 in in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 137, 1980.

Souslin Set

The continuous image of a POLISH SPACE, also called an ANALYTIC SET.

see also ANALYTIC SET, POLISH SPACE

Sous-Triple

A MULTIPERFECT NUMBER P_4. 36 sous-triples are known, and these are believed to comprise all sous-triples.

see also MULTIPERFECT NUMBER, SOUS-DOUBLE

Space

The concept of a space is an extremely general and important mathematical construct. Members of the space obey certain addition properties. Spaces which have been investigated and found to be of interest are usually named after one or more of their investigators. This practice unfortunately leads to names which give very little insight into the relevant properties of a given space.

One of the most general type of mathematical spaces is the TOPOLOGICAL SPACE.

see also AFFINE SPACE, BAIRE SPACE, BANACH SPACE, BASE SPACE, BERGMAN SPACE, BESOV SPACE, BOREL SPACE, CALABI-YAU SPACE, CELLULAR SPACE, CHU SPACE, DODECAHEDRAL SPACE, DRINFELD'S SYMMETRIC SPACE, EILENBERG-MAC LANE SPACE, EUCLIDEAN SPACE, FIBER SPACE, FINSLER SPACE,

FIRST-COUNTABLE SPACE, FRÉCHET SPACE, FUNCTION SPACE, G-SPACE, GREEN SPACE, HAUSDORFF SPACE, HEISENBERG SPACE, HILBERT SPACE, HYPERBOLIC SPACE, INNER PRODUCT SPACE, L_2-SPACE, LENS SPACE, LINE SPACE, LINEAR SPACE, LIOUVILLE SPACE, LOCALLY CONVEX SPACE, LOCALLY FINITE SPACE, LOOP SPACE, MAPPING SPACE, MEASURE SPACE, METRIC SPACE, MINKOWSKI SPACE, MÜNTZ SPACE, NON-EUCLIDEAN GEOMETRY, NORMED SPACE, PARACOMPACT SPACE, PLANAR SPACE, POLISH SPACE, PROBABILITY SPACE, PROJECTIVE SPACE, QUOTIENT SPACE, RIEMANN'S MODULI SPACE, RIEMANN SPACE, SAMPLE SPACE, STANDARD SPACE, STATE SPACE, STONE SPACE, TEICHMÜLLER SPACE, TENSOR SPACE, TOPOLOGICAL SPACE, TOPOLOGICAL VECTOR SPACE, TOTAL SPACE, VECTOR SPACE

Space of Closed Paths

see LOOP SPACE

Space Conic

see SKEW CONIC

Space Curve

A curve which may pass through any region of 3-D space, as contrasted to a PLANE CURVE which must lie in a single PLANE. Von Staudt (1847) classified space curves geometrically by considering the curve

$$\phi : I \to \mathbb{R}^3 \tag{1}$$

at $t_0 = 0$ and assuming that the parametric functions $\phi_i(t)$ for $i = 1, 2, 3$ are given by POWER SERIES which converge for small t. If the curve is contained in no PLANE for small t, then a coordinate transformation puts the parametric equations in the normal form

$$\phi_1(t) = t^{1+k_1} + \ldots \tag{2}$$

$$\phi_2(t) = t^{2+k_1+k_2} + \ldots \tag{3}$$

$$\phi_3(t) = t^{3+k_1+k_2+k_3} + \ldots \tag{4}$$

for integers k_1, k_2, $k_3 \geq 0$, called the local numerical invariants.

see also CURVE, CYCLIDE, FUNDAMENTAL THEOREM OF SPACE CURVES, HELIX, PLANE CURVE, SEIFERT'S SPHERICAL SPIRAL, SKEW CONIC, SPACE-FILLING FUNCTION, SPHERICAL SPIRAL, SURFACE, VIVIANI'S CURVE

References

do Carmo, M.; Fischer, G.; Pinkall, U.; and Reckziegel, H. "Singularities of Space Curves." §3.1 in *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, pp. 24–25, 1986.

Fine, H. B. "On the Singularities of Curves of Double Curvature." *Amer. J. Math.* **8**, 156–177, 1886.

Fischer, G. (Ed.). Plates 57–64 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, pp. 58–59, 1986.

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Griffiths, P. and Harris, J. *Principles of Algebraic Geometry.* New York: Wiley, 1978.

Saurel, P. "On the Singularities of Tortuous Curves." *Ann. Math.* **7**, 3–9, 1905.

Staudt, C. von. *Geometrie der Lage.* Nürnberg, Germany, 1847.

Wiener, C. "Die Abhängigkeit der Rückkehrelemente der Projektion einer unebenen Curve von deren der Curve selbst." *Z. Math. & Phys.* **25**, 95–97, 1880.

Space Diagonal

The LINE SEGMENT connecting opposite VERTICES (i.e., two VERTICES which do not share a common face) in a PARALLELEPIPED or other similar solid.

see also DIAGONAL (POLYGON), DIAGONAL (POLYHEDRON), EULER BRICK

Space Distance

The maximum distance in 3-D can occur no more than $2n - 2$ times. Also, there exists a fixed number c such that no distance determined by a set of n points in 3-D space occurs more than $cn^{5/3}$ times. The maximum distance can occur no more than $\left\lfloor \frac{1}{4}n^2 \right\rfloor$ times in 4-D, where $\lfloor x \rfloor$ is the FLOOR FUNCTION.

References

Honsberger, R. *Mathematical Gems II.* Washington, DC: Math. Assoc. Amer., pp. 122–123, 1976.

Space Division

The number of regions into which space can be divided by n SPHERES is

$$N = \tfrac{1}{3}n(n^2 - 3n + 8),$$

giving 2, 4, 8, 16, 30, 52, 84, ... (Sloane's A046127).

see also PLANE DIVISION

References

Sloane, N. J. A. Sequence A046127 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Space-Filling Curve

see SPACE-FILLING FUNCTION

Space-Filling Function

A "CURVE" (i.e., a continuous map of a 1-D INTERVAL) into a 2-D area (a PLANE-FILLING FUNCTION) or a 3-D volume.

see also HILBERT CURVE, PEANO CURVE, PEANO-GOSPER CURVE, PLANE-FILLING CURVE, SIERPIŃSKI CURVE, SPACE-FILLING POLYHEDRON

References

Pappas, T. "Paradoxical Curve–Space-Filling Curve." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./ Tetra, p. 208, 1989.

Platzman, L. K. and Bartholdi, J. J. "Spacefilling Curves and the Planar Travelling Salesman Problem." *J. Assoc. Comput. Mach.* **46**, 719–737, 1989.

Wagon, S. "A Spacefilling Curve." §6.3 in *Mathematica in Action.* New York: W. H. Freeman, pp. 196–209, 1991.

Space-Filling Polyhedron

A space-filling polyhedron is a POLYHEDRON which can be used to generate a TESSELLATION of space. There exists one 16-sided space-filling POLYHEDRON, but it is unknown if this is the unique 16-sided space-filler. The CUBE, RHOMBIC DODECAHEDRON, and TRUNCATED OCTAHEDRON are space-fillers, as are the ELONGATED DODECAHEDRON and hexagonal PRISM. These five solids are all "primary" PARALLELOHEDRA (Coxeter 1973).

P. Schmitt discovered a nonconvex aperiodic polyhedral space-filler around 1990, and a convex POLYHEDRON known as the SCHMITT-CONWAY BIPRISM which fills space only aperiodically was found by J. H. Conway in 1993 (Eppstein).

see also CUBE, ELONGATED DODECAHEDRON, KELLER'S CONJECTURE, PARALLELOHEDRON, PRISM, RHOMBIC DODECAHEDRON, SCHMITT-CONWAY BIPRISM, TESSELLATION, TILING, TRUNCATED OCTAHEDRON

References

Coxeter, H. S. M. *Regular Polytopes, 3rd ed.* New York: Dover, pp. 29–30, 1973.

Critchlow, K. *Order in Space: A Design Source Book.* New York: Viking Press, 1970.

Devlin, K. J. "An Aperiodic Convex Space-filler is Discovered." *Focus: The Newsletter of the Math. Assoc. Amer.* **13**, 1, Dec. 1993.

Eppstein, D. "Re: Aperiodic Space-Filling Tile?." `http://www.ics.uci.edu/~eppstein/junkyard/biprism.html`.

Holden, A. *Shapes, Space, and Symmetry.* New York: Dover, pp. 154–163, 1991.

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Tutton, A. E. H. *Crystallography and Practical Crystal Measurement, 2nd ed.* London: Lubrecht & Cramer, pp. 567 and 723, 1964.

Space Groups

The space groups in 2-D are called WALLPAPER GROUPS. In 3-D, the space groups are the symmetry GROUPS possible in a crystal lattice with the translation symmetry element. There are 230 space groups in $\mathbb{R}^3$, although 11 are MIRROR IMAGES of each other. They are listed by HERMANN-MAUGUIN SYMBOL in Cotton (1990).

see also HERMANN-MAUGUIN SYMBOL, LATTICE GROUPS, POINT GROUPS, WALLPAPER GROUPS

References

Arfken, G. "Crystallographic Point and Space Groups." *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, p. 248–249, 1985.

Buerger, M. J. *Elementary Crystallography.* New York: Wiley, 1956.

Cotton, F. A. *Chemical Applications of Group Theory, 3rd ed.* New York: Wiley, pp. 250–251, 1990.

Span (Geometry)

The largest possible distance between two points for a finite set of points.

see also JUNG'S THEOREM

Span (Link)

The span of an unoriented LINK diagram (also called the SPREAD) is the difference between the highest and lowest degrees of its BRACKET POLYNOMIAL. The span is a topological invariant of a knot. If a KNOT K has a reduced alternating projection of n crossings, then the span of K is $4n$.

see also LINK

Span (Polynomial)

The difference between the highest and lowest degrees of a POLYNOMIAL.

Span (Set)

For a SET S, the span is defined by $\max S - \min S$, where max is the MAXIMUM and min is the MINIMUM.

References

Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 207, 1994.

Span (Vectors)

The span of SUBSPACE generated by VECTORS $\mathbf{v}_1$ and $\mathbf{v}_2 \in \mathbb{V}$ is

$$\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) \equiv \{r\mathbf{v}_1 + s\mathbf{v}_2 : r, s \in \mathbb{R}\}.$$

Sparse Matrix

A MATRIX which has only a small number of NONZERO elements.

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Sparse Linear Systems." §2.7 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 63–82, 1992.

Spearman Rank Correlation

The Spearman rank correlation is defined by

$$r' \equiv \frac{\sum xy}{\sqrt{\sum x^2 \sum y^2}} = 1 - 6\sum \frac{d^2}{N(N^2-1)}. \quad (1)$$

The VARIANCE, KURTOSIS, and higher order MOMENTS are

$$\sigma^2 = \frac{1}{N-1} \quad (2)$$

$$\gamma_2 = -\frac{114}{25N} - \frac{6}{5N^2} - \ldots \quad (3)$$

$$\gamma_3 = \gamma_5 = \ldots = 0. \quad (4)$$

Student was the first to obtain the VARIANCE. The Spearman rank correlation is an R-ESTIMATE.

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 634–637, 1992.

Special Curve

see PLANE CURVE, SPACE CURVE

Special Function

see FUNCTION

Special Linear Group

The special linear group $SL_n(q)$ is the MATRIX GROUP corresponding to the set of $n \times n$ COMPLEX MATRICES having DETERMINANT +1. It is a SUBGROUP of the GENERAL LINEAR GROUP $GL_n(q)$ and is also a LIE GROUP.

see also GENERAL LINEAR GROUP, SPECIAL ORTHOGONAL GROUP, SPECIAL UNITARY GROUP

References

Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; and Wilson, R. A. "The Groups $GL_n(q)$, $SL_n(q)$, $PGL_n(q)$, and $PSL_n(q) = L_n(q)$." §2.1 in *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford, England: Clarendon Press, p. x, 1985.

Special Matrix

A matrix whose entries satisfy

$$a_{ij} = \begin{cases} 0 & \text{if } j > i+1 \\ -1 & \text{if } j = i+1 \\ 0 \text{ or } 1 & \text{if } j \leq i. \end{cases}$$

There are 2^{n-1} special MINIMAL MATRICES of size $n \times n$.

References

Knuth, D. E. "Problem 10470." *Amer. Math. Monthly* **102**, 655, 1995.

Special Orthogonal Group

The special orthogonal group $SO_n(q)$ is the SUBGROUP of the elements of GENERAL ORTHOGONAL GROUP $GO_n(q)$ with DETERMINANT 1.

see also GENERAL ORTHOGONAL GROUP, SPECIAL LINEAR GROUP, SPECIAL UNITARY GROUP

References

Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; and Wilson, R. A. "The Groups $GO_n(q)$, $SO_n(q)$, $PGO_n(q)$, and $PSO_n(q)$, and $O_n(q)$." §2.4 in *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford, England: Clarendon Press, pp. xi–xii, 1985.

Special Point

A POINT which does not lie on at least one ORDINARY LINE.

see also ORDINARY POINT

References

Guy, R. K. "Unsolved Problems Come of Age." *Amer. Math. Monthly* **96**, 903–909, 1989.

Special Series Theorem

If the difference between the order and the dimension of a series is less than the GENUS (CURVE), then the series is special.

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 253, 1959.

Special Unitary Group

The special unitary group $SU_n(q)$ is the set of $n \times n$ UNITARY MATRICES with DETERMINANT +1 (having $n^2 - 1$ independent parameters). $SU(2)$ is HOMEOMORPHIC with the ORTHOGONAL GROUP $O_3^+(2)$. It is also called the UNITARY UNIMODULAR GROUP and is a LIE GROUP. The special unitary group can be represented by the MATRIX

$$U(a,b) = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}, \quad (1)$$

where $a^*a + b^*b = 1$ and a, b are the CAYLEY-KLEIN PARAMETERS. The special unitary group may also be represented by the MATRIX

$$U(\xi,\eta,\zeta) = \begin{bmatrix} e^{i\xi}\cos\eta & e^{i\zeta}\sin\eta \\ -e^{-i\zeta}\sin\eta & e^{-i\xi}\cos\eta \end{bmatrix}, \quad (2)$$

or the matrices

$$U_x(\tfrac{1}{2}\phi) = \begin{bmatrix} \cos(\frac{1}{2}\phi) & i\sin(\frac{1}{2}\phi) \\ i\sin(\frac{1}{2}\phi) & \cos(\frac{1}{2}\phi) \end{bmatrix} \quad (3)$$

$$U_y(\tfrac{1}{2}\beta) = \begin{bmatrix} \cos(\frac{1}{2}\beta) & \sin(\frac{1}{2}\beta) \\ -\sin(\frac{1}{2}\beta) & \cos(\frac{1}{2}\beta) \end{bmatrix} \quad (4)$$

$$U_z(\xi) = \begin{bmatrix} e^{i\xi} & 0 \\ 0 & e^{-i\xi} \end{bmatrix}. \quad (5)$$

The order $2j+1$ representation is

$$\begin{aligned} U_{p,q}{}^{(j)}(\alpha,\beta,\gamma) &= \sum_m \frac{(-1)^{m-q-p}\sqrt{(j+p)!(j-p)!(j+q)!(j-q)!}}{(j-p-m)!(j+q-m)!(m+p-q)!m!} \\ &\quad\times e^{iq\alpha}\cos^{2j+q-p-2m}(\tfrac{1}{2}\beta)\sin^{p+2m-q}(\tfrac{1}{2}\beta)e^{ip\gamma}. \qquad (6) \end{aligned}$$

The summation is terminated by putting $1/(-N)! = 0$. The CHARACTER is given by

$$\begin{aligned} \chi^{(j)}(\alpha) &= \begin{cases} 1+2\cos\alpha+\ldots+2\cos(j\alpha) \\ 2[\cos(\frac{1}{2}\alpha)+\cos(\frac{3}{2}\alpha)+\ldots+\cos(j\alpha)] \end{cases} \\ &= \begin{cases} \frac{\sin[(j+\frac{1}{2})\alpha]}{\sin(\frac{1}{2}\alpha)} & \text{for } j=0,1,2,\ldots \\ \frac{\sin[(j+\frac{1}{2})\alpha]}{\sin(\frac{1}{2}\alpha)} & \text{for } j=\frac{1}{2},\frac{3}{2},\ldots. \end{cases} \qquad (7) \end{aligned}$$

see also ORTHOGONAL GROUP, SPECIAL LINEAR GROUP, SPECIAL ORTHOGONAL GROUP

References

Arfken, G. "Special Unitary Group, $SU(2)$ and $SU(2)$–O_3^+ Homomorphism." *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 253–259, 1985.

Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; and Wilson, R. A. "The Groups $GU_n(q)$, $SU_n(q)$, $PGU_n(q)$, and $PSU_n(q) = U_n(q)$." §2.2 in *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford, England: Clarendon Press, p. x, 1985.

Species

A species of structures is a rule F which

1. Produces, for each finite set U, a finite set $F[U]$,
2. Produces, for each bijection $\sigma: U \to V$, a function

$$F[\sigma]: F[U] \to F[V].$$

The functions $F[\sigma]$ should further satisfy the following functorial properties:

1. For all bijections $\sigma: U \to V$ and $\tau: V \to W$,

$$F[\tau\circ\sigma] = F[\tau]\circ F[\sigma],$$

2. For the IDENTITY MAP $\mathrm{Id}_U: U \to U$,

$$F[\underset{U}{\mathrm{Id}}] = \underset{F[U]}{\mathrm{Id}}.$$

An element $\sigma \in F[U]$ is called an F-structure on U (or a structure of species F on U). The function $F[\sigma]$ is called the transport of F-structures along σ.

References

Bergeron, F.; Labelle, G.; and Leroux, P. *Combinatorial Species and Tree-Like Structures.* Cambridge, England: Cambridge University Press, p. 5, 1998.

Specificity

The probability that a STATISTICAL TEST will be negative for a negative statistic.

see also SENSITIVITY, STATISTICAL TEST, TYPE I ERROR, TYPE II ERROR

Spectral Norm

The NATURAL NORM induced by the L_2-NORM. Let $\mathsf{A}^\dagger$ be the ADJOINT of the SQUARE MATRIX A, so that $\mathsf{A}^\dagger = a_{ji}^*$, then

$$\begin{aligned} \|\mathsf{A}\|_2 &= (\text{maximum eigenvalue of } \mathsf{A}^\dagger\mathsf{A})^{1/2} \\ &= \max_{\|x\|_2\neq 0}\frac{\|\mathsf{A}\mathbf{x}\|_2}{\|\mathbf{x}\|_2}. \end{aligned}$$

see also L_2-NORM, MATRIX NORM

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, pp. 1115, 1979.

Strang, G. §6.2 and 7.2 in *Linear Algebra and Its Applications, 4th ed.* New York: Academic Press, 1980.

Spectral Power Density

$$P_y(\nu) \equiv \lim_{T\to\infty}\frac{2}{T}\left|\int_{-T/2}^{T/2}[y(t)-\bar{y}]e^{-2\pi i\nu t}\,dt\right|^2,$$

so

$$\begin{aligned} \int_0^\infty P_y(\nu)\,d\nu &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}[y(t)-\bar{y}]^2\,dt \\ &= \left\langle (y-\bar{y})^2\right\rangle = \sigma_y{}^2. \end{aligned}$$

see also POWER SPECTRUM

Spectral Radius

Let A be an $n\times n$ MATRIX with COMPLEX or REAL elements with EIGENVALUES $\lambda_1, \ldots, \lambda_n$. Then the spectral radius $\rho(\mathsf{A})$ of A is

$$\rho(\mathsf{A}) = \max_{1\leq i\leq n}|\lambda_i|.$$

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, pp. 1115–1116, 1979.

Spectral Rigidity

The mean square deviation of the best local fit straight line to a staircase cumulative spectral density over a normalized energy scale.

References

Ott, E. *Chaos in Dynamical Systems.* New York: Cambridge University Press, p. 341, 1993.

Spectral Theorem

Let H be a HILBERT SPACE, $B(H)$ the set of BOUNDED linear operators from H to itself, and $\sigma(T)$ the SPECTRUM of T. Then if $T \in B(H)$ and T is normal, there exists a unique resolution of the identity E on the Borel subsets of $\sigma(T)$ which satisfies

$$T = \underset{\sigma(T)}{\mathrm{int}}\, \lambda\, dE(\lambda).$$

Furthermore, every projection $E(\omega)$ COMMUTES with every $S \in B(H)$ which COMMUTES with T.

References

Rudin, W. Theorem 12.23 in *Functional Analysis, 2nd ed.* New York: McGraw-Hill, 1991.

Spectrum (Operator)

Let T be an OPERATOR on a HILBERT SPACE. The spectrum $\sigma(T)$ of T is the set of λ such that $(T - \lambda I)$ is not invertible on all of the HILBERT SPACE, where the λs are COMPLEX NUMBERS and I is the IDENTITY OPERATOR. The definition can also be stated in terms of the resolvent of an operator

$$\rho(T) = \{\lambda : (T - \lambda I) \text{ is invertible}\},$$

and then the spectrum is defined to be the complement of $\rho(T)$ in the COMPLEX PLANE. The reason for doing this is that it is easy to demonstrate that $\rho(T)$ is an OPEN SET, which shows that the spectrum is closed.

see also HILBERT SPACE

Spectrum Sequence

A spectrum sequence is a SEQUENCE formed by successive multiples of a REAL NUMBER a rounded down to the nearest INTEGER $s_n = \lfloor na \rfloor$. If a is IRRATIONAL, the spectrum is called a BEATTY SEQUENCE.

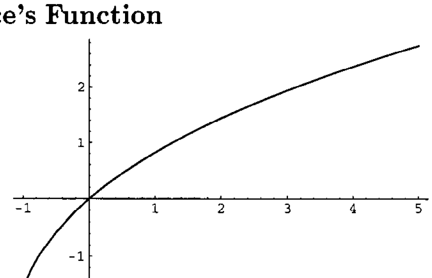
see also BEATTY SEQUENCE, LAGRANGE SPECTRUM, MARKOV SPECTRUM

Speed

The SCALAR $|\mathbf{v}|$ equal to the magnitude of the VELOCITY $\mathbf{v}$.

see also ANGULAR VELOCITY, VELOCITY

Spence's Function



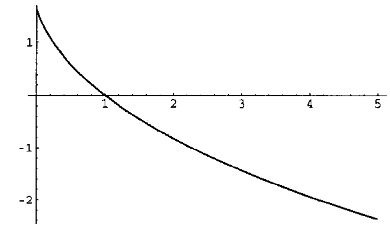
$$F(x) \equiv \int_0^x \frac{\ln(1+t)}{t}\, dt.$$

see also SPENCE'S INTEGRAL

References

Berestetskii, V. B.; Lifschitz, E. M.; and Ditaevskii, L. P. *Quantum Electrodynamics, 2nd ed.* Oxford, England: Pergamon Press, p. 596, 1982.

Spence's Integral



$$F(x) = \mathrm{Li}_2(1-x) = \int_{1-x}^{0} \frac{\ln(1-t)}{t}\, dt,$$

where $\mathrm{Li}_2(x)$ is the DILOGARITHM.

see also SPENCE'S FUNCTION

Spencer's 15-Point Moving Average

A MOVING AVERAGE using 15 points having weights -3, -6, -5, 3, 21, 46, 67, 74, 67, 46, 21, 3, -5, -6, and -3. It is sometimes used by actuaries.

see also MOVING AVERAGE

References

Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics, Pt. 1, 3rd ed.* Princeton, NJ: Van Nostrand, p. 223, 1962.

Sperner's Theorem

The MAXIMUM CARDINALITY of a collection of SUBSETS of a t-element SET T, none of which contains another, is the BINOMIAL COEFFICIENT $\binom{t}{\lfloor t/2 \rfloor}$, where $\lfloor x \rfloor$ is the FLOOR FUNCTION.

see also CARDINALITY

Sphenocorona

see JOHNSON SOLID

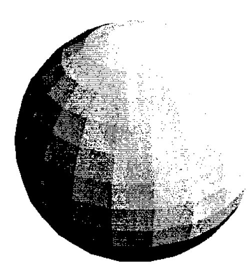
Sphenoid

see DISPHENOID

Sphenomegacorona

see JOHNSON SOLID

Sphere



A sphere is defined as the set of all points in $\mathbb{R}^3$ which are a distance r (the "RADIUS") from a given point (the "CENTER"). Twice the RADIUS is called the DIAMETER, and pairs of points on opposite sides of a DIAMETER are called ANTIPODES. The term "sphere" technically refers to the outer surface of a "BUBBLE," which is denoted $\mathbb{S}^2$. However, in common usage, the word *sphere* is also used to mean the UNION of a sphere and its INTERIOR (a "solid sphere"), where the INTERIOR is called a BALL. The SURFACE AREA of the sphere and VOLUME of the BALL of RADIUS r are given by

$$S = 4\pi r^2 \tag{1}$$
$$V = \tfrac{4}{3}\pi r^3 \tag{2}$$

(Beyer 1987, p. 130). In *On the Sphere and Cylinder* (ca. 225 BC), Archimedes became the first to derive these equations (although he expressed π in terms of the sphere's circular cross-section). The fact that

$$\frac{V_{\text{sphere}}}{V_{\text{circumscribed cylinder}} - V_{\text{sphere}}} = \frac{\frac{4}{3}}{2-\frac{4}{3}} = \frac{\frac{4}{3}}{\frac{2}{3}} = 2 \tag{3}$$

was also known to Archimedes.

Any cross-section through a sphere is a CIRCLE (or, in the degenerate case where the slicing PLANE is tangent to the sphere, a point). The size of the CIRCLE is maximized when the PLANE defining the cross-section passes through a DIAMETER.

The equation of a sphere of RADIUS r is given in CARTESIAN COORDINATES by

$$x^2 + y^2 + z^2 = r^2, \tag{4}$$

which is a special case of the ELLIPSOID

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{5}$$

and SPHEROID

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1. \tag{6}$$

A sphere may also be specified in SPHERICAL COORDINATES by

$$x = \rho\cos\theta\sin\phi \tag{7}$$
$$y = \rho\sin\theta\sin\phi \tag{8}$$
$$z = \rho\cos\phi, \tag{9}$$

where θ is an azimuthal coordinate running from 0 to 2π (LONGITUDE), ϕ is a polar coordinate running from 0 to π (COLATITUDE), and ρ is the RADIUS. Note that there are several other notations sometimes used in which the symbols for θ and ϕ are interchanged or where r is used instead of ρ. If ρ is allowed to run from 0 to a given RADIUS r, then a solid BALL is obtained. Converting to "standard" parametric variables $a = \rho$, $u = \theta$, and $v = \phi$ gives the first FUNDAMENTAL FORMS

$$E = a\sin^2 v \tag{10}$$
$$F = 0 \tag{11}$$
$$G = a, \tag{12}$$

second FUNDAMENTAL FORMS

$$e = a^2\sin^2 v \tag{13}$$
$$f = 0 \tag{14}$$
$$g = a^2, \tag{15}$$

AREA ELEMENT

$$dA = a\sin v, \tag{16}$$

GAUSSIAN CURVATURE

$$K = \frac{1}{a^2}, \tag{17}$$

and MEAN CURVATURE

$$H = \frac{1}{a}. \tag{18}$$

A sphere may also be represented parametrically by letting $u \equiv r\cos\phi$, so

$$x = \sqrt{r^2-u^2}\cos\theta \tag{19}$$
$$y = \sqrt{r^2-u^2}\sin\theta \tag{20}$$
$$z = u, \tag{21}$$

where θ runs from 0 to 2π and u runs from $-r$ to r.

Given two points on a sphere, the shortest path on the surface of the sphere which connects them (the SPHERE GEODESIC) is an ARC of a CIRCLE known as a GREAT CIRCLE. The equation of the sphere with points (x_1, y_1, z_1) and (x_2, y_2, z_2) lying on a DIAMETER is given by

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2) = 0. \tag{22}$$

Four points are sufficient to uniquely define a sphere. Given the points (x_i, y_i, z_i) with $i = 1, 2, 3$, and 4, the sphere containing them is given by the beautiful DETERMINANT equation

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ {x_1}^2+{y_1}^2+{z_1}^2 & x_1 & y_1 & z_1 & 1 \\ {x_2}^2+{y_2}^2+{z_2}^2 & x_2 & y_2 & z_2 & 1 \\ {x_3}^2+{y_3}^2+{z_3}^2 & x_3 & y_3 & z_3 & 1 \\ {x_4}^2+{y_4}^2+{z_4}^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0 \tag{23}$$

(Beyer 1987, p. 210).

The generalization of a sphere in n dimensions is called a HYPERSPHERE. An n-D HYPERSPHERE can be specified by the equation

$$x_1{}^2 + x_2{}^2 + \ldots + x_n{}^2 = r^2. \tag{24}$$

The distribution of ANGLES for random rotation of a sphere is

$$P(\theta) = \frac{2}{\pi}\sin^2(\tfrac{1}{2}\theta), \tag{25}$$

giving a MEAN of $\pi/2 + 2/\pi$.

To pick a random point on the surface of a sphere, let u and v be random variates on $[0,1]$. Then

$$\theta = 2\pi u \tag{26}$$

$$\phi = \cos^{-1}(2v-1). \tag{27}$$

This works since the SOLID ANGLE is

$$d\Omega = \sin\phi\, d\theta\, d\phi = d\theta\, d(\cos\phi). \tag{28}$$

Another easy way to pick a random point on a SPHERE is to generate three gaussian random variables x, y, and z. Then the distribution of the vectors

$$\frac{1}{\sqrt{x^2+y^2+z^2}}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{29}$$

is uniform over the surface $\mathbb{S}^2$. Another method is to pick z from a UNIFORM DISTRIBUTION over $[-1,1]$ and θ from a UNIFORM DISTRIBUTION over $[0,2\pi)$. Then the points

$$\begin{bmatrix} \sqrt{1-z^2}\cos\theta \\ \sqrt{1-z^2}\sin\theta \\ z \end{bmatrix} \tag{30}$$

are uniformly distributed over $\mathbb{S}^2$.

Pick four points on a sphere. What is the probability that the TETRAHEDRON having these points as VERTICES contains the CENTER of the sphere? In the 1-D case, the probability that a second point is on the opposite side of 1/2 is 1/2. In the 2-D case, pick two points. In order for the third to form a TRIANGLE containing the CENTER, it must lie in the quadrant bisected by a LINE SEGMENT passing through the center of the CIRCLE and the bisector of the two points. This happens for one QUADRANT, so the probability is 1/4. Similarly, for a sphere the probability is one OCTANT, or 1/8.

Pick two points at random on a unit sphere. The first one can be assigned the coordinate (0, 0, 1) without loss of generality. The second point can be given the coordinates $(\sin\phi, 0\cos\phi)$ with $\theta \equiv 0$ since all points with the same ϕ are rotationally identical. The distance between the two points is then

$$r = \sqrt{\sin^2\phi + (1-\cos\phi)^2} = \sqrt{2-\cos\phi} = 2\sin(\tfrac{1}{2}\phi). \tag{31}$$

Because the surface AREA element is

$$d\Omega = \sin\phi\, d\theta\, d\phi, \tag{32}$$

the probability that two points are a distance r apart is

$$\begin{aligned} P_\phi(r) &= \frac{\int_0^\pi \delta(\phi - r)\sin\phi\, d\phi}{\int_0^\pi \sin\phi\, d\phi} \\ &= \tfrac{1}{2}\int_0^\pi \delta[r - 2\sin(\tfrac{1}{2}\phi)]\sin\phi\, d\phi. \end{aligned} \tag{33}$$

The DELTA FUNCTION contributes when

$$\tfrac{1}{2}r = \sin(\tfrac{1}{2}\phi) \tag{34}$$

$$\phi = 2\sin^{-1}(\tfrac{1}{2}r), \tag{35}$$

so

$$\begin{aligned} P_\phi(r) &= \tfrac{1}{2}\sin[2\sin^{-1}(\tfrac{1}{2}r)] = \sin[\sin^{-1}(\tfrac{1}{2}r)]\cos[\sin^{-1}(\tfrac{1}{2}r)] \\ &= \tfrac{1}{2}r\sqrt{1-(\tfrac{1}{2}r)^2} = \tfrac{1}{4}r\sqrt{4-r^2}. \end{aligned} \tag{36}$$

However, we need

$$P_r(r)\, dr = P_\phi(r)\frac{d\phi}{dr}\, dr, \tag{37}$$

and

$$\begin{aligned} \tfrac{1}{2}dr &= \tfrac{1}{2}\cos(\tfrac{1}{2}\phi)\, d\phi = \tfrac{1}{2}\sqrt{1-\sin^2(\tfrac{1}{2}\phi)}\, d\phi \\ &= \tfrac{1}{2}\sqrt{1-(\tfrac{1}{2}r)^2}\, d\phi = \tfrac{1}{4}\sqrt{4-r^2}\, d\phi \end{aligned} \tag{38}$$

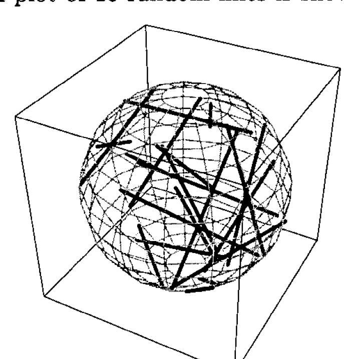
so

$$\frac{d\phi}{dr} = \frac{2}{\sqrt{4-r^2}}, \tag{39}$$

and

$$P_r(r) = \tfrac{1}{4}r\sqrt{4-r^2}\frac{2}{\sqrt{4-r^2}} = \tfrac{1}{2}r \tag{40}$$

for $r \in [0,2]$. Somewhat surprisingly, the largest distances are the most common, contrary to most people's intuition. A plot of 15 random lines is shown below.



The MOMENTS about zero are

$$\mu'_n = \langle r^n \rangle = \int_0^2 r^n\,dr = \frac{2^{n+1}}{2+n}, \tag{41}$$

giving the first few as

$$\mu'_1 = \tfrac{4}{3} \tag{42}$$
$$\mu'_2 = 2 \tag{43}$$
$$\mu'_3 = \tfrac{16}{5} \tag{44}$$
$$\mu'_4 = \tfrac{16}{3}. \tag{45}$$

Moments about the MEAN are

$$\mu = \tfrac{4}{3} \tag{46}$$
$$\mu_2 = \sigma^2 = \tfrac{2}{9} \tag{47}$$
$$\mu_3 = -\tfrac{8}{135} \tag{48}$$
$$\mu_4 = \tfrac{16}{135}, \tag{49}$$

so the SKEWNESS and KURTOSIS are

$$\gamma_1 = \tfrac{4}{5}\sqrt{2} \tag{50}$$
$$\gamma_2 = -\tfrac{5}{3}. \tag{51}$$

see also BALL, BING'S THEOREM, BUBBLE, CIRCLE, DANDELIN SPHERES, DIAMETER, ELLIPSOID, EXOTIC SPHERE, FEJES TÓTH'S PROBLEM, HYPERSPHERE, LIEBMANN'S THEOREM, LIOUVILLE'S SPHERE-PRESERVING THEOREM, MIKUSIŃSKI'S PROBLEM, NOISE SPHERE, OBLATE SPHEROID, OSCULATING SPHERE, PARALLELIZABLE, PROLATE SPHEROID, RADIUS, SPACE DIVISION, SPHERE PACKING, TENNIS BALL THEOREM

References

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Geometry Center. "The Sphere." `http://www.geom.umn.edu/zoo/toptype/sphere/`.

Sphere-Cylinder Intersection

see CYLINDER-SPHERE INTERSECTION

Sphere Embedding

A 4-sphere has POSITIVE CURVATURE, with

$$R^2 = x^2 + y^2 + z^2 + w^2 \tag{1}$$

$$2x\frac{dx}{dw} + 2y\frac{dy}{dw} + 2z\frac{dz}{dw} + 2w = 0. \tag{2}$$

Since

$$\mathbf{r} \equiv x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}, \tag{3}$$

$$dw = -\frac{x\,dx + y\,dy + z\,dz}{w} = -\frac{\mathbf{r}\cdot d\mathbf{r}}{\sqrt{R^2 - r^2}}. \tag{4}$$

To stay on the surface of the sphere,

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 + dw^2 \\ &= dx^2 + dy^2 + dz^2 + \frac{r^2\,dr^2}{R^2 - r^2} \\ &= dr^2 + r^2\,d\Omega^2 + \frac{dr^2}{\frac{R^2}{r^2} - 1} \\ &= dr^2\left(1 + \frac{1}{\frac{R^2}{r^2} - 1}\right) + r^2\,d\Omega^2 \\ &= dr^2\left(\frac{\frac{R^2}{r^2}}{\frac{R^2}{r^2} - 1}\right) + r^2\,d\Omega^2 \\ &= \frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2\,d\Omega^2. \end{aligned} \tag{5}$$

With the addition of the so-called expansion parameter, this is the Robertson-Walker line element.

Sphere Eversion

Smale (1958) proved that it is mathematically possible to turn a SPHERE inside-out without introducing a sharp crease at any point. This means there is a regular homotopy from the standard embedding of the 2-SPHERE in EUCLIDEAN 3-space to the mirror-reflection embedding such that at every stage in the homotopy, the sphere is being IMMERSED in EUCLIDEAN SPACE. This result is so counterintuitive and the proof so technical that the result remained controversial for a number of years.

In 1961, Arnold Shapiro devised an explicit eversion but did not publicize it. Phillips (1966) heard of the result and, in trying to reproduce it, actually devised an independent method of his own. Yet another eversion was devised by Morin, which became the basis for the movie by Max (1977). Morin's eversion also produced explicit algebraic equations describing the process. The original method of Shapiro was subsequently published by Francis and Morin (1979).

see also EVERSION, SPHERE

References

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Phillips, A. "Turning a Surface Inside Out." *Sci. Amer.* **214**, 112–120, Jan. 1966.

Smale, S. "A Classification of Immersions of the Two-Sphere." *Trans. Amer. Math. Soc.* **90**, 281–290, 1958.

Sphere Geodesic

see GREAT CIRCLE

Sphere Packing

Let η denote the PACKING DENSITY, which is the fraction of a VOLUME filled by identical packed SPHERES. In 2-D (CIRCLE PACKING), there are two periodic packings for identical CIRCLES: square lattice and hexagonal lattice. Fejes Tóth (1940) proved that the hexagonal lattice is indeed the densest of *all* possible plane packings (Conway and Sloane 1993, pp. 8–9).

In 3-D, there are three periodic packings for identical spheres: cubic lattice, face-centered cubic lattice, and hexagonal lattice. It was hypothesized by Kepler in 1611 that close packing (cubic or hexagonal) is the densest possible (has the greatest η), and this assertion is known as the KEPLER CONJECTURE. The problem of finding the densest packing of spheres (not necessarily periodic) is therefore known as the KEPLER PROBLEM. The KEPLER CONJECTURE is intuitively obvious, but the proof still remains elusive. However, Gauss (1831) did prove that the face-centered cubic is the densest *lattice* packing in 3-D (Conway and Sloane 1993, p. 9). This result has since been extended to HYPERSPHERE PACKING.

In 3-D, face-centered cubic close packing and hexagonal close packing (which is distinct from hexagonal lattice), both give

$$\eta = \frac{\pi}{3\sqrt{2}} \approx 74.048\%. \tag{1}$$

For packings in 3-D, C. A. Rogers (1958) showed that

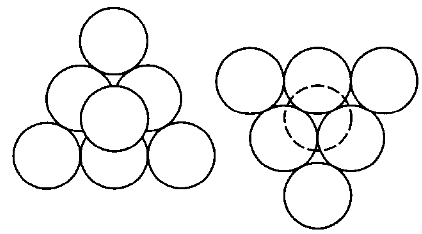
$$\eta < \sqrt{18}\,(\cos^{-1}\tfrac{1}{3} - \tfrac{1}{3}\pi) \approx 77.96355700\% \tag{2}$$

(Le Lionnais 1983). This was subsequently improved to 77.844% (Lindsey 1986), then 77.836% (Muder 1988). However, Rogers (1958) remarks that "many mathematicians believe, and all physicists know" that the actual answer is 74.05% (Conway and Sloane 1993, p. 3).

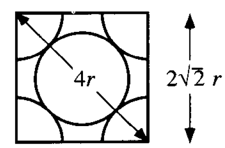
"Random" close packing in 3-D gives only $\eta \approx 64\%$ (Jaeger and Nagel 1992).

The PACKING DENSITIES for several packing types are summarized in the following table.

Packing	η (exact)	η (approx.)
square lattice (2-D)	$\frac{\pi}{4}$	0.7854
hexagonal lattice (2-D)	$\frac{\pi}{2\sqrt{3}}$	0.9069
cubic lattice	$\frac{\pi}{6}$	0.5236
hexagonal lattice	$\frac{\pi}{3\sqrt{3}}$	0.6046
face-centered cubic lattice	$\frac{\pi}{3\sqrt{2}}$	0.7405
random	—	0.6400



For cubic close packing, pack six SPHERES together in the shape of an EQUILATERAL TRIANGLE and place another SPHERE on top to create a TRIANGULAR PYRAMID. Now create another such grouping of seven SPHERES and place the two PYRAMIDS together facing in opposite directions. A CUBE emerges. Consider a face of the CUBE, illustrated below.



The "unit cell" cube contains eight 1/8-spheres (one at each VERTEX) and six HEMISPHERES. The total VOLUME of SPHERES in the unit cell is

$$\begin{aligned} V_{\text{spheres in unit cell}} &= (8\cdot\tfrac{1}{8} + 6\cdot\tfrac{1}{2})\frac{4\pi}{3}r^3 \\ &= 4\cdot\frac{4\pi}{3}r^3 = \tfrac{16}{3}\pi r^3. \end{aligned} \tag{3}$$

The diagonal of the face is $4r$, so each side is $2\sqrt{2}\,r$. The VOLUME of the unit cell is therefore

$$V_{\text{unit cell}} = (2\sqrt{2}\,r)^3 = 16\sqrt{2}\,r^3. \tag{4}$$

The PACKING DENSITY is therefore

$$\eta_{\text{CCP}} = \frac{\frac{16}{3}\pi r^2}{16\sqrt{2}r^3} = \frac{\pi}{3\sqrt{2}} \tag{5}$$

(Conway and Sloane 1993, p. 2).

Hexagonal close packing must give the same values, since sliding one sheet of SPHERES cannot affect the volume they occupy. To verify this, construct a 3-D diagram containing a hexagonal unit cell with three layers. Both the top and the bottom contain six 1/6-SPHERES and one HEMISPHERE. The total number of spheres in these two rows is therefore

$$2(6\tfrac{1}{6} + 1\tfrac{1}{2}) = 3. \tag{6}$$

The VOLUME of SPHERES in the middle row cannot be simply computed using geometry. However, symmetry requires that the piece of the SPHERE which is cut off is exactly balanced by an extra piece on the other side. There are therefore three SPHERES in the middle layer, for a total of six, and a total VOLUME

$$V_{\text{spheres in unit cell}} = 6\cdot\frac{4\pi}{3}r^3(3+3) = 8\pi r^3. \tag{7}$$

The base of the HEXAGON is made up of 6 EQUILATERAL TRIANGLES with side lengths $2r$. The unit cell base AREA is therefore

$$A_{\text{unit cell}} = 6[\tfrac{1}{2}(2r)(\sqrt{3}\,r)] = 6\sqrt{3}\,r^2. \tag{8}$$

The height is the same as that of two TETRAHEDRA length $2r$ on a side, so

$$h_{\text{unit cell}} = 2\left(2r\sqrt{\frac{2}{3}}\right), \tag{9}$$

giving

$$\eta_{\text{HCP}} = \frac{8\pi r^3}{(6\sqrt{3}\,r^2)\left(4r\sqrt{\frac{2}{3}}\right)} = \frac{\pi}{3\sqrt{2}} \tag{10}$$

(Conway and Sloane 1993, pp. 7 and 9).

If we had actually wanted to compute the VOLUME of SPHERE inside and outside the HEXAGONAL PRISM, we could use the SPHERICAL CAP equation to obtain

$$\begin{aligned} V_\subset &= \tfrac{1}{3}\pi h^2(3r-h) = \frac{1}{3}\pi r^3 \frac{1}{3}\left(3 - \frac{1}{\sqrt{3}}\right) \\ &= \frac{1}{9}\pi r^3\left(3 - \frac{\sqrt{3}}{3}\right) = \tfrac{1}{27}\pi r^3(9-\sqrt{3}) \qquad (11) \\ V_\supset &= \pi r^3\left[\tfrac{4}{3} - \tfrac{1}{27}(9-\sqrt{3})\right] = \tfrac{1}{27}\pi r^3(36-9+\sqrt{3}) \\ &= \tfrac{1}{27}\pi r^3(27+\sqrt{3}). \qquad (12) \end{aligned}$$

The rigid packing with *lowest* density known has $\eta \approx 0.0555$ (Gardner 1966). To be RIGID, each SPHERE must touch at least four others, and the four contact points cannot be in a single HEMISPHERE or all on one equator.

If spheres packed in a cubic lattice, face-centered cubic lattice, and hexagonal lattice are allowed to expand, they form cubes, hexagonal prisms, and rhombic dodecahedra. Compressing a random packing gives polyhedra with an average of 13.3 faces (Coxeter 1958, 1961).

For sphere packing inside a CUBE, see Goldberg (1971) and Schaer (1966).

see also CANNONBALL PROBLEM, CIRCLE PACKING, DODECAHEDRAL CONJECTURE, HEMISPHERE, HERMITE CONSTANTS, HYPERSPHERE, HYPERSPHERE PACKING, KEPLER CONJECTURE, KEPLER PROBLEM, KISSING NUMBER, LOCAL DENSITY, LOCAL DENSITY CONJECTURE, SPHERE

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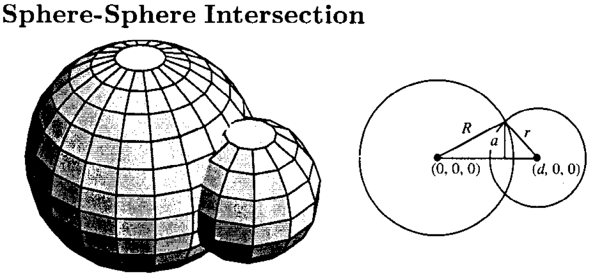
Sphere Point Picking

see FEJES TÓTH'S PROBLEM

Sphere-Sphere Intersection



Let two spheres of RADII R and r be located along the x-AXIS centered at $(0,0,0)$ and $(d,0,0)$, respectively. Not surprisingly, the analysis is very similar to the case of the CIRCLE-CIRCLE INTERSECTION. The equations of the two SPHERES are

$$x^2 + y^2 + z^2 = R^2 \tag{1}$$
$$(x-d)^2 + y^2 + z^2 = r^2. \tag{2}$$

Combining (1) and (2) gives

$$(x-d)^2 + (R^2 - x^2) = r^2. \tag{3}$$

Multiplying through and rearranging give

$$x^2 - 2dx + d^2 - x^2 = r^2 - R^2. \tag{4}$$

Solving for x gives

$$x = \frac{d^2 - r^2 + R^2}{2d}. \tag{5}$$

The intersection of the SPHERES is therefore a curve lying in a PLANE parallel to the yz-plane at a single x-coordinate. Plugging this back into (1) gives

$$y^2 + z^2 = R^2 - x^2 = R^2 - \left(\frac{d^2 - r^2 + R^2}{2d}\right)^2 = \frac{4d^2R^2 - (d^2 - r^2 + R^2)^2}{4d^2}, \tag{6}$$

which is a CIRCLE with RADIUS

$$\begin{aligned} a &= \frac{1}{2d}\sqrt{4d^2R^2 - (d^2 - r^2 + R^2)^2} \\ &= \frac{1}{2d}[(-d + r - R)(-d - r + R) \\ &\quad \times [(-d + r + R)(d + r + R)]^{1/2}. \end{aligned} \tag{7}$$

The VOLUME of the 3-D LENS common to the two spheres can be found by adding the two SPHERICAL CAPS. The distances from the SPHERES' centers to the bases of the caps are

$$d_1 = x \tag{8}$$
$$d_2 = d - x, \tag{9}$$

so the heights of the caps are

$$h_1 = R - d_1 = \frac{(r - R + d)(r + R - d)}{2d} \tag{10}$$
$$h_2 = r - d_2 = \frac{(R - r + d)(R + r - d)}{2d}. \tag{11}$$

The VOLUME of a SPHERICAL CAP of height h' for a SPHERE of RADIUS R' is

$$V(R', h') = \tfrac{1}{3}\pi h'^2(3R' - h'). \tag{12}$$

Letting $R_1 = R$ and $R_2 = r$ and summing the two caps gives

$$\begin{aligned} V &= V(R_1, h_1) + V(R_2, h_2) \\ &= \frac{\pi(R + r - d)^2(d^2 + 2dr - 3r^2 + 2dR + 6rR - 3R^2)}{12d}. \end{aligned} \tag{13}$$

This expression gives $V = 0$ for $d = r + R$ as it must. In the special case $r = R$, the VOLUME simplifies to

$$V = \tfrac{1}{12}\pi(4R + d)(2R - d)^2. \tag{14}$$

see also APPLE, CIRCLE-CIRCLE INTERSECTION, DOUBLE BUBBLE, LENS, SPHERE

Sphere with Tunnel

Find the tunnel between two points A and B on a gravitating SPHERE which gives the shortest transit time under the force of gravity. Assume the SPHERE to be nonrotating, of RADIUS a, and with uniform density ρ. Then the standard form EULER-LAGRANGE DIFFERENTIAL EQUATION in polar coordinates is

$$r_{\phi\phi}(r^3 - ra^2) + {r_\phi}^2(2a^2 - r^2) + a^2r^2 = 0, \tag{1}$$

along with the boundary conditions $r(\phi = 0) = r_0$, $r_\phi(\phi = 0) = 0$, $r(\phi = \phi_A) = a$, and $r(\phi = \phi_B) = a$. Integrating once gives

$${r_\phi}^2 = \frac{a^2r^2}{{r_0}^2}\frac{r^2 - {r_0}^2}{a^2 - r^2}. \tag{2}$$

But this is the equation of a HYPOCYCLOID generated by a CIRCLE of RADIUS $\frac{1}{2}(a - r_0)$ rolling inside the CIRCLE of RADIUS a, so the tunnel is shaped like an arc of a HYPOCYCLOID. The transit time from point A to point B is

$$T = \pi\sqrt{\frac{a^2 - {r_0}^2}{ag}}, \tag{3}$$

where

$$g = \frac{GM}{a^2} = \tfrac{4}{3}\pi\rho G a \tag{4}$$

is the surface gravity with G the universal gravitational constant.

Spherical Bessel Differential Equation

Take the HELMHOLTZ DIFFERENTIAL EQUATION

$$\nabla^2 F + k^2 F = 0 \tag{1}$$

in SPHERICAL COORDINATES. This is just LAPLACE'S EQUATION in SPHERICAL COORDINATES with an additional term,

$$\begin{aligned} &\frac{d^2R}{dr^2}\Phi\Theta + \frac{2}{r}\frac{dR}{dr} + \frac{1}{r^2\sin^2\phi}\frac{d^2\Theta}{d\theta^2}\Phi R \\ &\quad + \frac{\cos\phi}{r^2\sin\phi}\frac{d\Phi}{d\phi}\Theta R + \frac{1}{r^2}\frac{d^2\Phi}{d\phi^2} + k^2 R\Phi\Theta = 0. \end{aligned} \tag{2}$$

Multiply through by $r^2/R\Phi\Theta$,

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} + k^2r^2 + \frac{1}{\Theta\sin^2\phi}\frac{d^2\Theta}{d\theta^2} + \frac{\cos\phi}{\Phi\sin\phi}\frac{d\Phi}{d\phi} + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0. \quad (3)$$

This equation is separable in R. Call the separation constant $n(n+1)$,

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} + k^2r^2 = n(n+1). \quad (4)$$

Now multiply through by R,

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + [k^2r^2 - n(n+1)]R = 0. \quad (5)$$

This is the SPHERICAL BESSEL DIFFERENTIAL EQUATION. It can be transformed by letting $x \equiv kr$, then

$$r\frac{dR(r)}{dr} = kr\frac{dR(r)}{k\,dr} = kr\frac{dR(r)}{d(kr)} = x\frac{dR(r)}{dx}. \quad (6)$$

Similarly,

$$r^2\frac{d^2R(r)}{dr^2} = x^2\frac{d^2R(r)}{dx^2}, \quad (7)$$

so the equation becomes

$$x^2\frac{d^2R}{dx^2} + 2x\frac{dR}{dx} + [x^2 - n(n+1)]R = 0. \quad (8)$$

Now look for a solution of the form $R(r) = Z(x)x^{-1/2}$, denoting a derivative with respect to x by a prime,

$$R' = Z'x^{-1/2} - \tfrac{1}{2}Zx^{-3/2} \quad (9)$$

$$\begin{aligned} R'' &= Z''x^{-1/2} - \tfrac{1}{2}Z'x^{-3/2} - \tfrac{1}{2}Z'x^{-3/2} \\ &\quad - \tfrac{1}{2}(-\tfrac{3}{2})Zx^{-5/2} \\ &= Z''x^{-1/2} - Z'x^{-3/2} + \tfrac{3}{4}Zx^{-5/2}, \end{aligned} \quad (10)$$

so

$$x^2(Z''x^{-1/2} - Z'x^{-3/2} + \tfrac{3}{4}Zx^{-5/2}) + 2x(Z'x^{-1/2} - \tfrac{1}{2}Zx^{-3/2}) + [x^2 - n(n+1)]Zx^{-1/2} = 0 \quad (11)$$

$$x^2(Z'' - Z'x^{-1} + \tfrac{3}{4}Zx^{-2}) + 2x(Z' - \tfrac{1}{2}Zx^{-1}) + [x^2 - n(n+1)]Z = 0 \quad (12)$$

$$x^2Z'' + (-x+2x)Z' + [\tfrac{3}{4} - 1 + x^2 - n(n+1)]Z = 0 \quad (13)$$

$$\begin{aligned} x^2Z'' + xZ' + [x^2 - (n^2 + n + \tfrac{1}{4})]Z &= 0 \\ x^2Z'' + xZ' + [x^2 - (n + \tfrac{1}{2})^2]Z &= 0. \end{aligned} \quad (14)$$

But the solutions to this equation are BESSEL FUNCTIONS of half integral order, so the normalized solutions to the original equation are

$$R(r) \equiv A\frac{J_{n+1/2}(kr)}{\sqrt{kr}} + B\frac{Y_{n+1/2}(kr)}{\sqrt{kr}} \quad (15)$$

which are known as SPHERICAL BESSEL FUNCTIONS. The two types of solutions are denoted $j_n(x)$ (SPHERICAL BESSEL FUNCTION OF THE FIRST KIND) or $n_n(x)$ (SPHERICAL BESSEL FUNCTION OF THE SECOND KIND), and the general solution is written

$$R(r) = A'j_n(kr) + B'n_n(kr), \quad (16)$$

where

$$j_n(z) \equiv \sqrt{\frac{\pi}{2}}\frac{J_{n+1/2}(z)}{\sqrt{z}} \quad (17)$$

$$n_n(z) \equiv \sqrt{\frac{\pi}{2}}\frac{Y_{n+1/2}(z)}{\sqrt{z}}. \quad (18)$$

see also SPHERICAL BESSEL FUNCTION, SPHERICAL BESSEL FUNCTION OF THE FIRST KIND, SPHERICAL BESSEL FUNCTION OF THE SECOND KIND

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Spherical Bessel Function

A solution to the SPHERICAL BESSEL DIFFERENTIAL EQUATION. The two types of solutions are denoted $j_n(x)$ (SPHERICAL BESSEL FUNCTION OF THE FIRST KIND) or $n_n(x)$ (SPHERICAL BESSEL FUNCTION OF THE SECOND KIND).

see also SPHERICAL BESSEL FUNCTION OF THE FIRST KIND, SPHERICAL BESSEL FUNCTION OF THE SECOND KIND

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Spherical Bessel Functions." §10.1 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 437–442, 1972.

Arfken, G. "Spherical Bessel Functions." §11.7 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 622–636, 1985.

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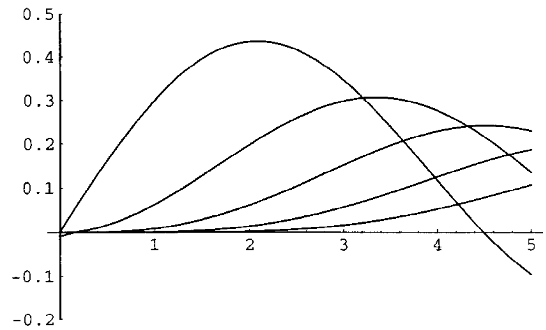
Spherical Bessel Function of the First Kind



$$
\begin{aligned}
j_n(x) &\equiv \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x) \\
&= 2^n x^n \sum_{s=0}^{\infty} \frac{(-1)^s (s-n)!}{s!(2s+2n+1)!} x^{2s} \\
&= \frac{x^n}{(2n+1)!!} \left[1 - \frac{\frac{1}{2}x^2}{1!(2n+3)} \right. \\
&\quad \left. + \frac{\left(\frac{1}{2}x^2\right)^2}{2!(2n+3)(2n+5)} + \ldots \right] \\
&= (-1)^n x^n \left(\frac{d}{x\,dx} \right)^n \frac{\sin x}{x}.
\end{aligned}
$$

The first few functions are

$$
\begin{aligned}
j_0(x) &= \frac{\sin x}{x} \\
j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x} \\
j_2(x) &= \left(\frac{3}{x^3} - \frac{1}{x} \right) \sin x - \frac{3}{x^2} \cos x.
\end{aligned}
$$

see also POISSON INTEGRAL REPRESENTATION, RAYLEIGH'S FORMULAS

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Spherical Bessel Functions." §10.1 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 437–442, 1972.

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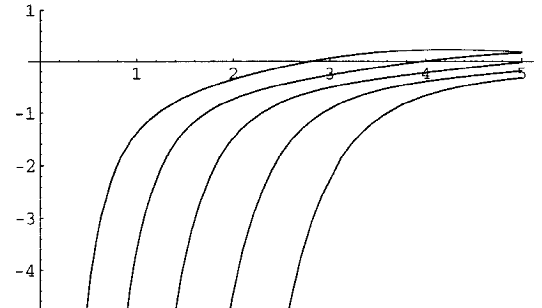
Spherical Bessel Function of the Second Kind



$$
\begin{aligned}
n_n(x) &\equiv \sqrt{\frac{\pi}{2x}} Y_{n+1/2}(x) \\
&= \frac{(-1)^{n+1}}{2^n x^{n+1}} \sum_{n=0}^{\infty} \frac{(-1)^s (s-n)!}{s!(2s-2n)!} x^{2s} \\
&= -\frac{(2n-1)!!}{x^{n+1}} \left[1 - \frac{\frac{1}{2}x^2}{1!(1-2n)} \right. \\
&\quad \left. + \frac{\left(\frac{1}{2}x^2\right)^2}{2!(1-2n)(3-2n)} + \ldots \right] \\
&= (-1)^{n+1} \sqrt{\frac{\pi}{2x}} J_{-n-1/2}(x).
\end{aligned}
$$

The first few functions are

$$
\begin{aligned}
n_0(x) &= -\frac{\cos x}{x} \\
n_1(x) &= -\frac{\cos x}{x^2} - \frac{\sin x}{x} \\
n_2(x) &= -\left(\frac{3}{x^3} - \frac{1}{x} \right) \cos x - \frac{3}{x^2} \sin x.
\end{aligned}
$$

see also RAYLEIGH'S FORMULAS

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Spherical Bessel Functions." §10.1 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 437–442, 1972.

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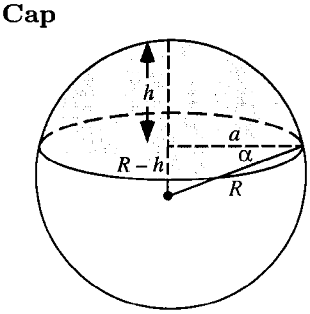
Spherical Bessel Function of the Third Kind

see SPHERICAL HANKEL FUNCTION OF THE FIRST KIND, SPHERICAL HANKEL FUNCTION OF THE SECOND KIND

Spherical Cap



A spherical cap is the region of a SPHERE which lies above (or below) a given PLANE. If the PLANE passes through the CENTER of the SPHERE, the cap is a HEMISPHERE. Let the SPHERE have RADIUS R, then the VOLUME of a spherical cap of height h and base RADIUS a is given by the equation of a SPHERICAL SEGMENT (which is a spherical cut by a second PLANE)

$$V_{\text{spherical segment}} = \tfrac{1}{6}\pi h(3a^2 + 3b^2 + h^2) \tag{1}$$

with $b = 0$, giving

$$V_{\text{cap}} = \tfrac{1}{6}\pi h(3a^2 + h^2). \tag{2}$$

Using the PYTHAGOREAN THEOREM gives

$$(R-h)^2 + a^2 = R^2, \tag{3}$$

which can be solved for a^2 as

$$a^2 = 2Rh - h^2, \tag{4}$$

and plugging this in gives the equivalent formula

$$V_{\text{cap}} = \tfrac{1}{3}\pi h^2(3R - h). \tag{5}$$

In terms of the so-called CONTACT ANGLE (the angle between the normal to the sphere at the bottom of the cap and the base plane)

$$R - h = R\sin\theta \tag{6}$$

$$\alpha \equiv \sin^{-1}\left(\frac{R-h}{R}\right), \tag{7}$$

so

$$V_{\text{cap}} = \tfrac{1}{3}\pi R^3(2 - 3\sin\alpha + \sin^3\alpha). \tag{8}$$

Consider a cylindrical box enclosing the cap so that the top of the box is tangent to the top of the SPHERE. Then the enclosing box has VOLUME

$$\begin{aligned} V_{\text{box}} &= \pi a^2 h = \pi(R\cos\alpha)[R(1-\sin\alpha)] \\ &= \pi R^3(1 - \sin\alpha - \sin^2\alpha + \sin^3\alpha), \end{aligned} \tag{9}$$

so the hollow volume between the cap and box is given by

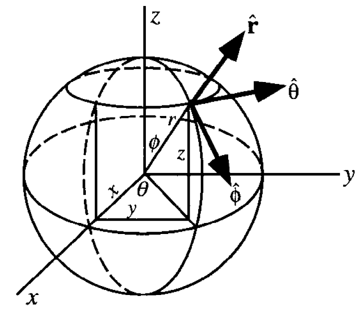
$$V_{\text{box}} - V_{\text{cap}} = \tfrac{1}{3}\pi R^3(1 - 3\sin^2\alpha + 2\sin^3\alpha). \tag{10}$$

If a second PLANE cuts the cap, the resulting SPHERICAL FRUSTUM is called a SPHERICAL SEGMENT. The SURFACE AREA of the spherical cap is given by the same equation as for a general ZONE:

$$S_{\text{cap}} = 2\pi Rh. \tag{11}$$

see also CONTACT ANGLE, DOME, FRUSTUM, HEMISPHERE, SOLID OF REVOLUTION, SPHERE, SPHERICAL SEGMENT, TORISPHERICAL DOME, ZONE

Spherical Coordinates



A system of CURVILINEAR COORDINATES which is natural for describing positions on a SPHERE or SPHEROID. Define θ to be the azimuthal ANGLE in the xy-PLANE from the x-AXIS with $0 \leq \theta < 2\pi$ (denoted λ when referred to as the LONGITUDE), ϕ to be the polar ANGLE from the z-AXIS with $0 \leq \phi \leq \pi$ (COLATITUDE, equal to $\phi = 90° - \delta$ where δ is the LATITUDE), and r to be distance (RADIUS) from a point to the ORIGIN.

Unfortunately, the convention in which the symbols θ and ϕ are reversed is frequently used, especially in physics, leading to unnecessary confusion. The symbol ρ is sometimes also used in place of r. Arfken (1985) uses (r, ϕ, θ), whereas Beyer (1987) uses (ρ, θ, ϕ). Be very careful when consulting the literature.

In this work, the symbols for the azimuthal, polar, and radial coordinates are taken as θ, ϕ, and r, respectively. Note that this definition provides a logical extension of the usual POLAR COORDINATES notation, with θ remaining the ANGLE in the xy-PLANE and ϕ becoming the ANGLE out of the PLANE.

$$r = \sqrt{x^2 + y^2 + z^2} \tag{1}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{2}$$

$$\phi = \sin^{-1}\left(\frac{\sqrt{x^2+y^2}}{r}\right) = \cos^{-1}\left(\frac{z}{r}\right), \tag{3}$$

where $r \in [0, \infty)$, $\theta \in [0, 2\pi)$, and $\phi \in [0, \pi]$. In terms of CARTESIAN COORDINATES,

$$x = r\cos\theta\sin\phi \tag{4}$$

$$y = r\sin\theta\sin\phi \tag{5}$$

$$z = r\cos\phi. \tag{6}$$

The SCALE FACTORS are

$$h_r = 1 \tag{7}$$

$$h_\theta = r\sin\phi \tag{8}$$

$$h_\phi = r, \tag{9}$$

so the METRIC COEFFICIENTS are

$$g_{rr} = 1 \quad (10)$$

$$g_{\theta\theta} = r^2 \sin^2\phi \quad (11)$$

$$g_{\phi\phi} = r^2. \quad (12)$$

The LINE ELEMENT is

$$d\mathbf{s} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}} + r\sin\phi\,d\theta\,\hat{\boldsymbol{\theta}}, \quad (13)$$

the AREA element

$$d\mathbf{a} = r^2 \sin\phi\,d\theta\,d\phi\,\hat{\mathbf{r}}, \quad (14)$$

and the VOLUME ELEMENT

$$dV = r^2 \sin\phi\,d\theta\,d\phi\,dr. \quad (15)$$

The JACOBIAN is

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right| = r^2 |\sin\phi|. \quad (16)$$

The POSITION VECTOR is

$$\mathbf{r} \equiv \begin{bmatrix} r\cos\theta\sin\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{bmatrix}, \quad (17)$$

so the UNIT VECTORS are

$$\hat{\mathbf{r}} \equiv \frac{\frac{d\mathbf{r}}{dr}}{\left|\frac{d\mathbf{r}}{dr}\right|} = \begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix} \quad (18)$$

$$\hat{\boldsymbol{\theta}} \equiv \frac{\frac{d\mathbf{r}}{d\theta}}{\left|\frac{d\mathbf{r}}{d\theta}\right|} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \quad (19)$$

$$\hat{\boldsymbol{\phi}} \equiv \frac{\frac{d\mathbf{r}}{d\phi}}{\left|\frac{d\mathbf{r}}{d\phi}\right|} = \begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ -\sin\phi \end{bmatrix}. \quad (20)$$

Derivatives of the UNIT VECTORS are

$$\frac{\partial\hat{\mathbf{r}}}{\partial r} = \mathbf{0} \quad (21)$$

$$\frac{\partial\hat{\boldsymbol{\theta}}}{\partial r} = \mathbf{0} \quad (22)$$

$$\frac{\partial\hat{\boldsymbol{\phi}}}{\partial r} = \mathbf{0} \quad (23)$$

$$\frac{\partial\hat{\mathbf{r}}}{\partial\theta} = \begin{bmatrix} -\sin\theta\sin\phi \\ \cos\theta\sin\phi \\ 0 \end{bmatrix} = \sin\phi\,\hat{\boldsymbol{\theta}} \quad (24)$$

$$\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} = \begin{bmatrix} -\cos\theta \\ -\sin\theta \\ 0 \end{bmatrix} = -\cos\phi\,\hat{\boldsymbol{\phi}} - \sin\phi\,\hat{\mathbf{r}} \quad (25)$$

$$\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\theta} = \begin{bmatrix} -\sin\theta\cos\phi \\ \cos\theta\cos\phi \\ 0 \end{bmatrix} = \cos\phi\,\hat{\boldsymbol{\theta}} \quad (26)$$

$$\frac{\partial\hat{\mathbf{r}}}{\partial\phi} = \begin{bmatrix} \cos\theta \\ \sin\theta\cos\phi \\ -\sin\phi \end{bmatrix} = \hat{\boldsymbol{\phi}} \quad (27)$$

$$\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\phi} = \mathbf{0} \quad (28)$$

$$\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi} = \begin{bmatrix} -\cos\theta\sin\phi \\ -\sin\theta\sin\phi \\ -\cos\phi \end{bmatrix} = -\hat{\mathbf{r}}. \quad (29)$$

The GRADIENT is

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\boldsymbol{\phi}}\frac{\partial}{\partial\phi} + \frac{1}{r\sin\phi}\hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta}, \quad (30)$$

so

$$\nabla_r\hat{\mathbf{r}} = \mathbf{0} \quad (31)$$

$$\nabla_r\hat{\boldsymbol{\theta}} = \mathbf{0} \quad (32)$$

$$\nabla_r\hat{\boldsymbol{\phi}} = \hat{\mathbf{0}} \quad (33)$$

$$\nabla_\theta\hat{\mathbf{r}} = \frac{\sin\phi\,\hat{\boldsymbol{\theta}}}{r\sin\phi} = \frac{1}{r}\hat{\boldsymbol{\theta}} \quad (34)$$

$$\nabla_\theta\hat{\boldsymbol{\theta}} = -\frac{\cos\phi\,\hat{\boldsymbol{\phi}} + \sin\phi\,\hat{\mathbf{r}}}{r\sin\phi} = -\frac{\cot\phi}{r}\hat{\boldsymbol{\phi}} - \frac{1}{r}\hat{\mathbf{r}} \quad (35)$$

$$\nabla_\theta\hat{\boldsymbol{\phi}} = \frac{\cos\phi\,\hat{\boldsymbol{\phi}}}{r\sin\phi} = \frac{1}{r}\cot\phi\,\hat{\boldsymbol{\theta}}. \quad (36)$$

Now, since the CONNECTION COEFFICIENTS are given by $\Gamma^i_{jk} = \hat{\mathbf{x}}_i \cdot (\nabla_k \hat{\mathbf{x}}_j)$,

$$\Gamma^\theta = \begin{bmatrix} 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{\cot\phi}{r} & 0 \end{bmatrix} \quad (37)$$

$$\Gamma^\phi = \begin{bmatrix} 0 & 0 & \frac{1}{r} \\ 0 & -\frac{\cot\phi}{r} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (38)$$

$$\Gamma^r = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{r} & 0 \\ 0 & 0 & -\frac{1}{r} \end{bmatrix}. \quad (39)$$

The DIVERGENCE is

$$\begin{aligned}\nabla\cdot\mathbf{F} &= A^k{}_{,k} + \Gamma^k_{jk}A^j \\ &= [A^r{}_{,r} + (\Gamma^r_{rr}A^r + \Gamma^r_{\theta r}A^\theta + \Gamma^r_{\phi r}A^\phi] \\ &\quad +[A^\theta{}_{,\theta} + (\Gamma^\theta_{r\theta}A^r + \Gamma^\theta_{\theta\theta}A^\theta + \Gamma^\theta_{\phi\theta}A^\phi)] \\ &\quad +[A^\phi{}_{,\phi} + (\Gamma^\phi_{r\phi}A^r + \Gamma^\phi_{\theta\phi}A^\theta + \Gamma^\phi_{\phi\phi}A^\phi)] \\ &= \frac{1}{g_r}\frac{\partial A^r}{\partial r} + \frac{1}{g_\theta}\frac{\partial A^\theta}{\partial\theta} + \frac{1}{g_\phi}\frac{\partial A^\phi}{\partial\phi} + (0+0+0) \\ &\quad + \left(\frac{1}{r}A^r + 0 + \frac{\cot\phi}{r}A^\phi\right) + \left(\frac{1}{r}A^r + 0 + 0\right) \\ &= \frac{\partial}{\partial r}A^r + \frac{2}{r}A^r + \frac{1}{r\sin\phi}\frac{\partial}{\partial\theta}A^\theta + \frac{1}{r}\frac{\partial}{\partial\phi}A^\phi + \frac{\cot\phi}{r}A^\phi,\end{aligned} \quad (40)$$

or, in VECTOR notation,

$$\begin{aligned}\nabla\cdot\mathbf{F} &= \left(\frac{2}{r} + \frac{\partial}{\partial r}\right)F_r + \left(\frac{1}{r}\frac{\partial}{\partial\phi} + \frac{\cot\phi}{r}\right)F_\phi + \frac{1}{\sin\phi}\frac{\partial F_\theta}{\partial\theta} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi F_\phi) + \frac{1}{r\sin\phi}\frac{\partial F_\theta}{\partial\theta}.\end{aligned} \quad (41)$$

The COVARIANT DERIVATIVES are given by

$$A_{j;k} = \frac{1}{g_{kk}}\frac{\partial A_j}{\partial x_k} - \Gamma^i_{jk}A_i, \quad (42)$$

so

$$A_{r;r} = \frac{\partial A_r}{\partial r} - \Gamma^i_{rr} A_i = \frac{\partial A_r}{\partial r} \tag{43}$$

$$A_{r;\theta} = \frac{1}{r \sin\phi} \frac{\partial A_r}{\partial \theta} - \Gamma^i_{r\theta} = \frac{1}{r\sin\phi}\frac{\partial A_r}{\partial\theta} - \Gamma_{r\theta} A_\theta$$
$$= \frac{1}{r\sin\phi}\frac{\partial A_r}{\partial\phi} - \frac{A_\theta}{r} \tag{44}$$

$$A_{r;\phi} = \frac{1}{r}\frac{\partial A_r}{\partial\phi} - \Gamma^i_{r\phi} A_i = \frac{1}{r}\frac{\partial A_r}{\partial\phi} - \Gamma^\phi_{r\phi} A_\phi$$
$$= \frac{1}{r}\left(\frac{\partial A_r}{\partial\phi} - A_\phi\right) \tag{45}$$

$$A_{\theta;r} = \frac{\partial A_\theta}{\partial r} - \Gamma^i_{\theta r} A_i = \frac{\partial A_\theta}{\partial r} \tag{46}$$

$$A_{\theta;\theta} = \frac{1}{r\sin\phi}\frac{\partial A_\phi}{\partial\theta} - \Gamma^i_{\theta\theta} A_i$$
$$= \frac{1}{r\sin\phi} \partial A_\theta \partial\theta - \Gamma^\phi_{\theta\theta} A_\phi - \Gamma^r_{\theta\theta} A_r$$
$$= \frac{1}{r\sin\phi}\frac{\partial A_\theta}{\partial\theta} + \frac{\cot\phi}{r} A_\phi + \frac{A_r}{r} \tag{47}$$

$$A_{\theta;\phi} = \frac{1}{r}\frac{\partial A_\theta}{\partial r} - \Gamma^i_{\phi r} A_i \frac{\partial A_\theta}{\partial\phi} \tag{48}$$

$$A_{\phi;r} = \frac{\partial A_\phi}{\partial r} - \Gamma^i_{\phi r} A_i = \frac{\partial A_\phi}{r} \tag{49}$$

$$A_{\phi;\theta} = \frac{1}{r\sin\phi}\frac{\partial A_\phi}{\partial\theta} - \Gamma^i_{\phi\theta} A_i = \frac{1}{r\sin\phi}\frac{\partial A_\phi}{\partial\theta} - \Gamma^\theta_{\phi\theta}$$
$$= \frac{1}{r\sin\phi}\frac{\partial A_\phi}{\partial\theta} - \frac{\cot\phi}{r} A_\theta \tag{50}$$

$$A_{\phi;\phi} = \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} - \Gamma^i_{\phi\phi} A_i = \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} - \Gamma^r_{\phi\phi} A_r$$
$$= \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{A_r}{r}. \tag{51}$$

The COMMUTATION COEFFICIENTS are given by

$$c^\mu_{\alpha\beta}\vec{e}_\mu = [\vec{e}_\alpha, \vec{e}_\beta] = \nabla_\alpha \vec{e}_\beta - \nabla_\beta \vec{e}_\alpha \tag{52}$$

$$[\hat{\mathbf{r}}, \hat{\mathbf{r}}] = [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}}] = [\hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\phi}}] = \mathbf{0}, \tag{53}$$

so $c^\alpha_{rr} = c^\alpha_{\theta\theta} = c^\alpha_{\phi\phi} = 0$, where $\alpha = r, \theta, \phi$.

$$[\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}] = -[\hat{\boldsymbol{\theta}}, \hat{\mathbf{r}}] = \nabla_r \hat{\boldsymbol{\theta}} - \nabla_\theta \hat{\mathbf{r}} = \mathbf{0} - \frac{1}{r}\hat{\boldsymbol{\theta}} = -\frac{1}{r}\hat{\boldsymbol{\theta}}, \tag{54}$$

so $c^\theta_{r\theta} = -c^\theta_{\theta r} = -\frac{1}{r}$, $c^r_{r\theta} = c^\phi_{r\theta} = 0$.

$$[\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}] = -[\hat{\boldsymbol{\phi}}, \hat{\mathbf{r}}] = \mathbf{0} - \frac{1}{r}\hat{\boldsymbol{\phi}} = -\frac{1}{r}\hat{\boldsymbol{\phi}}, \tag{55}$$

so $c^\phi_{r\phi} = -c^\phi_{\phi r} = \frac{1}{r}$.

$$[\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}] = -[\hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\theta}}] = \frac{1}{r}\cot\phi\hat{\boldsymbol{\theta}} - \mathbf{0} = \frac{1}{r}\cot\phi\hat{\boldsymbol{\theta}}, \tag{56}$$

so

$$c^\theta_{\theta\phi} = -c^\theta_{\phi\theta} = \frac{1}{r}\cot\phi. \tag{57}$$

Summarizing,

$$c^r = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{58}$$

$$c^\theta = \begin{bmatrix} 0 & -\frac{1}{r} & 0 \\ \frac{1}{r} & 0 & \frac{1}{r}\cot\phi \\ 0 & -\frac{1}{r}\cot\phi & 0 \end{bmatrix} \tag{59}$$

$$c^\phi = \begin{bmatrix} 0 & 0 & -\frac{1}{r} \\ 0 & 0 & 0 \\ \frac{1}{r} & 0 & 0 \end{bmatrix}. \tag{60}$$

Time derivatives of the POSITION VECTOR are

$$\dot{\mathbf{r}} = \begin{bmatrix} \cos\theta\sin\phi\,\dot{r} - r\sin\theta\sin\phi\,\dot{\theta} + r\cos\theta\cos\phi\,\dot{\phi} \\ \sin\theta\sin\phi\,\dot{r} + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi} \\ \cos\phi\,\dot{r} - r\sin\phi\,\dot{\phi} \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix} \dot{r} + r\sin\phi \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \dot{\theta}$$
$$+ r \begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ -\sin\phi \end{bmatrix} \dot{\phi}$$
$$= \dot{r}\,\hat{\mathbf{r}} + r\sin\phi\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\,\dot{\phi}\,\hat{\boldsymbol{\phi}}. \tag{61}$$

The SPEED is therefore given by

$$v \equiv |\dot{\mathbf{r}}| = \sqrt{\dot{r}^2 + r^2\sin^2\phi\dot{\theta}^2 + r^2\dot{\phi}^2}. \tag{62}$$

The ACCELERATION is

$$\ddot{x} = (-\sin\theta\sin\phi\dot{\theta}\dot{r} + \cos\theta\cos\phi\dot{r}\dot{\phi} + \cos\theta\sin\phi\ddot{r})$$
$$- (\sin\theta\sin\phi\dot{r}\dot{\theta} + r\cos\theta\sin\phi\dot{\theta}^2 + r\sin\theta\cos\phi\dot{\theta}\dot{\phi}$$
$$+ r\sin\theta\sin\phi\ddot{\theta}) + (\cos\theta\cos\phi\dot{r}\dot{\phi} - r\sin\theta\cos\dot{\theta}\dot{\phi}$$
$$- r\cos\theta\sin\phi\dot{\phi}^2 + r\cos\theta\cos\phi\ddot{\phi})$$
$$= -2\sin\theta\sin\phi\dot{\theta}\dot{r} + 2\cos\theta\cos\phi\dot{r}\dot{\phi} - 2r\sin\theta\cos\phi\dot{\theta}\dot{\phi}$$
$$+ \cos\theta\sin\phi\ddot{r} - r\sin\theta\sin\phi\ddot{\theta} + r\cos\theta\cos\phi\ddot{\phi}$$
$$- r\cos\theta\sin\phi(\dot{\theta}^2 + \dot{\phi}^2) \tag{63}$$

$$\ddot{y} = (\sin\theta\sin\phi\ddot{r} + r\cos\theta\sin\phi\dot{\theta} + r\cos\phi\sin\theta\dot{\phi})$$
$$+ (\cos\theta\sin\phi\dot{r}\dot{\theta} - r\sin\theta\sin\phi\dot{\theta}^2 + r\cos\theta\cos\phi\dot{\theta}\dot{\phi}$$
$$+ r\cos\theta\sin\phi\ddot{\theta}) + (\sin\theta\cos\phi\dot{r}\dot{\phi} + r\cos\theta\cos\phi\dot{\theta}\dot{\phi}$$
$$- r\sin\theta\sin\phi\dot{\phi}^2 + r\sin\theta\cos\phi\ddot{\phi})$$
$$= 2\cos\theta\sin\phi\dot{\theta}\dot{r} + 2\sin\theta\cos\phi\dot{r}\dot{\phi} + 2r\cos\theta\cos\phi\dot{\theta}\dot{\phi}$$
$$+ \sin\theta\sin\phi\ddot{r} + r\cos\theta\sin\phi\ddot{\theta} + r\sin\theta\cos\phi\ddot{\phi}$$
$$- r\sin\theta\sin\phi(\dot{\theta}^2 + \dot{\phi}^2) \tag{64}$$

$$\ddot{z} = (\cos\phi\ddot{r} - \sin\phi\dot{r}\dot{\phi}) - (\dot{r}\sin\phi\dot{\phi} + r\cos\phi\dot{\phi}^2 + r\sin\phi\ddot{\phi})$$
$$= -r\cos\phi\dot{\phi}^2 + \cos\phi\ddot{r} - 2\sin\phi\dot{\phi}\dot{r} - r\sin\phi\ddot{\phi}. \tag{65}$$

Plugging these in gives

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\phi}^2)\begin{bmatrix}\cos\theta\sin\phi\\ \sin\theta\sin\phi\\ \cos\phi\end{bmatrix} + (2r\cos\phi\dot{\theta}\dot{\phi} + r\sin\phi\ddot{\theta})\begin{bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{bmatrix} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\begin{bmatrix}\cos\theta\cos\phi\\ \sin\theta\cos\phi\\ -\sin\phi\end{bmatrix} - r\sin\phi\dot{\theta}^2\begin{bmatrix}\cos\theta\\ \sin\theta\\ 0\end{bmatrix}, \tag{66}$$

but

$$\sin\phi\hat{\mathbf{r}} + \cos\phi\hat{\boldsymbol{\phi}} = \begin{bmatrix}\cos\theta\sin^2\phi + \cos\theta\cos^2\phi\\ \sin\theta\sin^2\phi + \sin\theta\cos^2\phi\\ 0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\\ 0\end{bmatrix}, \tag{67}$$

so

$$\begin{aligned}\ddot{\mathbf{r}} &= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (2r\cos\phi\dot{\theta}\dot{\phi} + 2\sin\phi\dot{\theta}\dot{r} + r\sin\phi\ddot{\theta})\hat{\boldsymbol{\theta}}\\ &\quad + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\boldsymbol{\phi}} - r\sin\phi\dot{\theta}^2(\sin\phi\hat{\mathbf{r}} + \cos\phi\hat{\boldsymbol{\phi}})\\ &= (\ddot{r} - r\dot{\phi}^2 - r\sin^2\phi\dot{\theta}^2)\hat{\mathbf{r}}\\ &\quad + (2\sin\phi\dot{\theta}\dot{r} + 2r\cos\phi\dot{\theta}\dot{\phi} + r\sin\phi\ddot{\theta})\hat{\boldsymbol{\theta}}\\ &\quad + (2\dot{r}\dot{\phi} + r\ddot{\phi} - r\sin\phi\cos\phi\dot{\theta}^2)\hat{\boldsymbol{\phi}}.\end{aligned} \tag{68}$$

Time DERIVATIVES of the UNIT VECTORS are

$$\dot{\hat{\mathbf{r}}} = \begin{bmatrix}-\sin\theta\sin\phi\,\dot{\theta} + \cos\theta\cos\phi\,\dot{\phi}\\ \cos\theta\sin\phi\,\dot{\theta} + \sin\theta\cos\phi\,\dot{\phi}\\ -\sin\phi\,\dot{\phi}\end{bmatrix} = \sin\phi\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{\phi}\hat{\boldsymbol{\phi}} \tag{69}$$

$$\dot{\hat{\boldsymbol{\theta}}} = \begin{bmatrix}-\cos\theta\dot{\theta}\\ -\sin\theta\dot{\theta}\\ 0\end{bmatrix} = -\dot{\theta}\begin{bmatrix}\cos\theta\\ \sin\theta\\ 0\end{bmatrix} = -\dot{\theta}(\sin\phi\hat{\mathbf{r}} + \cos\phi\hat{\boldsymbol{\phi}}) \tag{70}$$

$$\dot{\hat{\boldsymbol{\phi}}} = \begin{bmatrix}-\sin\theta\cos\phi\,\dot{\theta} - \cos\theta\sin\phi\,\dot{\phi}\\ \cos\theta\cos\phi\,\dot{\theta} - \sin\theta\sin\phi\,\dot{\phi}\\ -\cos\phi\,\dot{\phi}\end{bmatrix} = -\dot{\phi}\hat{\mathbf{r}} + \cos\phi\dot{\theta}\hat{\boldsymbol{\theta}}. \tag{71}$$

The CURL is

$$\nabla\times\mathbf{F} = \frac{1}{r\sin\phi}\left[\frac{\partial}{\partial\phi}(\sin\phi F_\theta) - \frac{\partial F_\phi}{\partial\theta}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\phi}\frac{\partial F_r}{\partial\theta} - \frac{\partial}{\partial r}(rF_\theta)\right]\hat{\boldsymbol{\phi}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\phi) - \frac{\partial F_r}{\partial\phi}\right]\hat{\boldsymbol{\theta}}. \tag{72}$$

The LAPLACIAN is

$$\begin{aligned}\nabla^2 &\equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right)\\ &= \frac{1}{r^2}\left(r^2\frac{\partial^2}{\partial r^2} + 2r\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin\phi}\left(\cos\phi\frac{\partial}{\partial\phi} + \sin\phi\frac{\partial^2}{\partial\phi^2}\right)\\ &= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{\cos\phi}{r^2\sin\phi}\frac{\partial}{\partial\phi} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}.\end{aligned} \tag{73}$$

The vector LAPLACIAN is

$$\nabla^2\mathbf{v} = \begin{bmatrix}\frac{1}{r}\frac{\partial^2(rv_r)}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2} + \frac{\cot\theta}{r^2}\frac{\partial v_r}{\theta} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi} - \frac{2v_r}{r^2} - \frac{2\cot\theta}{r^2}v_\theta\\ \frac{1}{r}\frac{\partial^2(rv_\theta)}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2} + \frac{\cot\theta}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{2}{r^2}\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2\sin^2\theta}\\ \frac{1}{r}\frac{\partial^2(rv_\phi)}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\phi a}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2} + \frac{\cot\theta}{r^2}\frac{\partial v_\phi}{\partial\theta} + \frac{2}{r^2}\frac{\partial v_r}{\partial\phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial\phi} - \frac{v_\phi}{r^2\sin^2\theta}\end{bmatrix}. \tag{74}$$

To express PARTIAL DERIVATIVES with respect to Cartesian axes in terms of PARTIAL DERIVATIVES of the spherical coordinates,

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}r\cos\theta\sin\phi\\ r\sin\theta\sin\phi\\ r\cos\phi\end{bmatrix} \tag{75}$$

$$\begin{aligned}\begin{bmatrix}dx\\ dy\\ dz\end{bmatrix} &= \begin{bmatrix}\cos\theta\sin\phi\,dr - r\sin\theta\sin\phi\,d\theta + r\cos\theta\cos\phi\,d\phi\\ \sin\theta\sin\phi\,dr + r\sin\phi\cos\theta\,d\theta + r\sin\theta\cos\phi\,d\phi\\ \cos\phi\,dr - r\sin\phi\,d\phi\end{bmatrix}\\ &= \begin{bmatrix}\cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi\\ \sin\theta\sin\phi & r\sin\phi\cos\theta & r\sin\theta\cos\phi\\ \cos\phi & 0 & -r\sin\phi\end{bmatrix}\begin{bmatrix}dr\\ d\theta\\ d\phi\end{bmatrix}.\end{aligned} \tag{76}$$

Upon inversion, the result is

$$\begin{bmatrix}dr\\ d\theta\\ d\phi\end{bmatrix} = \begin{bmatrix}\cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi\\ -\frac{\sin\theta}{r\sin\phi} & \frac{\cos\theta}{r\sin\phi} & 0\\ \frac{\cos\theta\cos\phi}{r} & \frac{\sin\theta\cos\phi}{r} & -\frac{\sin\phi}{r}\end{bmatrix}\begin{bmatrix}dx\\ dy\\ dz\end{bmatrix}. \tag{77}$$

The Cartesian PARTIAL DERIVATIVES in spherical coordinates are therefore

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}\\ &= \cos\theta\sin\phi\frac{\partial}{\partial r} - \frac{\sin\theta}{r\sin\phi}\frac{\partial}{\partial\theta} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\phi}\end{aligned} \tag{78}$$

$$\begin{aligned}\frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}\\ &= \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta}{r\sin\phi}\frac{\partial}{\partial\theta} + \frac{\sin\theta\cos\phi}{r}\frac{\partial}{\partial\phi}\end{aligned} \tag{79}$$

$$\begin{aligned}\frac{\partial}{\partial z} &= \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi}\\ &= \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\end{aligned} \tag{80}$$

(Gasiorowicz 1974, pp. 167–168).

The HELMHOLTZ DIFFERENTIAL EQUATION is separable in spherical coordinates.

see also COLATITUDE, GREAT CIRCLE, HELMHOLTZ DIFFERENTIAL EQUATION—SPHERICAL COORDINATES, LATITUDE, LONGITUDE, OBLATE SPHEROIDAL COORDINATES, PROLATE SPHEROIDAL COORDINATES

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Spherical Design

X is a spherical t-design in E IFF it is possible to exactly determine the average value on E of any POLYNOMIAL f of degree at most t by sampling f at the points of X. In other words,

$$\frac{1}{\text{volume } E}\int_E f(\xi)\,d\xi = \frac{1}{|X|}\sum_{x\in X} f(x).$$

References

Colbourn, C. J. and Dinitz, J. H. (Eds.) "Spherical t-Designs." Ch. 44 in *CRC Handbook of Combinatorial Designs.* Boca Raton, FL: CRC Press, pp. 462–466, 1996.

Spherical Excess

The difference between the sum of the angles of a SPHERICAL TRIANGLE and 180°.

see also ANGULAR DEFECT, DESCARTES TOTAL ANGULAR DEFECT, GIRARD'S SPHERICAL EXCESS FORMULA, L'HUILIER'S THEOREM, SPHERICAL TRIANGLE

Spherical Frustum

see SPHERICAL SEGMENT

Spherical Geometry

The study of figures on the surface of a SPHERE (such as the SPHERICAL TRIANGLE and SPHERICAL POLYGON), as opposed to the type of geometry studied in PLANE GEOMETRY or SOLID GEOMETRY.

see also PLANE GEOMETRY, SOLID GEOMETRY, SPHERICAL TRIGONOMETRY, THURSTON'S GEOMETRIZATION CONJECTURE

Spherical Hankel Function of the First Kind

$$h_n^{(1)}(x) \equiv \sqrt{\frac{\pi}{2x}}H_{n+1/2}^{(1)}(x) = j_n(x) + in_n(x),$$

where $H^{(1)}(x)$ is the HANKEL FUNCTION OF THE FIRST KIND and $j_n(x)$ and $n_n(x)$ are the SPHERICAL BESSEL FUNCTIONS OF THE FIRST and SECOND KINDS. Explicitly, the first few are

$$\begin{aligned}h_0^{(1)}(x) &= \frac{1}{x}(\sin x - i\cos x) = -\frac{i}{x}e^{ix}\\ h_1^{(1)}(x) &= e^{ix}\left(-\frac{1}{x} - \frac{i}{x^2}\right)\\ h_2^{(1)}(x) &= e^{ix}\left(\frac{i}{x} - \frac{3}{x^2} - \frac{3i}{x^3}\right).\end{aligned}$$

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Spherical Bessel Functions." §10.1 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 437–442, 1972.

Spherical Hankel Function of the Second Kind

$$h_n^{(2)}(x) \equiv \sqrt{\frac{\pi}{2x}}H_{n+1/2}^{(2)}(x) = j_n(x) - in_n(x),$$

where $H^{(2)}(x)$ is the HANKEL FUNCTION OF THE SECOND KIND and $j_n(x)$ and $n_n(x)$ are the SPHERICAL BESSEL FUNCTIONS OF THE FIRST and SECOND KINDS. Explicitly, the first is

$$h_0^{(2)}(x) = \frac{1}{x}(\sin x + i\cos x) = \frac{i}{x}e^{-ix}.$$

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Spherical Bessel Functions." §10.1 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 437–442, 1972.

Spherical Harmonic

The spherical harmonics $Y_l^m(\theta,\phi)$ are the angular portion of the solution to LAPLACE'S EQUATION in SPHERICAL COORDINATES where azimuthal symmetry is not present. Some care must be taken in identifying the notational convention being used. In the below equations, θ is taken as the azimuthal (longitudinal) coordinate, and ϕ as the polar (latitudinal) coordinate (opposite the notation of Arfken 1985).

$$Y_l^m(\theta,\phi) \equiv \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\phi)e^{im\theta}, \qquad (1)$$

where $m = -l, -1+1, \ldots, 0, \ldots, l$ and the normalization is chosen such that

$$\int_0^{2\pi}\int_0^{\pi} Y_l^m Y_{l'}^{m'\,*}\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\int_{-1}^{1} Y_l^m Y_{l'}^{m'\,*}\,d(\cos\phi)\,d\theta = \delta_{mm'}\delta_{ll'}, \qquad (2)$$

where δ_{mn} is the KRONECKER DELTA. Sometimes, the CONDON-SHORTLEY PHASE $(-1)^m$ is prepended to the definition of the spherical harmonics.

Integrals of the spherical harmonics are given by

$$\int Y_{l_1}^{m_1}Y_{l_2}^{m_2}Y_{l_3}^{m_3}\,d\Omega = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \times \begin{pmatrix} l_1 & l_2 & l_3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\end{pmatrix}, \qquad (3)$$

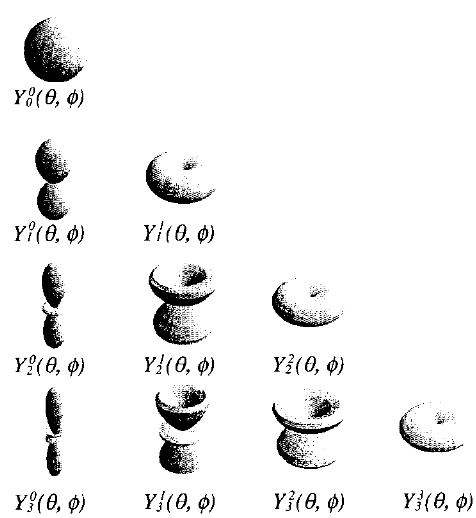
where $\begin{pmatrix} l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\end{pmatrix}$ is a WIGNER $3j$-SYMBOL (which is related to the CLEBSCH-GORDON COEFFICIENTS). The spherical harmonics obey

$$Y_l^{-l} = \frac{1}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}\sin^l\phi e^{-il\theta} \qquad (4)$$

$$Y_l^0 = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\phi) \qquad (5)$$

$$Y_l^{-m} = (-1)^m Y_l^{m*}, \qquad (6)$$

where $P_l(x)$ is a LEGENDRE POLYNOMIAL.



The above illustrations show $|Y_l^m(\theta,\phi)|$ (top) and $\Re[Y_l^m(\theta,\phi)]$ and $\Im[Y_l^m(\theta,\phi)]$ (bottom). The first few spherical harmonics are

$$Y_0^0 = \frac{1}{2}\frac{1}{\sqrt{\pi}}$$

$$Y_1^{-1} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\phi\,e^{-i\theta}$$

$$Y_1^0 = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\phi$$

$$Y_1^1 = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\phi\,e^{i\theta}$$

$$Y_2^{-2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\phi\,e^{-2i\theta}$$

$$Y_2^{-1} = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\phi\cos\phi\,e^{-i\theta}$$

$$Y_2^0 = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\phi - 1)$$

$$Y_2^1 = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\phi\cos\phi\,e^{i\theta}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\phi\,e^{2i\theta}$$

$$Y_3^{-3} = \frac{1}{8}\sqrt{\frac{35}{\pi}}\sin^3\phi\,e^{-3i\theta}$$

$$Y_3^{-2} = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\phi\cos\phi\,e^{-2i\theta}$$

$$Y_3^{-1} = \frac{1}{8}\sqrt{\frac{21}{\pi}}\sin\phi(5\cos^2\phi - 1)e^{-i\theta}$$

$$Y_3^0 = \frac{1}{4}\sqrt{\frac{7}{\pi}}(5\cos^3\phi - 3\cos\phi)$$

$$Y_3^1 = -\frac{1}{8}\sqrt{\frac{21}{\pi}}\sin\phi(5\cos^2\phi - 1)e^{i\theta}$$

$$Y_3^2 = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\phi\cos\phi\,e^{2i\theta}$$

$$Y_3^3 = -\frac{1}{8}\sqrt{\frac{35}{\pi}}\sin^3\phi\,e^{3i\theta}.$$

Written in terms of CARTESIAN COORDINATES,

$$e^{i\theta} = \frac{x+iy}{\sqrt{x^2+y^2}} \quad (7)$$

$$\phi = \sin^{-1}\left(\sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}}\right) \quad (8)$$

$$= \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right), \quad (9)$$

so

$$Y_0^0 = \frac{1}{2}\frac{1}{\sqrt{\pi}} \quad (10)$$

$$Y_1^0 = \frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{z}{\sqrt{x^2+y^2+z^2}} \quad (11)$$

$$Y_1^1 = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{x+iy}{\sqrt{x^2+y^2+z^2}} \quad (12)$$

$$Y_2^0 = \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(\frac{3z^2}{x^2+y^2+z^2}-1\right) \quad (13)$$

$$Y_2^1 = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\frac{z(x+iy)}{x^2+y^2+z^2} \quad (14)$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\frac{(x+iy)^2}{x^2+y^2+z^2}. \quad (15)$$

These can be separated into their REAL and IMAGINARY PARTS

$$Y_l^{m\,s}(\theta,\phi) \equiv P_l^m(\cos\phi)\sin(m\theta) \quad (16)$$

$$Y_l^{m\,c}(\theta,\phi) \equiv P_l^m(\cos\phi)\cos(m\theta). \quad (17)$$

The ZONAL HARMONICS are defined to be those of the form

$$P_n^m(\cos\theta). \quad (18)$$

The TESSERAL HARMONICS are those of the form

$$\sin(m\phi)P_n^m(\cos\theta) \quad (19)$$

$$\cos(m\phi)P_n^m(\cos\theta) \quad (20)$$

for $n \neq m$. The SECTORIAL HARMONICS are of the form

$$\sin(m\phi)P_m^m(\cos\theta) \quad (21)$$

$$\cos(m\phi)P_m^m(\cos\theta). \quad (22)$$

The spherical harmonics form a COMPLETE ORTHONORMAL BASIS, so an arbitrary REAL function $f(\theta,\phi)$ can be expanded in terms of COMPLEX spherical harmonics

$$f(\theta,\phi) \equiv \sum_{l=0}^{\infty}\sum_{m=-1}^{l} A_l^m Y_l^m(\theta,\phi), \quad (23)$$

or REAL spherical harmonics

$$f(\theta,\phi) \equiv \sum_{l=0}^{\infty}\sum_{m=0}^{l}[C_l^m Y_l^{m\,c}(\theta,\phi)\sin(m\theta) + S_l^m Y_l^{m\,s}(\theta,\phi)]. \quad (24)$$

see also CORRELATION COEFFICIENT, SPHERICAL HARMONIC ADDITION THEOREM, SPHERICAL HARMONIC CLOSURE RELATIONS, SPHERICAL VECTOR HARMONIC

References

Arfken, G. "Spherical Harmonics." §12.6 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 680–685, 1985.

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Sansone, G. "Harmonic Polynomials and Spherical Harmonics," "Integral Properties of Spherical Harmonics and the Addition Theorem for Legendre Polynomials," and "Completeness of Spherical Harmonics with Respect to Square Integrable Functions." §3.18–3.20 in *Orthogonal Functions, rev. English ed.* New York: Dover, pp. 253–272, 1991.

Sternberg, W. and Smith, T. L. *The Theory of Potential and Spherical Harmonics, 2nd ed.* Toronto: University of Toronto Press, 1946.

Spherical Harmonic Addition Theorem

A FORMULA also known as the LEGENDRE ADDITION THEOREM which is derived by finding GREEN'S FUNCTIONS for the SPHERICAL HARMONIC expansion and equating them to the generating function for LEGENDRE POLYNOMIALS. When γ is defined by

$$\cos\gamma \equiv \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi_1 - \phi_2,$$

$$P_n(\cos\gamma) = \frac{4\pi}{2n+1}\sum_{m=-n}^{n}(-1)^m Y_m^n(\theta_1,\phi_1)Y_{-m}^n(\theta_2,\phi_2)$$

$$= \frac{4\pi}{2n+1}\sum_{m=-n}^{n} Y_m^n(\theta_1,\phi_1)Y_m^{n\,*}(\theta_2,\phi_2)$$

$$= P_n(\cos\theta_1)P_n(\cos\theta_2)$$

$$+2\sum_{m=-n}^{n}\frac{(n-m)!}{(n+m)!}P_m^n(\cos\theta_1)P_m^n(\cos\theta_2)\cos[m(\phi_1-\phi_2)].$$

References

Arfken, G. "The Addition Theorem for Spherical Harmonics." §12.8 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 693–695, 1985.

Spherical Harmonic Closure Relations

The sum of the absolute squares of the SPHERICAL HARMONICS $Y_l^m(\theta,\phi)$ over all values of m is

$$\sum_{m=-l}^{l} |Y_l^m(\theta,\phi)|^2 = \frac{2l+1}{4\pi}.$$

The double sum over m and l is given by

$$\begin{aligned}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} & Y_l^m(\theta_1,\phi_1)Y_l^{m*}(\theta_2,\phi_2)\\ &= \frac{1}{\sin\theta_1}\delta(\theta_1-\theta_2)\delta(\phi_1-\phi_2)\\ &= \delta(\cos\theta_1-\cos\theta_2)\delta(\phi_1-\phi_2),\end{aligned}$$

where $\delta(x)$ is the DELTA FUNCTION.

Spherical Harmonic Tensor

A tensor defined in terms of the TENSORS which satisfy the DOUBLE CONTRACTION RELATION.

see also DOUBLE CONTRACTION RELATION, SPHERICAL HARMONIC

Spherical Helix

The TANGENT INDICATRIX of a CURVE OF CONSTANT PRECESSION is a spherical helix. The equation of a spherical helix on a SPHERE with RADIUS r making an ANGLE θ with the z-axis is

$$x(\psi) = \tfrac{1}{2}r(1+\cos\theta)\cos\psi - \tfrac{1}{2}r(1-\cos\theta)\cos\left(\frac{1+\cos\theta}{1-\cos\theta}\psi\right) \quad (1)$$

$$y(\psi) = \tfrac{1}{2}r(1+\cos\theta)\sin\psi - \tfrac{1}{2}r(1-\cos\theta)\sin\left(\frac{1+\cos\theta}{1-\cos\theta}\psi\right) \quad (2)$$

$$z(\psi) = r\sin\theta\cos\left(\frac{\cos\theta}{1-\cos\theta}\psi\right). \quad (3)$$

The projection on the xy-plane is an EPICYCLOID with RADII

$$a = r\cos\theta \quad (4)$$

$$b = r\sin^2(\tfrac{1}{2}\theta). \quad (5)$$

see also HELIX, LOXODROME, SPHERICAL SPIRAL

References

Scofield, P. D. "Curves of Constant Precession." *Amer. Math. Monthly* **102**, 531–537, 1995.

Spherical Point System

How can n points be distributed on a SPHERE such that they maximize the minimum distance between any pair of points? This is FEJES TÓTH'S PROBLEM.

see also FEJES TÓTH'S PROBLEM

Spherical Polygon

A closed geometric figure on the surface of a SPHERE which is formed by the ARCS of GREAT CIRCLES. The spherical polygon is a generalization of the SPHERICAL TRIANGLE. If θ is the sum of the RADIAN ANGLES of a spherical polygon on a SPHERE of RADIUS r, then the AREA is

$$S = [\theta-(n-2)\pi]r^2.$$

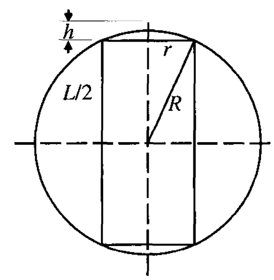
see also GREAT CIRCLE, SPHERICAL TRIANGLE

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 131, 1987.

Spherical Ring

A SPHERE with a CYLINDRICAL HOLE cut so that the centers of the CYLINDER and SPHERE coincide, also called a NAPKIN RING.



The volume of the entire CYLINDER is

$$V_{\rm cyl} = \pi LR^2, \quad (1)$$

and the VOLUME of the upper segment is

$$V_{\rm seg} = \tfrac{1}{6}\pi h(3R^2+h^2), \quad (2)$$

where

$$R = \sqrt{r^2-\tfrac{1}{4}L^2} \quad (3)$$

$$h = r-\tfrac{1}{2}L, \quad (4)$$

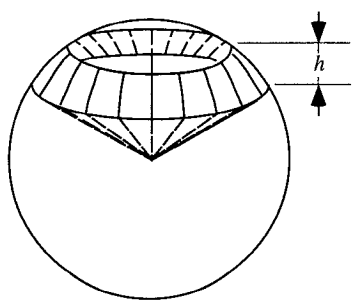
so the VOLUME removed upon drilling of a CYLINDRICAL hole is

$$\begin{aligned}V_{\rm rem} &= V_{\rm cyl}+2V_{\rm seg} = \pi[LR^2+\tfrac{1}{3}h(3R^2+h^2)]\\ &= \pi(LR^2+hR^2+\tfrac{1}{3}h^3)\\ &= \pi[L(r^2-\tfrac{1}{4}L^2)+(r-\tfrac{1}{2}L)(r^2-\tfrac{1}{4}L^2)\\ &\quad +\tfrac{1}{3}(r-\tfrac{1}{2}L)^3]\\ &= \pi[Lr^2-\tfrac{1}{4}L^3+(r^3-\tfrac{1}{2}r^2L-\tfrac{1}{4}RL^2+\tfrac{1}{8}L^3)\\ &\quad +\tfrac{1}{3}(r^3-\tfrac{3}{2}r^2L+\tfrac{3}{4}rL^2-\tfrac{1}{8}L^3)]\\ &= \pi[\tfrac{4}{3}r^3+(1-\tfrac{1}{2}-\tfrac{1}{2})r^2L+(-\tfrac{1}{4}+\tfrac{1}{4})RL^2\\ &\quad +L^3(-\tfrac{1}{4}+\tfrac{1}{8}-\tfrac{1}{24})]\\ &= \tfrac{4}{3}\pi r^3-\tfrac{1}{6}\pi L^3 = \tfrac{1}{6}\pi(8r^3-L^3), \end{aligned} \quad (5)$$

so

$$V_{\text{left}} = V_{\text{sphere}} - V_{\text{rem}} = \tfrac{4}{3}\pi r^3 - (\tfrac{4}{3}\pi r^3 - \tfrac{1}{6}\pi L^3) = \tfrac{1}{6}\pi L^3. \tag{6}$$

Spherical Sector



The VOLUME of a spherical sector, depicted above, is given by

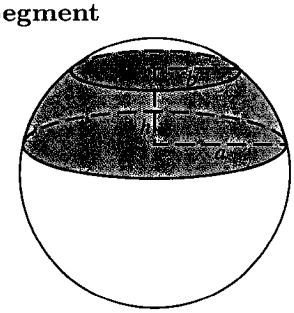
$$V = \tfrac{2}{3}\pi R^2 h,$$

where h is the vertical height of the upper and lower curves.

see also CYLINDRICAL SEGMENT, SPHERE, SPHERICAL CAP, SPHERICAL SEGMENT, ZONE

References
Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 131, 1987.

Spherical Segment



A spherical segment is the solid defined by cutting a SPHERE with a pair of PARALLEL PLANES. It can be thought of as a SPHERICAL CAP with the top truncated, and so it corresponds to a SPHERICAL FRUSTUM. The surface of the spherical segment (excluding the bases) is called a ZONE.

Call the RADIUS of the SPHERE R and the height of the segment (the distance from the plane to the top of SPHERE) h. Let the RADII of the lower and upper bases be denoted a and b, respectively. Call the distance from the center to the start of the segment d, and the height from the bottom to the top of the segment h. Call the RADIUS parallel to the segment r, and the height above the center y. Then $r^2 = R^2 - y^2$,

$$\begin{aligned} V &= \int_d^{d+h} \pi r^2\,dy = \pi \int_d^{d+h} (R^2 - y^2)\,dy \\ &= \pi \left[R^2 y - \tfrac{1}{3}y^3\right]_d^{d+h} = \pi\{R^2 h - \tfrac{1}{3}[(d+h)^3 - d^3]\} \\ &= \pi[R^2 h - \tfrac{1}{3}(d^3 + 3d^2 h + 3h^2 d + h^3 - d^3)] \\ &= \pi(R^2 h - d^2 h - h^2 d - \tfrac{1}{3}h^3) \\ &= \pi h(R^2 - d^2 - hd - \tfrac{1}{3}h^2). \end{aligned} \tag{1}$$

Using

$$a^2 = R^2 - d^2 \tag{2}$$

$$b^2 = R^2 - (d+h)^2 = R^2 - d^2 - 2dh - h^2, \tag{3}$$

gives

$$a^2 + b^2 = 2R^2 - 2d^2 - 2dh - h^2 \tag{4}$$

$$R^2 - d^2 - dh = \tfrac{1}{2}(a^2 + b^2 + h^2), \tag{5}$$

so

$$\begin{aligned} V &= \pi h[\tfrac{1}{2}(a^2 + b^2 + h^2) - \tfrac{1}{3}h^2] = \pi h(\tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 + \tfrac{1}{6}h^2) \\ &= \tfrac{1}{6}\pi h(3a^2 + 3b^2 + h^2). \end{aligned} \tag{6}$$

The surface area of the ZONE (which excludes the top and bottom bases) is given by

$$S = 2\pi R h. \tag{7}$$

see also ARCHIMEDES' PROBLEM, FRUSTUM, HEMISPHERE, SPHERE, SPHERICAL CAP, SPHERICAL SECTOR, SURFACE OF REVOLUTION, ZONE

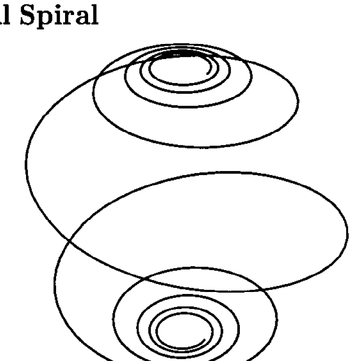
References
Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 130, 1987.

Spherical Shell

A generalization of an ANNULUS to 3-D. A spherical shell is the intersection of two concentric BALLS of differing RADII.

see also ANNULUS, BALL, CHORD, SPHERE, SPHERICAL HELIX

Spherical Spiral



The path taken by a ship which travels from the south pole to the north pole of a SPHERE while keeping a fixed (but not RIGHT) ANGLE with respect to the meridians. The curve has an infinite number of loops since the separation of consecutive revolutions gets smaller and smaller near the poles. It is given by the parametric equations

$$x = \cos t \cos c$$
$$y = \sin t \cos c$$
$$z = -\sin c,$$

where

$$c \equiv \tan^{-1}(at)$$

and a is a constant.

see also MERCATOR PROJECTION, SEIFERT'S SPHERICAL SPIRAL

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 162, 1993.

Lauwerier, H. "Spherical Spiral." In *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, pp. 64–66, 1991.

Spherical Symmetry

Let $\mathbf{A}$ and $\mathbf{B}$ be constant VECTORS. Define

$$Q \equiv 3(\mathbf{A}\cdot\hat{\mathbf{r}})(\mathbf{B}\cdot\hat{\mathbf{r}}) - \mathbf{A}\cdot\mathbf{B}.$$

Then the average of Q over a spherically symmetric surface or volume is

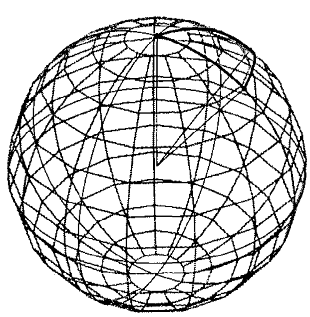
$$\langle Q\rangle = \left\langle 3\cos^2\theta - 1\right\rangle(\mathbf{A}\cdot\mathbf{B}) = 0,$$

since $\left\langle 3\cos^2\theta - 1\right\rangle = 0$ over the sphere.

Spherical Tessellation

see TRIANGULAR SYMMETRY GROUP

Spherical Triangle



A spherical triangle is a figure formed on the surface of a sphere by three great circular arcs intersecting pairwise in three vertices. The spherical triangle is the spherical analog of the planar TRIANGLE. Let a spherical triangle have ANGLES α, β, and γ and RADIUS r. Then the AREA of the spherical triangle is

$$K = r^2[(\alpha+\beta+\gamma)-\pi].$$

The sum of the angles of a spherical triangle is between $180°$ and $540°$. The amount by which it exceeds $180°$ is called the SPHERICAL EXCESS and is denoted E or Δ.

The study of angles and distances of figures on a sphere is known as SPHERICAL TRIGONOMETRY.

see also COLUNAR TRIANGLE, GIRARD'S SPHERICAL EXCESS FORMULA, L'HUILIER'S THEOREM, SPHERICAL POLYGON, SPHERICAL TRIGONOMETRY

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 79, 1972.

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 131 and 147–150, 1987.

Spherical Trigonometry

Define a SPHERICAL TRIANGLE on the surface of a unit SPHERE, centered at a point O, with vertices A, B, and C. Define ANGLES $a \equiv \angle BOC$, $b \equiv \angle COA$, and $c \equiv \angle AOB$. Let the ANGLE between PLANES AOB and AOC be α, the ANGLE between PLANES BOC and AOB be β, and the ANGLE between PLANES BOC and AOC be γ. Define the VECTORS

$$\mathbf{a} \equiv \overrightarrow{OA} \tag{1}$$
$$\mathbf{b} \equiv \overrightarrow{OB} \tag{2}$$
$$\mathbf{c} \equiv \overrightarrow{OC}. \tag{3}$$

Then

$$\begin{aligned}(\hat{\mathbf{a}}\times\hat{\mathbf{b}})\cdot(\hat{\mathbf{a}}\times\hat{\mathbf{c}}) &= (|\hat{\mathbf{a}}|\,|\hat{\mathbf{b}}|\sin c)(|\hat{\mathbf{a}}|\,|\hat{\mathbf{c}}|\sin b)\cos\alpha \\ &= \sin b\sin c\cos\alpha.\end{aligned} \tag{4}$$

Equivalently,

$$\begin{aligned}(\hat{\mathbf{a}}\times\hat{\mathbf{b}})\cdot(\hat{\mathbf{a}}\times\hat{\mathbf{c}}) &= \hat{\mathbf{a}}\cdot[\hat{\mathbf{b}}\times(\hat{\mathbf{a}}\times\hat{\mathbf{c}})] \\ &= \hat{\mathbf{a}}\cdot[\hat{\mathbf{a}}(\hat{\mathbf{b}}\cdot\hat{\mathbf{c}}) - \hat{\mathbf{c}}(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})] \\ &= (\hat{\mathbf{b}}\cdot\hat{\mathbf{c}}) - (\hat{\mathbf{a}}\cdot\hat{\mathbf{c}})(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}) \\ &= \cos a - \cos c\cos b.\end{aligned} \tag{5}$$

Since these two expressions are equal, we obtain the identity

$$\cos a = \cos b\cos c + \sin b\sin c\cos\alpha \tag{6}$$

The identity

$$\begin{aligned}\sin\alpha &= \frac{|(\hat{\mathbf{a}}\times\hat{\mathbf{b}})\times(\hat{\mathbf{a}}\times\hat{\mathbf{c}})|}{|\hat{\mathbf{a}}\times\hat{\mathbf{b}}||\hat{\mathbf{a}}\times\hat{\mathbf{c}}|} = -\frac{|\hat{\mathbf{a}}[\hat{\mathbf{b}},\hat{\mathbf{a}},\hat{\mathbf{c}}] + \hat{\mathbf{b}}[\hat{\mathbf{a}},\hat{\mathbf{a}},\hat{\mathbf{c}}]|}{\sin b\sin c} \\ &= \frac{[\hat{\mathbf{a}},\hat{\mathbf{b}},\hat{\mathbf{c}}]}{\sin b\sin c},\end{aligned} \tag{7}$$

where $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is the SCALAR TRIPLE PRODUCT, gives a spherical analog of the LAW OF SINES,

$$\frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b} = \frac{\sin\gamma}{\sin c} = \frac{6\,\mathrm{Vol}(OABC)}{\sin a \sin b \sin c}, \tag{8}$$

where $\mathrm{Vol}(OABC)$ is the VOLUME of the TETRAHEDRON. From (7) and (8), it follows that

$$\sin a \cos\beta = \cos b \sin c - \sin b \cos c \cos\alpha \tag{9}$$

$$\cos a \cos\gamma = \sin a \cot b - \sin\gamma \cot\beta. \tag{10}$$

These are the fundamental equalities of spherical trigonometry.

There are also spherical analogs of the LAW OF COSINES for the sides of a spherical triangle,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \tag{11}$$

$$\cos b = \cos c \cos a + \sin c \sin a \cos B \tag{12}$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C, \tag{13}$$

and the angles of a spherical triangle,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \tag{14}$$

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b \tag{15}$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c \tag{16}$$

(Beyer 1987), as well as the LAW OF TANGENTS

$$\frac{\tan[\frac{1}{2}(a-b)]}{\tan[\frac{1}{2}(a+b)]} = \frac{\tan[\frac{1}{2}(A-B)]}{\tan[\frac{1}{2}(A+B)]}. \tag{17}$$

Let

$$s \equiv \tfrac{1}{2}(a+b+c) \tag{18}$$

$$S \equiv \tfrac{1}{2}(A+B+C), \tag{19}$$

then the half-angle formulas are

$$\tan(\tfrac{1}{2}A) = \frac{k}{\sin(s-a)} \tag{20}$$

$$\tan(\tfrac{1}{2}B) = \frac{k}{\sin(s-b)} \tag{21}$$

$$\tan(\tfrac{1}{2}C) = \frac{k}{\sin(s-c)}, \tag{22}$$

where

$$k^2 = \frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s} = \tan^2 r, \tag{23}$$

and the half-side formulas are

$$\tan(\tfrac{1}{2}a) = K\cos(S-A) \tag{24}$$

$$\tan(\tfrac{1}{2}b) = K\cos(S-B) \tag{25}$$

$$\tan(\tfrac{1}{2}c) = K\cos(S-C), \tag{26}$$

where

$$K^2 = -\frac{\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)} = \tan^2 R, \tag{27}$$

where R is the RADIUS of the SPHERE on which the spherical triangle lies.

Additional formulas include the HAVERSINE formulas

$$\mathrm{hav}\, a = \mathrm{hav}(b-c) + \sin b \sin c \sin(s-c) \tag{28}$$

$$\mathrm{hav}\, A = \frac{\sin(s-b)\sin(s-c)}{\sin b \sin c} \tag{29}$$

$$= \frac{\mathrm{hav}\, a - \mathrm{hav}(b-c)}{\sin b \sin c} \tag{30}$$

$$= \mathrm{hav}[\pi - (B+C)] + \sin B \sin C\, \mathrm{hav}\, a, \tag{31}$$

GAUSS'S FORMULAS

$$\frac{\sin[\frac{1}{2}(a-b)]}{\sin(\frac{1}{2}c)} = \frac{\sin[\frac{1}{2}(A-B)]}{\cos(\frac{1}{2}C)} \tag{32}$$

$$\frac{\sin[\frac{1}{2}(a+b)]}{\sin(\frac{1}{2}c)} = \frac{\cos[\frac{1}{2}(A-B)]}{\sin(\frac{1}{2}C)} \tag{33}$$

$$\frac{\cos[\frac{1}{2}(a-b)]}{\cos(\frac{1}{2}c)} = \frac{\sin[\frac{1}{2}(A+B)]}{\cos(\frac{1}{2}C)} \tag{34}$$

$$\frac{\cos[\frac{1}{2}(a+b)]}{\cos(\frac{1}{2}c)} = \frac{\cos[\frac{1}{2}(A+B)]}{\sin(\frac{1}{2}C)}, \tag{35}$$

and NAPIER'S ANALOGIES

$$\frac{\sin[\frac{1}{2}(A-B)]}{\sin[\frac{1}{2}(A+B)]} = \frac{\tan[\frac{1}{2}(a-b)]}{\tan(\frac{1}{2}c)} \tag{36}$$

$$\frac{\cos[\frac{1}{2}(A-B)]}{\cos[\frac{1}{2}(A+B)]} = \frac{\tan[\frac{1}{2}(a+b)]}{\tan(\frac{1}{2}c)} \tag{37}$$

$$\frac{\sin[\frac{1}{2}(a-b)]}{\sin[\frac{1}{2}(a+b)]} = \frac{\tan[\frac{1}{2}(A-B)]}{\cot(\frac{1}{2}C)} \tag{38}$$

$$\frac{\cos[\frac{1}{2}(a-b)]}{\cos[\frac{1}{2}(a+b)]} = \frac{\tan[\frac{1}{2}(A+B)]}{\cot(\frac{1}{2}C)} \tag{39}$$

(Beyer 1987).

see also ANGULAR DEFECT, DESCARTES TOTAL ANGULAR DEFECT, GAUSS'S FORMULAS, GIRARD'S SPHERICAL EXCESS FORMULA, LAW OF COSINES, LAW OF SINES, LAW OF TANGENTS, L'HUILIER'S THEOREM, NAPIER'S ANALOGIES, SPHERICAL EXCESS, SPHERICAL GEOMETRY, SPHERICAL POLYGON, SPHERICAL TRIANGLE

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 131 and 147–150, 1987.

Danby, J. M. *Fundamentals of Celestial Mechanics, 2nd ed., rev. ed.* Richmond, VA: Willmann-Bell, 1988.

Smart, W. M. *Text-Book on Spherical Astronomy, 6th ed.* Cambridge, England: Cambridge University Press, 1960.

Spherical Vector Harmonic

see VECTOR SPHERICAL HARMONIC

Spheroid

A spheroid is an ELLIPSOID

$$\frac{r^2\cos^2\theta\sin^2\phi}{a^2}+\frac{r^2\sin^2\theta\sin^2\phi}{b^2}+\frac{r^2\cos^2\phi}{c^2}=1 \quad (1)$$

with two SEMIMAJOR AXES equal. Orient the ELLIPSE so that the a and b axes are equal, then

$$\frac{r^2\cos^2\theta\sin^2\phi}{a^2}+\frac{r^2\sin^2\theta\sin^2\phi}{a^2}+\frac{r^2\cos^2\phi}{c^2}=1 \quad (2)$$

$$\frac{r^2\sin^2\phi}{a^2}+\frac{r^2\cos^2\phi}{c^2}=1, \quad (3)$$

where a is the equatorial RADIUS and c is the polar RADIUS. Here ϕ is the colatitude, so take $\delta\equiv\pi/2-\phi$ to express in terms of latitude.

$$\frac{r^2\cos^2\delta}{a^2}+\frac{r^2\sin^2\delta}{c^2}=1. \quad (4)$$

Rewriting $\cos^2\delta=1-\sin^2\delta$ gives

$$\frac{r^2}{a^2}+r^2\sin^2\delta\left(\frac{1}{c^2}-\frac{1}{a^2}\right)=1 \quad (5)$$

$$r^2\left(1+a^2\sin^2\delta\frac{a^2-c^2}{c^2a^2}\right)=r^2\left(1+\sin^2\delta\frac{a^2-c^2}{c^2}\right)=a^2, \quad (6)$$

so

$$r=a\left(1+\sin^2\delta\frac{a^2-c^2}{c^2}\right)^{-1/2}. \quad (7)$$

If $a>c$, the spheroid is OBLATE. If $a<c$, the spheroid is PROLATE. If $a=c$, the spheroid degenerates to a SPHERE.

see also DARWIN-DE SITTER SPHEROID, ELLIPSOID, OBLATE SPHEROID, PROLATE SPHEROID

Spheroidal Harmonic

A spheroidal harmonic is a special case of the ELLIPSOIDAL HARMONIC which satisfies the differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dS}{dx}\right]+\left(\lambda-c^2x^2-\frac{m^2}{1-x^2}\right)S=0$$

on the interval $-1\le x\le 1$.

see also ELLIPSOIDAL HARMONIC

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "A Worked Example: Spheroidal Harmonics." §17.4 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 764–773, 1992.

Spheroidal Wavefunction

Whittaker and Watson (1990, p. 403) define the internal and external spheroidal wavefunctions as

$$S_{mn}^{(1)}=2\pi\frac{(n-m)!}{(n+m)!}P_n^m(ir)P_n^m(\cos\theta)\begin{matrix}\cos\\ \sin\end{matrix}(m\phi)$$

$$S_{mn}^{(2)}=2\pi\frac{(n-m)!}{(n+m)!}Q_n^m(ir)Q_n^m(\cos\theta)\begin{matrix}\cos\\ \sin\end{matrix}(m\phi).$$

see also ELLIPSOIDAL HARMONIC, OBLATE SPHEROIDAL WAVE FUNCTION, PROLATE SPHEROIDAL WAVE FUNCTION, SPHERICAL HARMONIC

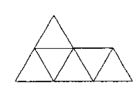
References

Abramowitz, M. and Stegun, C. A. (Eds.). "Spheroidal Wave Functions." Ch. 21 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 751–759, 1972.

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 642–644, 1953.

Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis, 4th ed.* Cambridge, England: Cambridge University Press, 1990.

Sphinx

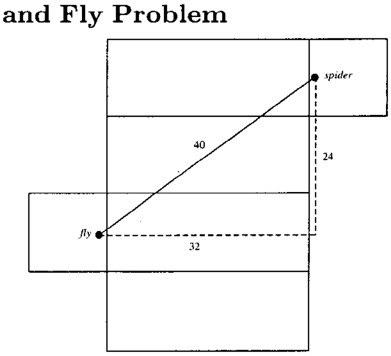


A 6-POLYIAMOND named for its resemblance to the Great Sphinx of Egypt.

References

Golomb, S. W. *Polyominoes: Puzzles, Patterns, Problems, and Packings, 2nd ed.* Princeton, NJ: Princeton University Press, p. 92, 1994.

Spider and Fly Problem



In a rectangular room (a CUBOID) with dimensions $30'\times 12'\times 12'$, a spider is located in the middle of one $12'\times 12'$ wall one foot away from the ceiling. A fly is in the middle of the opposite wall one foot away from the floor. If the fly remains stationary, what is the shortest distance the spider must crawl to capture the fly? The answer, $40'$, can be obtained by "flattening" the walls as illustrated above.

References

Pappas, T. "The Spider & the Fly Problem." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, pp. 218 and 233, 1989.

Spider Lines

see EPITROCHOID

Spiegeldrieck

see FUHRMANN TRIANGLE

Spieker Center

The center of the SPIEKER CIRCLE. It is the CENTROID of the PERIMETER of the original TRIANGLE. The third BROCARD POINT is COLLINEAR with the Spieker center and the ISOTOMIC CONJUGATE POINT of its INCENTER.

see also BROCARD POINTS, CENTROID (TRIANGLE), INCENTER, ISOTOMIC CONJUGATE POINT, PERIMETER, SPIEKER CIRCLE, TAYLOR CENTER

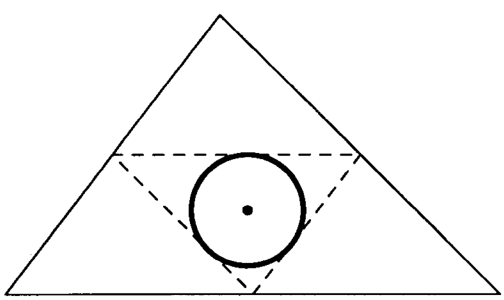
References

Casey, J. *A Treatise on the Analytical Geometry of the Point, Line, Circle, and Conic Sections, Containing an Account of Its Most Recent Extensions, with Numerous Examples, 2nd ed., rev. enl.* Dublin: Hodges, Figgis, & Co., p. 81, 1893.

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 226–229 and 249, 1929.

Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.

Spieker Circle



The INCIRCLE of the MEDIAL TRIANGLE. The center of the Spieker circle is called the SPIEKER CENTER.

see also INCIRCLE, MEDIAL TRIANGLE, SPIEKER CENTER

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 226–228, 1929.

Spigot Algorithm

An ALGORITHM which generates digits of a quantity one at a time without using or requiring previously computed digits. Amazingly, spigot ALGORITHMS are known for both PI and e.

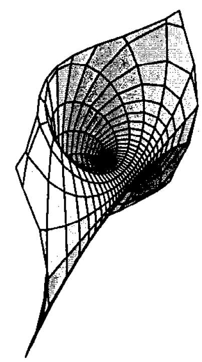
Spijker's Lemma

The image on the RIEMANN SPHERE of any CIRCLE under a COMPLEX rational mapping with NUMERATOR and DENOMINATOR having degrees no more than n has length no longer than $2n\pi$.

References

Edelman, A. and Kostlan, E. "How Many Zeros of a Random Polynomial are Real?" *Bull. Amer. Math. Soc.* **32**, 1–37, 1995.

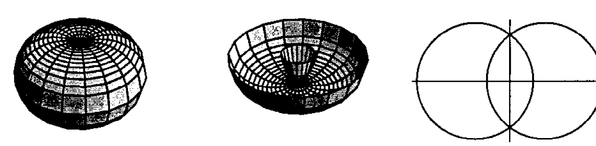
Spindle Cyclide



The inversion of a SPINDLE TORUS. If the inversion center lies on the torus, then the spindle cyclide degenerates to a PARABOLIC SPINDLE CYCLIDE.

see also CYCLIDE, HORN CYCLIDE, PARABOLIC CYCLIDE, RING CYCLIDE, SPINDLE TORUS, TORUS

Spindle Torus



One of the three STANDARD TORI given by the parametric equations

$$x = (c + a\cos v)\cos u$$
$$y = (c + a\cos v)\sin u$$
$$z = a\sin v$$

with $c < a$. The exterior surface is called an APPLE and the interior surface a LEMON. The above left figure shows a spindle torus, the middle a cutaway, and the right figure shows a cross-section of the spindle torus through the xz-plane.

see also APPLE, CYCLIDE, HORN TORUS, LEMON, PARABOLIC SPINDLE CYCLIDE, RING TORUS, SPINDLE CYCLIDE, STANDARD TORI, TORUS

References

Gray, A. "Tori." §11.4 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 218–220, 1993.

Pinkall, U. "Cyclides of Dupin." §3.3 in *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, pp. 28–30, 1986.

Spinode

see also ACNODE, CRUNODE, CUSP, TACNODE

Spinor

A two-component COMPLEX column VECTOR. Spinors are used in physics to represent particles with half-integral spin (i.e., Fermions).

References

Lounesto, P. "Counterexamples to Theorems Published and Proved in Recent Literature on Clifford Algebras, Spinors, Spin Groups, and the Exterior Algebra." `http://www.hit.fi/~lounesto/counterexamples.htm`.

Morse, P. M. and Feshbach, H. "The Lorentz Transformation, Four-Vectors, Spinors." §1.7 in *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 93–107, 1953.

Spira Mirabilis

see LOGARITHMIC SPIRAL

Spiral

In general, a spiral is a curve with $\tau(s)/\kappa(s)$ equal to a constant for all s, where τ is the TORSION and κ is the CURVATURE.

see also ARCHIMEDES' SPIRAL, CIRCLE INVOLUTE, CONICAL SPIRAL, CORNU SPIRAL, COTES' SPIRAL, DAISY, EPISPIRAL, FERMAT'S SPIRAL, HYPERBOLIC SPIRAL, LOGARITHMIC SPIRAL, MICE PROBLEM, NIELSEN'S SPIRAL, PHYLLOTAXIS, POINSOT'S SPIRALS, POLYGONAL SPIRAL, SPHERICAL SPIRAL

References

Eppstein, D. "Spirals." `http://www.ics.uci.edu/~eppstein/junkyard/spiral.html`.

Lauwerier, H. *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, pp. 54–66, 1991.

Lockwood, E. H. "Spirals." Ch. 22 in *A Book of Curves.* Cambridge, England: Cambridge University Press, pp. 172–175, 1967.

Yates, R. C. "Spirals." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 206–216, 1952.

Spiral Point

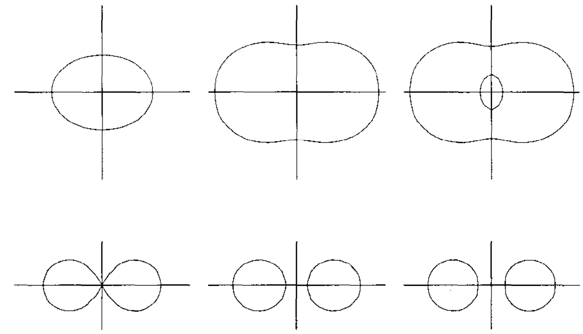
A FIXED POINT for which the EIGENVALUES are COMPLEX CONJUGATES.

see also STABLE SPIRAL POINT, UNSTABLE SPIRAL POINT

References

Tabor, M. "Classification of Fixed Points." §1.4.b in *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, pp. 22–25, 1989.

Spiric Section



A curve with Cartesian equation

$$(r^2 - a^2 + c^2 + x^2 + y^2) = 4r^2(x^2 + c^2).$$

Around 150 BC, Menaechmus constructed CONIC SECTIONS by cutting a CONE by a PLANE. Two hundred years later, the Greek mathematician Perseus investigated the curves obtained by cutting a TORUS by a PLANE which is PARALLEL to the line through the center of the HOLE of the TORUS (MacTutor).

In the FORMULA of the curve given above, the TORUS is formed from a CIRCLE of RADIUS a whose center is rotated along a CIRCLE of RADIUS r. The value of c gives the distance of the cutting PLANE from the center of the TORUS.

When $c = 0$, the curve consists of two CIRCLES of RADIUS a whose centers are at $(r, 0)$ and $(-r, 0)$. If $c = r + a$, the curve consists of one point (the origin), while if $c > r + a$, no point lies on the curve. The above curves have $(a, b, r) = (3, 4, 2)$, (3, 1, 2) (3, 0.8, 2), (3, 1, 4), (3, 1, 4.5), and (3, 0, 4.5).

References

MacTutor History of Mathematics Archive. "Spiric Sections." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Spiric.html`.

Spirograph

A HYPOTROCHOID generated by a fixed point on a CIRCLE rolling inside a fixed CIRCLE. It has parametric equations,

$$x = (R + r)\cos\theta - (r + \rho)\cos\left(\frac{R + r}{r}\theta\right) \tag{1}$$

$$y = (R + r)\sin\theta - (r + \rho)\sin\left(\frac{R + r}{r}\theta\right), \tag{2}$$

where R is the radius of the fixed circle, r is the radius of the rotating circle, and ρ is the offset of the edge of the rotating circle. The figure closes only if R, r, and ρ are RATIONAL. The equations can also be written

$$x = x_0[m\cos t + a\cos(nt)] - y_0[m\sin t - a\sin(nt)] \tag{3}$$

$$y = y_0[m\cos t + a\cos(nt)] + x_0[m\sin t - a\sin(nt)], \tag{4}$$

where the outer wheel has radius 1, the inner wheel a radius p/q, the pen is placed a units from the center, the beginning is at θ radians above the x-axis, and

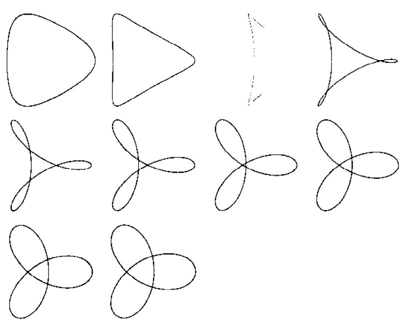
$$m \equiv \frac{q-p}{q} \quad (5)$$

$$n \equiv \frac{q-p}{p} \quad (6)$$

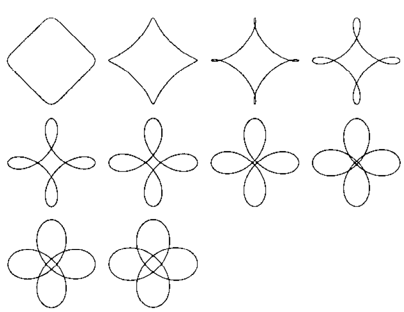
$$x_0 \equiv \cos\theta \quad (7)$$

$$y_0 \equiv \sin\theta. \quad (8)$$

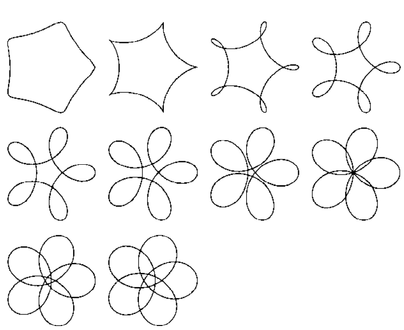
The following curves are for $a = i/10$, with $i = 1, 2, \ldots, 10$, and $\theta = 0$.



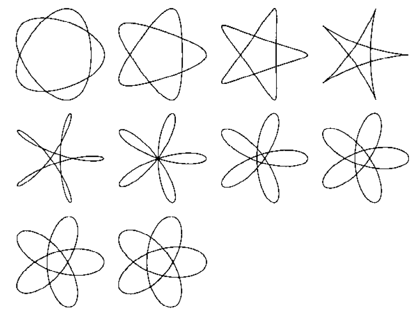
$(p, q) = (1, 3)$



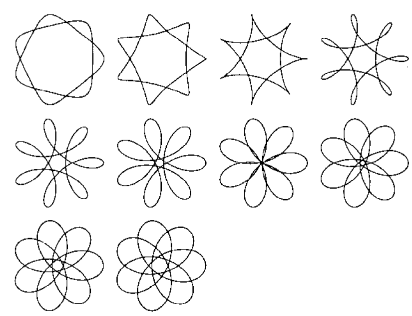
$(p, q) = (1, 4)$



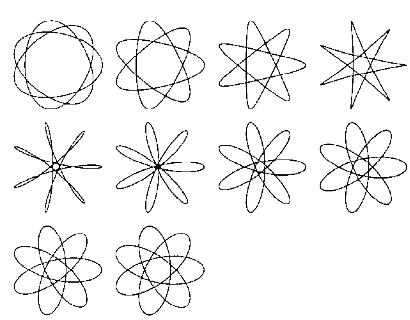
$(p, q) = (1, 5)$



$(p, q) = (2, 5)$

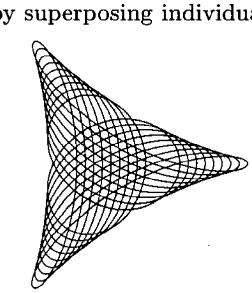


$(p, q) = (2, 7)$



$(p, q) = (3, 7)$

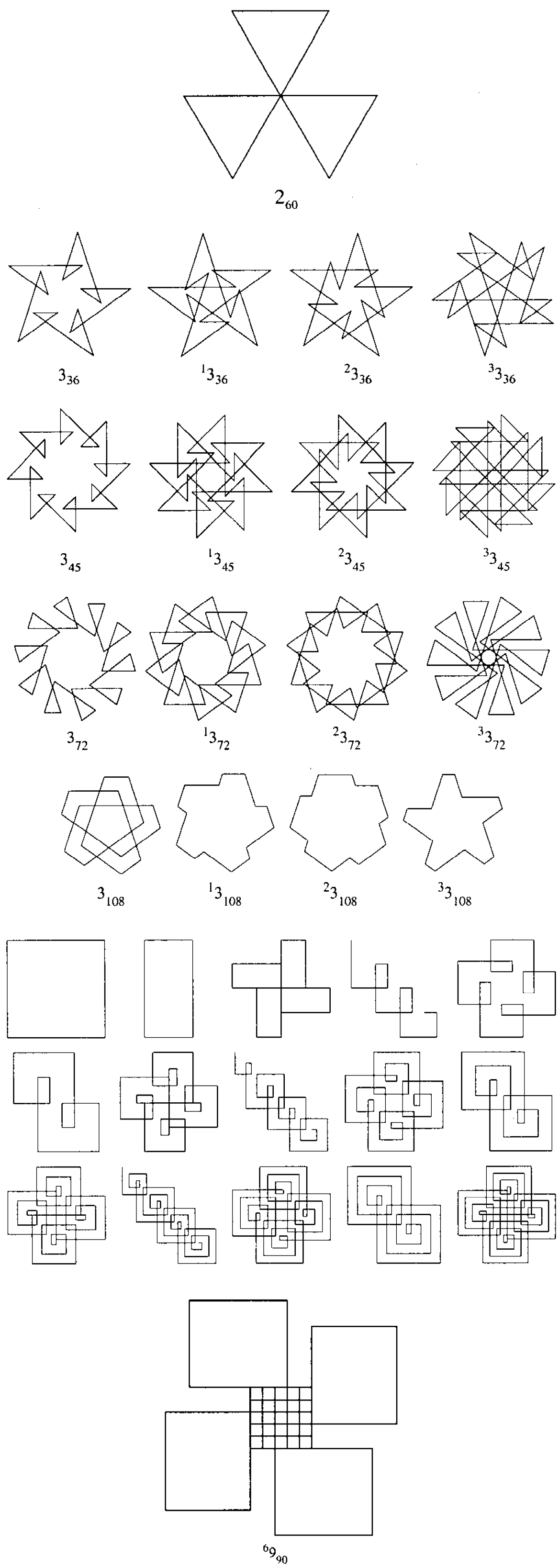
Additional attractive designs such as the following can also be made by superposing individual spirographs.



see also EPITROCHOID, HYPOTROCHOID, MAURER ROSE, SPIROLATERAL

Spirolateral

A figure formed by taking a series of steps of length 1, 2, ..., n, with an angle θ turn after each step. The symbol for a spirolateral is ${}^{a_1,\ldots,a_k}n_\theta$, where the a_is indicate that turns are in the $-\theta$ direction for these steps.



see also MAURER ROSE, SPIROGRAPH

References

Gardner, M. "Worm Paths." Ch. 17 in *Knotted Doughnuts and Other Mathematical Entertainments.* New York: W. H. Freeman, 1986.

Odds, F. C. "Spirolaterals." *Math. Teacher* **66**, 121–124, 1973.

Spline

An interpolating POLYNOMIAL which uses information from neighboring points to obtain a degree of global smoothness.

see also B-SPLINE, BÉZIER SPLINE, CUBIC SPLINE, NURBS CURVE

References

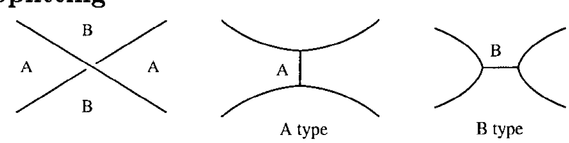
Bartels, R. H.; Beatty, J. C.; and Barsky, B. A. *An Introduction to Splines for Use in Computer Graphics and Geometric Modelling.* San Francisco, CA: Morgan Kaufmann, 1987.

de Boor, C. *A Practical Guide to Splines.* New York: Springer-Verlag, 1978.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Interpolation and Extrapolation." Ch. 3 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 99–122, 1992.

Späth, H. *One Dimensional Spline Interpolation Algorithms.* Wellesley, MA: A. K. Peters, 1995.

Splitting



Splitting Algorithm

A method for computing a UNIT FRACTION. This method always terminates (Beeckmans 1993).

References

Beeckmans, L. "The Splitting Algorithm for Egyptian Fractions." *J. Number Th.* **43**, 173–185, 1993.

Sponge

A sponge is a solid which can be parameterized by INTEGERS p, q, and n which satisfy the equation

$$2\sin\left(\frac{\pi}{p}\right)\sin\left(\frac{\pi}{q}\right)=\cos\left(\frac{\pi}{k}\right).$$

The possible sponges are $\{p,q|k\}=\{6,6|3\}$, $\{6,4|4\}$, $\{4,6|4\}$, $\{3,6|6\}$, and $\{4,4|\infty\}$ (Ball and Coxeter 1987).

see also HONEYCOMB, MENGER SPONGE, SIERPIŃSKI SPONGE, TETRIX

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 152, 1987.

Cromwell, P. R. *Polyhedra.* New York: Cambridge University Press, p. 79, 1997.

Sporadic Group

One of the 26 finite SIMPLE GROUPS. The most complicated is the MONSTER GROUP. A summary, as given by Conway *et al.* (1985), is given below.

Sym	Name	Order	M	A
M_{11}	Mathieu	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1	1
M_{12}	Mathieu	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2	2
M_{22}	Mathieu	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	12	2
M_{23}	Mathieu	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	1	1
M_{24}	Mathieu	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	1	1
$J_2 = HJ$	Janko	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2	2
Suz	Suzuki	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	6	2
HS	Higman-Sims	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	2	2
McL	McLaughlin	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	3	2
Co_3	Conway	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	1	1
Co_2	Conway	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	1	1
Co_1	Conway	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$	2	1
He	Held	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	1	2
Fi_{22}	Fischer	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	6	2
Fi_{23}	Fischer	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	1	1
Fi'_{24}	Fischer	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17$ $\cdot 23 \cdot 29$	3	2
HN	Harada-Norton	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$	1	2
Th	Thompson	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	1	1
B	Baby Monster	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$ $\cdot 23 \cdot 31 \cdot 47$	2	1
M	Monster	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19$ $\cdot 23 \cdot \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$	1	1
J_1	Janko	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	1	1
$O'N$	O'Nan	$2^9 \cdot 3^4 \cdot 7^3 \cdot 5 \cdot 11 \cdot 19 \cdot 31$	3	2
J_3	Janko	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$	3	2
Ly	Lyons	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$	1	1
Ru	Rudvalis	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$	2	1
J_4	Janko	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31$ $\cdot 37 \cdot 43$	1	1

see also BABY MONSTER GROUP, CONWAY GROUPS, FISCHER GROUPS, HARADA-NORTON GROUP, HELD GROUP, HIGMAN-SIMS GROUP, JANKO GROUPS, LYONS GROUP, MATHIEU GROUPS, MCLAUGHLIN GROUP, MONSTER GROUP, O'NAN GROUP, RUDVALIS GROUP, SUZUKI GROUP, THOMPSON GROUP

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Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; and Wilson, R. A. *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford, England: Clarendon Press, p. viii, 1985.

Math. Intell. Cover of volume **2**, 1980.

Wilson, R. A. "ATLAS of Finite Group Representation." `http://for.mat.bham.ac.uk/atlas#spo.`

Sports

see also BASEBALL, BOWLING, CHECKERS, CHESS, GO

Sprague-Grundy Function

see NIM-VALUE

Sprague-Grundy Number

see NIM-VALUE

Sprague-Grundy Value

see NIM-VALUE

Spread (Link)

see SPAN (LINK)

Spread (Tree)

A TREE having an infinite number of branches and whose nodes are sequences generated by a set of rules.

see also FAN

Spun Knot

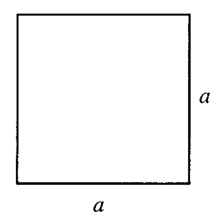
A 3-D KNOT spun about a plane in 4-D. Unlike SUSPENDED KNOTS, spun knots are smoothly embedded at the poles.

see also SUSPENDED KNOT, TWIST-SPUN KNOT

Squarable

An object which can be constructed by SQUARING is called squarable.

Square



The term square is sometimes used to mean SQUARE NUMBER. When used in reference to a geometric figure, however, it means a convex QUADRILATERAL with four equal sides at RIGHT ANGLES to each other, illustrated above.

The PERIMETER of a square with side length a is

$$L = 4a \tag{1}$$

and the AREA is

$$A = a^2. \tag{2}$$

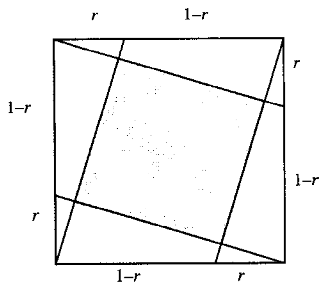
The INRADIUS r, CIRCUMRADIUS R, and AREA A can be computed directly from the formulas for a general regular POLYGON with side length a and $n = 4$ sides,

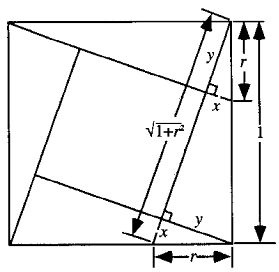
$$r = \tfrac{1}{2} a \cot\left(\frac{\pi}{4}\right) = \tfrac{1}{2} a \tag{3}$$

$$R = \tfrac{1}{2} a \csc\left(\frac{\pi}{4}\right) = \tfrac{1}{2}\sqrt{2}\, a \tag{4}$$

$$A = \tfrac{1}{4} n a^2 \cot\left(\frac{\pi}{4}\right) = a^2. \tag{5}$$

The length of the DIAGONAL of the UNIT SQUARE is $\sqrt{2}$, sometimes known as PYTHAGORAS'S CONSTANT.





The AREA of a square inscribed inside a UNIT SQUARE as shown in the above diagram can be found as follows. Label x and y as shown, then

$$x^2 + y^2 = r^2 \tag{6}$$

$$(\sqrt{1+r^2} - x)^2 + y^2 = 1. \tag{7}$$

Plugging (6) into (7) gives

$$(\sqrt{1+r^2} - x)^2 + (r^2 - x^2) = 1. \tag{8}$$

Expanding

$$x^2 - 2x\sqrt{1+r^2} + 1 + r^2 + r^2 - x^2 = 1 \tag{9}$$

and solving for x gives

$$x = \frac{r^2}{\sqrt{1+r^2}}. \tag{10}$$

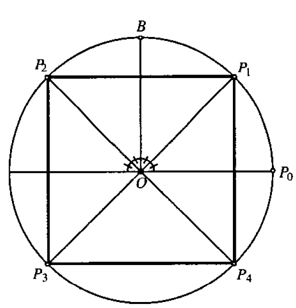
Plugging in for y yields

$$y = \sqrt{r^2 - x^2} = \frac{r}{\sqrt{1+r^2}}. \tag{11}$$

The area of the shaded square is then

$$A = (\sqrt{1+r^2} - x - y)^2 = \frac{(1-r)^2}{1+r^2} \tag{12}$$

(Detemple and Harold 1996).



Square

The STRAIGHTEDGE and COMPASS construction of the square is simple. Draw the line OP_0 and construct a circle having OP_0 as a radius. Then construct the perpendicular OB through O. Bisect P_0OB and $P_0'OB$ to locate P_1 and P_2, where P_0' is opposite P_0. Similarly, construct P_3 and P_4 on the other SEMICIRCLE. Connecting $P_1P_2P_3P_4$ then gives a square.

As shown by Schnirelmann, a square can be INSCRIBED in any closed convex planar curve (Steinhaus 1983). A square can also be CIRCUMSCRIBED about any closed curve (Steinhaus 1983).

An infinity of points in the interior of a square are known whose distances from three of the corners of a square are RATIONAL NUMBERS. Calling the distances a, b, and c where s is the side length of the square, these solutions satisfy

$$(s^2 + b^2 - a^2)^2 + (s^2 + b^2 - c^2)^2 = (2bs)^2 \tag{13}$$

(Guy 1994). In this problem, one of a, b, c, and s is DIVISIBLE by 3, one by 4, and one by 5. It is not known if there are points having distances from *all four* corners RATIONAL, but such a solution requires the additional condition

$$a^2 + c^2 = b^2 + d^2. \tag{14}$$

In this problem, s is DIVISIBLE by 4 and a, b, c, and d are ODD. If s is not DIVISIBLE by 3 (5), then two of a, b, c, and d are DIVISIBLE by 3 (5) (Guy 1994).

see also BROWKIN'S THEOREM, DISSECTION, DOUGLAS-NEUMANN THEOREM, FINSLER-HADWIGER THEOREM, LOZENGE, PERFECT SQUARE DISSECTION, PYTHAGORAS'S CONSTANT, PYTHAGOREAN SQUARE PUZZLE, RECTANGLE, SQUARE CUTTING, SQUARE NUMBER, SQUARE PACKING, SQUARE QUADRANTS, UNIT SQUARE, VON AUBEL'S THEOREM

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Guy, R. K. "Rational Distances from the Corners of a Square." §D19 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 181–185, 1994.
Steinhaus, H. *Mathematical Snapshots, 3rd American ed.* New York: Oxford University Press, p. 104, 1983.

Square Bracket Polynomial

A POLYNOMIAL which is not necessarily an invariant of a LINK. It is related to the DICHROIC POLYNOMIAL. It is defined by the SKEIN RELATIONSHIP

$$B_{L_+} = q^{-1/2} v B_{L_0} + B_{L_\infty}, \tag{1}$$

and satisfies

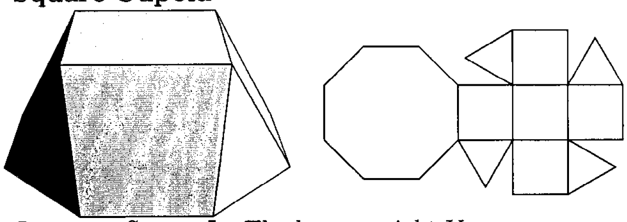
$$B_{\text{unknot}} = q^{1/2} \tag{2}$$

and

$$B_{L \cup \text{unknot}} = q^{1/2} B_L. \tag{3}$$

References
Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, pp. 235–241, 1994.

Square Cupola



JOHNSON SOLID J_4. The bottom eight VERTICES are

$$(\pm\tfrac{1}{2}(1+\sqrt{2}), \pm\tfrac{1}{2}, 0), (\pm\tfrac{1}{2}, \pm\tfrac{1}{2}(1+\sqrt{2}), 0),$$

and the top four VERTICES are

$$\left(\pm\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(0, \pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Square Curve

see SIERPIŃSKI CURVE

Square Cutting

The average number of regions into which N lines divide a SQUARE is

$$\tfrac{1}{16}N(N-1)\pi + N + 1$$

(Santaló 1976).

see also CIRCLE CUTTING

References

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/geom/geom.html`.

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Square-Free

see SQUAREFREE

Square Gyrobicupola

see JOHNSON SOLID

Square Integrable

A function $f(x)$ is said to be square integrable if

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx$$

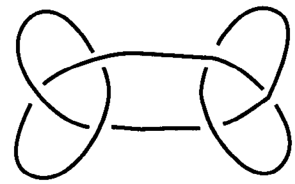
is finite.

see also INTEGRABLE, L_2-NORM, TITCHMARSH THEOREM

References

Sansone, G. "Square Integrable Functions." §1.1 in *Orthogonal Functions, rev. English ed.* New York: Dover, pp. 1–2, 1991.

Square Knot



A composite KNOT of six crossings consisting of a KNOT SUM of a TREFOIL KNOT and its MIRROR IMAGE. The GRANNY KNOT has the same ALEXANDER POLYNOMIAL $(x^2-x+1)^2$ as the square knot. The square knot is also called the REEF KNOT.

see also GRANNY KNOT, MIRROR IMAGE, TREFOIL KNOT

References

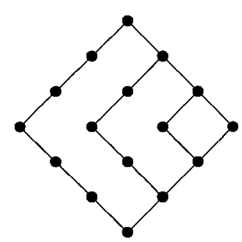
Owen, P. *Knots.* Philadelphia, PA: Courage, p. 50, 1993.

Square Matrix

A MATRIX for which horizontal and vertical dimensions are the same (i.e., an $n \times n$ MATRIX).

see also MATRIX

Square Number



A FIGURATE NUMBER of the form $m = n^2$, where n is an INTEGER. A square number is also called a PERFECT SQUARE. The first few square numbers are 1, 4, 9, 25, 36, 49, ... (Sloane's A000290). The GENERATING FUNCTION giving the square numbers is

$$\frac{x(x+1)}{(1-x)^3} = x + 4x^2 + 9x^3 + 16x^4 + \ldots. \tag{1}$$

The kth nonsquare number a_k is given by

$$a_n = n + \left\lfloor \tfrac{1}{2} + \sqrt{n} \right\rfloor, \tag{2}$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION, and the first few are 2, 3, 5, 6, 7, 8, 10, 11, ... (Sloane's A000037).

The only numbers which are simultaneously square and PYRAMIDAL (the CANNONBALL PROBLEM) are $P_1 = 1$ and $P_{24} = 4900$, corresponding to $S_1 = 1$ and $S_{70} = 4900$ (Dickson 1952, p. 25; Ball and Coxeter 1987, p. 59; Ogilvy 1988), as conjectured by Lucas (1875, 1876) and proved by Watson (1918). The CANNONBALL PROBLEM is equivalent to solving the DIOPHANTINE EQUATION

$$y^2 = \tfrac{1}{6}x(x+1)(2x+1) \tag{3}$$

(Guy 1994, p. 147).

The only numbers which are square and TETRAHEDRAL are $Te_1 = 1$, $Te_2 = 4$, and $Te_{48} = 19600$ (giving $S_1 = 1$, $S_2 = 4$, and $S_{140} = 19600$), as proved by Meyl (1878; cited in Dickson 1952, p. 25; Guy 1994, p. 147). In general, proving that only certain numbers are simultaneously figurate in two different ways is far from elementary.

To find the possible last digits for a square number, write $n = 10a+b$ for the number written in decimal NOTATION as ab_{10} ($a, b = 0, 1, \ldots, 9$). Then

$$n^2 = 100a^2 + 20ab + b^2, \tag{4}$$

so the last digit of n^2 is the same as the last digit of b^2. The following table gives the last digit of b^2 for $b = 0, 1, \ldots, 9$. As can be seen, the last digit can be only 0, 1, 4, 5, 6, or 9.

0	1	2	3	4	5	6	7	8	9
0	1	4	9	_6	_5	_6	_9	_4	_1

We can similarly examine the allowable last two digits by writing abc_{10} as

$$n = 100a + 10b + c, \tag{5}$$

so

$$\begin{aligned} n^2 &= (100a + 10b + c)^2 \\ &= 10^4a^2 + 2(1000ab + 100ac + 10bc) + 100b^2 + c^2 \\ &= (10^4a^2 + 2000ab + 100ac + 100b^2) + 20bc + c^2, \end{aligned} \tag{6}$$

so the last two digits are given by $20bc + c^2 = c(20b + c)$. But since the last digit must be 0, 1, 4, 5, 6, or 9, the following table exhausts all possible last two digits.

c	b									
	0	1	2	3	4	5	6	7	8	9
1	01	21	41	61	81	_01	_21	_41	_61	_81
4	16	96	_76	_56	_36	_16	_96	_76	_56	_36
5	25	_25	_25	_25	_25	_25	_25	_25	_25	_25
6	36	_56	_76	_96	_16	_36	_56	_76	_96	_16
9	81	_61	_41	_21	_01	_81	_61	_41	_21	_01

The only possibilities are 00, 01, 04, 09, 16, 21, 24, 25, 29, 36, 41, 44, 49, 56, 61, 64, 69, 76, 81, 84, 89, and 96, which can be summarized succinctly as 00, $e1$, $e4$, 25, $o6$, and $e9$, where e stands for an EVEN NUMBER and o for an ODD NUMBER. Additionally, unless the sum of the digits of a number is 1, 4, 7, or 9, it cannot be a square number.

The following table gives the possible residues mod n for square numbers for $n = 1$ to 20. The quantity $s(n)$ gives the number of distinct residues for a given n.

n	$s(n)$	x^2 (mod n)
2	2	0, 1
3	2	0, 1
4	2	0, 1
5	3	0, 1, 4
6	4	0, 1, 3, 4
7	4	0, 1, 2, 4
8	3	0, 1, 4
9	4	0, 1, 4, 7
10	6	0, 1, 4, 5, 6, 9
11	6	0, 1, 3, 4, 5, 9
12	4	0, 1, 4, 9
13	7	0, 1, 3, 4, 9, 10, 12
14	8	0, 1, 2, 4, 7, 8, 9, 11
15	6	0, 1, 4, 6, 9, 10
16	4	0, 1, 4, 9
17	9	0, 1, 2, 4, 8, 9, 13, 15, 16
18	8	0, 1, 4, 7, 9, 10, 13, 16
19	10	0, 1, 4, 5, 6, 7, 9, 11, 16, 17
20	6	0, 1, 4, 5, 9, 16

In general, the ODD squares are congruent to 1 (mod 8) (Conway and Guy 1996). Stangl (1996) gives an explicit formula by which the number of squares $s(n)$ in $\mathbb{Z}_n$ (i.e., mod n) can be calculated. Let p be an ODD PRIME. Then $s(n)$ is the MULTIPLICATIVE FUNCTION given by

$$s(2) = 2 \tag{7}$$

$$s(p) = \tfrac{1}{2}(p+1) \qquad (p \neq 2) \tag{8}$$

$$s(p^2) = \tfrac{1}{2}(p^2 - p + 2) \qquad (p \neq 2) \tag{9}$$

$$s(2^n) = \begin{cases} \frac{1}{3}(2^{n-1} + 4) & \text{for } n \text{ even} \\ \frac{1}{3}(2^{n-1} + 5) & \text{for } n \text{ odd} \end{cases} \tag{10}$$

$$s(p^n) = \begin{cases} \frac{p^{n+1}+p+2}{2(p+1)} & \text{for } n \geq 3 \text{ even} \\ \frac{p^{n+1}+2p+1}{2(p+1)} & \text{for } n \geq 3 \text{ odd.} \end{cases} \tag{11}$$

$s(n)$ is related to the number $q(n)$ of QUADRATIC RESIDUES in $\mathbb{Z}_n$ by

$$q(p^n) = s(p^n) - s(p^{n-2}) \tag{12}$$

for $n \geq 3$ (Stangl 1996).

For a perfect square n, $(n/p) = 0$ or 1 for all ODD PRIMES $p < n$ where (n/p) is the LEGENDRE SYMBOL. A number n which is not a perfect square but which satisfies this relationship is called a PSEUDOSQUARE.

The minimum number of squares needed to represent the numbers 1, 2, 3, ... are 1, 2, 3, 1, 2, 3, 4, 2, 1, 2, ... (Sloane's A002828), and the number of distinct ways to represent the numbers 1, 2, 3, ... in terms of squares are 1, 1, 1, 2, 2, 2, 2, 3, 4, 4, ... (Sloane's A001156). A brute-force algorithm for enumerating the square permutations of n is repeated application of the GREEDY ALGORITHM. However, this approach rapidly becomes impractical since the number of representations grows extremely rapidly with n, as shown in the following table.

n	Square Partitions
10	4
50	104
100	1116
150	6521
200	27482

Every POSITIVE integer is expressible as a SUM of (at most) $g(2) = 4$ square numbers (WARING'S PROBLEM). (Actually, the basis set is {0, 1, 4, 9, 16, 25, 36, 64, 81, 100, ...}, so 49 need never be used.) Furthermore, an infinite number of n require four squares to represent them, so the related quantity $G(2)$ (the least INTEGER n such that every POSITIVE INTEGER beyond a certain point requires $G(2)$ squares) is given by $G(2) = 4$.

Numbers expressible as the sum of two squares are those whose PRIME FACTORS are of the form $4k - 1$ taken to an EVEN POWER. Numbers expressible as the sum of three squares are those not of the form $4^k(8l + 7)$ for $k, l \geq 0$. The following table gives the first few numbers which require $N = 1$, 2, 3, and 4 squares to represent them as a sum.

N	Sloane	Numbers
1	000290	1, 4, 9, 16, 25, 36, 49, 64, 81, ...
2	000415	2, 5, 8, 10, 13, 17, 18, 20, 26, 29, ...
3	000419	3, 6, 11, 12, 14, 19, 21, 22, 24, 27, ...
4	004215	7, 15, 23, 28, 31, 39, 47, 55, 60, 63, ...

The FERMAT $4n + 1$ THEOREM guarantees that every PRIME of the form $4n+1$ is a sum of two SQUARE NUMBERS in only one way.

There are only 31 numbers which cannot be expressed as the sum of *distinct* squares: 2, 3, 6, 7, 8, 11, 12, 15, 18, 19, 22, 23, 24, 27, 28, 31, 32, 33, 43, 44, 47, 48, 60, 67, 72, 76, 92, 96, 108, 112, 128 (Sloane's A001422; Guy 1994). All numbers > 188 can be expressed as the sum of at most five distinct squares, and only

$$124 = 1 + 4 + 9 + 25 + 36 + 49 \tag{13}$$

and

$$188 = 1 + 4 + 9 + 25 + 49 + 100 \tag{14}$$

require six distinct squares (Bohman *et al.* 1979; Guy 1994, p. 136). In fact, 188 can also be represented using seven distinct squares:

$$188 = 1 + 4 + 9 + 25 + 36 + 49 + 64. \tag{15}$$

The following table gives the numbers which can be represented in W different ways as a sum of S squares. For example,

$$50 = 1^2 + 7^2 = 5^2 + 5^2$$

can be represented in two ways ($W = 2$) by two squares ($S = 2$).

S	W	Sloane	Numbers
1	1	000290	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
2	1	025284	2, 5, 8, 10, 13, 17, 18, 20, 25, 26, 29, ...
2	2	025285	50, 65, 85, 125, 130, 145, 170, 185, ...
3	1	025321	3, 6, 9, 11, 12, 14, 17, 18, 19, 21, 22, ...
3	2	025322	27, 33, 38, 41, 51, 57, 59, 62, 69, 74, ...
3	3	025323	54, 66, 81, 86, 89, 99, 101, 110, 114, ...
3	4	025324	129, 134, 146, 153, 161, 171, 189, ...
4	1	025357	4, 7, 10, 12, 13, 15, 16, 18, 19, 20, ...
4	2	025358	31, 34, 36, 37, 39, 43, 45, 47, 49, ...
4	3	025359	28, 42, 55, 60, 66, 67, 73, 75, 78, ...
4	4	025360	52, 58, 63, 70, 76, 84, 87, 91, 93, ...

The number of INTEGERS $< x$ which are squares or sums of two squares is

$$N(x) \sim kx(\ln x)^{-1/2}, \tag{16}$$

where

$$k = \sqrt{\frac{1}{2} \prod_{\substack{r=4n+3\\ r \text{ prime}}} (1 - r^{-2})^{-1}} \tag{17}$$

(Landau 1908; Le Lionnais 1983, p. 31). The product of four distinct NONZERO INTEGERS in ARITHMETIC PROGRESSION is square only for $(-3, -1, 1, 3)$, giving $(-3)(-1)(1)(3) = 9$ (Le Lionnais 1983, p. 53). It is possible to have three squares in ARITHMETIC PROGRESSION, but not four (Dickson 1952, pp. 435–440). If these numbers are r^2, s^2, and t^2, there are POSITIVE INTEGERS p and q such that

$$r = |p^2 - 2pq - q^2| \tag{18}$$

$$s = p^2 + q^2 \tag{19}$$

$$t = p^2 + 2pq - q^2, \tag{20}$$

where $(p, q) = 1$ and one of r, s, or t is EVEN (Dickson 1952, pp. 437–438). Every three-term progression of squares can be associated with a PYTHAGOREAN TRIPLE (X, Y, Z) by

$$X = \tfrac{1}{2}(r + t) \tag{21}$$

$$Y = \tfrac{1}{2}(t - r) \tag{22}$$

$$Z = s \tag{23}$$

(Robertson 1996).

CATALAN'S CONJECTURE states that 8 and 9 (2^3 and 3^2) are the only consecutive POWERS (excluding 0 and 1), i.e., the only solution to CATALAN'S DIOPHANTINE PROBLEM. This CONJECTURE has not yet been proved or refuted, although R. Tijdeman has proved that there can be only a finite number of exceptions should the CONJECTURE not hold. It is also known that 8 and 9 are the only consecutive CUBIC and square numbers (in either order).

A square number can be the concatenation of two squares, as in the case $16 = 4^2$ and $9 = 3^2$ giving $169 = 13^2$.

It is conjectured that, other than 10^{2n}, 4×10^{2n} and 9×10^{2n}, there are only a FINITE number of squares n^2 having exactly two distinct NONZERO DIGITS (Guy 1994, p. 262). The first few such n are 4, 5, 6, 7, 8, 9, 11, 12, 15, 21, ... (Sloane's A016070), corresponding to n^2 of 16, 25, 36, 49, 64, 81, 121, ... (Sloane's A016069).

The following table gives the first few numbers which, when squared, give numbers composed of only certain digits. The only known square number composed only of the digits 7, 8, and 9 is 9. Vardi (1991) considers numbers composed only of the square digits: 1, 4, and 9.

Digits	Sloane	n, n^2
1, 2, 3	030175	1, 11, 111, 36361, 363639, ...
	030174	1, 121, 12321, 1322122321, ...
1, 4, 6	027677	1, 2, 4, 8, 12, 31, 38, 108, ...
	027676	1, 4, 16, 64, 144, 441, 1444, ...
1, 4, 9	027675	1, 2, 3, 7, 12, 21, 38, 107, ...
	006716	1, 4, 9, 49, 144, 441, 1444, 11449, ...
2, 4, 8	027679	2, 22, 168, 478, 2878, 210912978, ...
	027678	4, 484, 28224, 228484, 8282884, ...
4, 5, 6	030177	2, 8, 216, 238, 258, 738, 6742, ...
	030176	4, 64, 46656, 56644, 66564, ...

BROWN NUMBERS are pairs (m, n) of INTEGERS satisfying the condition of BROCARD'S PROBLEM, i.e., such that

$$n! + 1 = m^2, \tag{24}$$

where $n!$ is a FACTORIAL. Only three such numbers are known: (5,4), (11,5), (71,7). Erdős conjectured that these are the only three such pairs.

Either $5x^2 + 4 = y^2$ or $5x^2 - 4 = y^2$ has a solution in POSITIVE INTEGERS IFF, for some n, $(x, y) = (F_n, L_n)$, where F_n is a FIBONACCI NUMBER and L_n is a LUCAS NUMBER (Honsberger 1985, pp. 114–118).

The smallest and largest square numbers containing the digits 1 to 9 are

$$11,826^2 = 139,854,276, \tag{25}$$

$$30,384^2 = 923,187,456. \tag{26}$$

The smallest and largest square numbers containing the digits 0 to 9 are

$$32,043^2 = 1,026,753,849, \tag{27}$$

$$99,066^2 = 9,814,072,356 \tag{28}$$

(Madachy 1979, p. 159). The smallest and largest square numbers containing the digits 1 to 9 twice each are

$$335,180,136^2 = 112,345,723,568,978,496 \tag{29}$$

$$999,390,432^2 = 998,781,235,573,146,624, \tag{30}$$

and the smallest and largest containing 1 to 9 three times are

$$10,546,200,195,312^2 = 111,222,338,559,598,866,946,777,344 \tag{31}$$

$$31,621,017,808,182^2 = 999,888,767,225,363,175,346,145,124 \tag{32}$$

(Madachy 1979, p. 159).

Madachy (1979, p. 165) also considers number which are equal to the sum of the squares of their two "halves" such as

$$1233 = 12^2 + 33^2 \tag{33}$$

$$8833 = 88^2 + 33^2 \tag{34}$$

$$10100 = 10^2 + 100^2 \tag{35}$$

$$5882353 = 588^2 + 2353^2, \tag{36}$$

in addition to a number of others.

see also ANTISQUARE NUMBER, BIQUADRATIC NUMBER, BROCARD'S PROBLEM, BROWN NUMBERS, CANNONBALL PROBLEM, CATALAN'S CONJECTURE, CENTERED SQUARE NUMBER, CLARK'S TRIANGLE, CUBIC NUMBER, DIOPHANTINE EQUATION, FERMAT $4n + 1$ THEOREM, GREEDY ALGORITHM, GROSS, LAGRANGE'S FOUR-SQUARE THEOREM, LANDAU-RAMANUJAN CONSTANT, PSEUDOSQUARE, PYRAMIDAL NUMBER, $r_k(n)$, SQUAREFREE, SQUARE TRIANGULAR NUMBER, WARING'S PROBLEM

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Square Orthobicupola

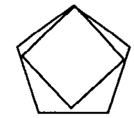
see JOHNSON SOLID

Square Packing

Find the minimum size SQUARE capable of bounding n equal SQUARES arranged in any configuration. The only packings which have been proven optimal are 2, 3, 5, and SQUARE NUMBERS (4, 9, ...). If $n = a^2 - a$ for some a, it is CONJECTURED that the size of the minimum bounding square is a for small n. The smallest n for which the CONJECTURE is known to be violated is 1560. The size is known to scale as k^b, where

$$\tfrac{1}{2}(3-\sqrt{3}) < b < \tfrac{1}{2}.$$

n	Exact	Decimal
1	1	1
2	2	2
3	2	2
4	2	2
5	$2+\frac{1}{2}\sqrt{2}$	2.707...
6	3	3
7	3	3
8	3	3
9	3	3
10	$3+\frac{1}{2}\sqrt{2}$	3.707...
11		3.877...
12	4	4
13	4	4
14	4	4
15	4	4
16	4	4
17	$4+\frac{1}{2}\sqrt{2}$	4.707...
18	$2(7+\sqrt{7})$	4.822...
19	$3+\frac{4}{3}\sqrt{2}$	4.885...
20	5	5
21	5	5
22	5	5
23	5	5
24	5	5
25	5	5
26		5.650...



The best packing of a SQUARE inside a PENTAGON, illustrated above, is 1.0673....

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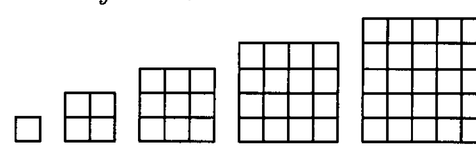
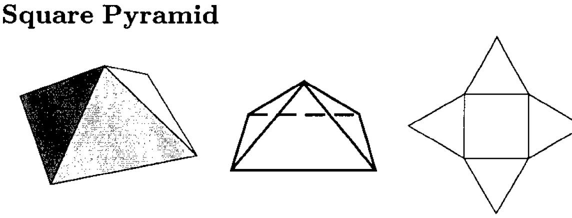
Square Polyomino



see also L-POLYOMINO, SKEW POLYOMINO, STRAIGHT POLYOMINO, T-POLYOMINO

Square Pyramid



A square pyramid is a PYRAMID with a SQUARE base. If the top of the pyramid is cut off by a PLANE, a square PYRAMIDAL FRUSTUM is obtained. If the four TRIANGLES of the square pyramid are EQUILATERAL, the square pyramid is the "regular" POLYHEDRON known as JOHNSON SOLID J_1 and, for side length a, has height

$$h = \tfrac{1}{2}\sqrt{2}\,a. \tag{1}$$

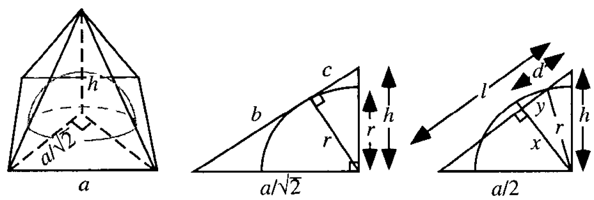
Using the equation for a general PYRAMID, the VOLUME of the "regular" is therefore

$$V = \tfrac{1}{3}hA_b = \tfrac{1}{6}\sqrt{2}\,a^3. \tag{2}$$

If the apex of the pyramid does not lie atop the center of the base, then the SLANT HEIGHT is given by

$$s = \sqrt{h^2 + \tfrac{1}{2}a^2}, \tag{3}$$

where h is the height and a is the length of a side of the base.



(a) (b) (c)

Consider a HEMISPHERE placed on the base of a square pyramid (having side lengths a and height h). Further, let the hemisphere be tangent to the four apex edges. Then what is the volume of the HEMISPHERE which is interior the pyramid (Cipra 1993)?

From Fig. (a), the CIRCUMRADIUS of the base is $a/\sqrt{2}$. Now find h in terms of r and a. Fig. (b) shows a CROSS-SECTION cut by the plane through the pyramid's apex, one of the base's vertices, and the base center. This figure gives

$$b = \sqrt{\tfrac{1}{2}a^2 - r^2} \tag{4}$$

$$c = \sqrt{h^2 - r^2}, \tag{5}$$

so the SLANT HEIGHT is

$$s = \sqrt{h^2 + \tfrac{1}{2}a^2} = b + c = \sqrt{\tfrac{1}{2}a^2 - r^2} + \sqrt{h^2 - r^2}. \tag{6}$$

Solving for h gives

$$h = \frac{ra}{\sqrt{a^2 - 2r^2}}. \tag{7}$$

We know, however, that the HEMISPHERE must be tangent to the sides, so $r = a/2$, and

$$h = \frac{\frac{1}{2}a}{\sqrt{a^2 - \frac{1}{2}a^2}}a = \frac{\frac{1}{2}}{\sqrt{\frac{1}{2}}}a = \tfrac{1}{2}\sqrt{2}\,a. \tag{8}$$

Fig. (c) shows a CROSS-SECTION through the center, apex, and midpoints of opposite sides. The PYTHAGOREAN THEOREM once again gives

$$l = \sqrt{\tfrac{1}{4}a^2 + h^2} = \sqrt{\tfrac{1}{4}a^2 + \tfrac{1}{2}a^2} = \tfrac{1}{2}\sqrt{3}\,a. \tag{9}$$

We now need to find x and y.

$$\sqrt{\tfrac{1}{4}a^2 - x^2} + d = l. \tag{10}$$

But we know l and h, and d is given by

$$d = \sqrt{h^2 - x^2}, \tag{11}$$

so

$$\sqrt{\tfrac{1}{4}a^2 - x^2} + \sqrt{\tfrac{1}{2}a^2 - x^2} = \tfrac{1}{2}\sqrt{3}\,a. \tag{12}$$

Solving gives

$$x = \tfrac{1}{6}\sqrt{6}\,a, \tag{13}$$

so

$$y = \sqrt{r^2 - x^2} = \sqrt{\tfrac{1}{4} - \tfrac{1}{6}}\,a = \sqrt{\frac{3-2}{12}}\,a = \frac{a}{2\sqrt{3}}. \tag{14}$$

We can now find the AREA of the SPHERICAL CAP as

$$V_{\text{cap}} = \tfrac{1}{6}\pi H(3A^2 + H^2), \tag{15}$$

where

$$A \equiv y = \frac{a}{2\sqrt{3}} \tag{16}$$

$$H \equiv r - x = \tfrac{1}{2}a - \frac{a}{\sqrt{6}} = a\left(\frac{1}{2} - \frac{1}{\sqrt{6}}\right), \tag{17}$$

so

$$\begin{aligned} V_{\text{cap}} &= \tfrac{1}{6}\pi a^3 \left[3\left(\frac{1}{12}\right) + \left(\frac{1}{2} - \frac{1}{\sqrt{6}}\right)^2\right]\left(\frac{1}{2} - \frac{1}{\sqrt{6}}\right) \\ &= \tfrac{1}{6}\pi a^3 \left[\frac{1}{4} + \left(\frac{1}{4} + \frac{1}{6} - \frac{1}{\sqrt{6}}\right)\right]\left(\frac{1}{2} - \frac{1}{\sqrt{6}}\right) \\ &= \tfrac{1}{6}\pi a^3 \left(\frac{2}{3} - \frac{1}{\sqrt{6}}\right)\left(\frac{1}{2} - \frac{1}{\sqrt{6}}\right) \\ &= \tfrac{1}{6}\pi a^3 \left(\frac{1}{3} - \frac{1}{2\sqrt{6}} - \frac{2}{3\sqrt{6}} + \frac{1}{6}\right) \\ &= \tfrac{1}{6}\pi a^3 \left(\frac{1}{2} - \frac{7}{6\sqrt{6}}\right). \end{aligned} \tag{18}$$

Therefore, the volume within the pyramid is

$$\begin{aligned} V_{\text{inside}} &= \tfrac{2}{3}\pi r^3 - 4V_{\text{cap}} = \tfrac{2}{3}\pi\tfrac{1}{8}a^3 - \tfrac{2}{3}\pi a^3\left(\frac{1}{2} - \frac{7}{6\sqrt{6}}\right) \\ &= \tfrac{2}{3}\pi a^3\left(\frac{1}{8} - \frac{1}{2} + \frac{7}{6\sqrt{6}}\right) = \tfrac{2}{3}\pi a^3\left(\frac{7}{6\sqrt{6}} - \frac{3}{8}\right) \\ &= \pi a^3\left(\frac{7}{9\sqrt{6}} - \frac{1}{4}\right). \end{aligned} \tag{19}$$

This problem appeared in the Japanese scholastic aptitude test (Cipra 1993).

see also SQUARE PYRAMIDAL NUMBER

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Square Pyramidal Number

A FIGURATE NUMBER of the form

$$P_n = \tfrac{1}{6}n(n+1)(2n+1), \qquad (1)$$

corresponding to a configuration of points which form a SQUARE PYRAMID, is called a square pyramidal number (or sometimes, simply a PYRAMIDAL NUMBER). The first few are 1, 5, 14, 30, 55, 91, 140, 204, ... (Sloane's A000330). They are sums of consecutive pairs of TETRAHEDRAL NUMBERS and satisfy

$$P_n = \tfrac{1}{3}(2n+1)T_n, \qquad (2)$$

where T_n is the nth TRIANGULAR NUMBER.

The only numbers which are simultaneously SQUARE and pyramidal (the CANNONBALL PROBLEM) are $P_1 = 1$ and $P_{24} = 4900$, corresponding to $S_1 = 1$ and $S_{70} = 4900$ (Dickson 1952, p. 25; Ball and Coxeter 1987, p. 59; Ogilvy 1988), as conjectured by Lucas (1875, 1876) and proved by Watson (1918). The proof is far from elementary, and is equivalent to solving the DIOPHANTINE EQUATION

$$y^2 = \tfrac{1}{6}x(x+1)(2x+1) \qquad (3)$$

(Guy 1994, p. 147). However, an elementary proof has also been given by a number of authors.

Numbers which are simultaneously TRIANGULAR and square pyramidal satisfy the DIOPHANTINE EQUATION

$$3(2y+1)^2 = 8x^3 + 12x^2 + 4x + 3. \qquad (4)$$

The only solutions are $x = -1$, 0, 1, 5, 6, and 85 (Guy 1994, p. 147). Beukers (1988) has studied the problem of finding numbers which are simultaneously TETRAHEDRAL and square pyramidal via INTEGER points on an ELLIPTIC CURVE. He finds that the only solution is the trivial $Te_1 = P_1 = 1$.

see also TETRAHEDRAL NUMBER

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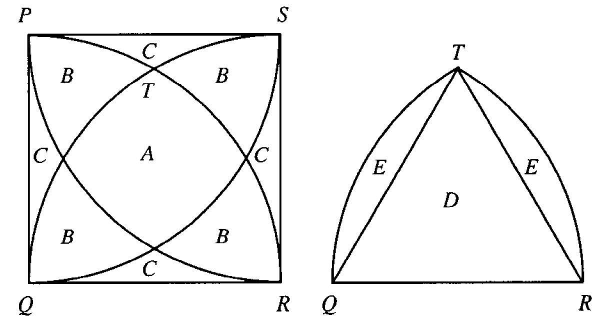
Square Quadrants



The areas of the regions illustrated above can be found from the equations

$$A + 4B + 4C = 1 \qquad (1)$$

$$A + 3B + 2C = \tfrac{1}{4}\pi. \qquad (2)$$

Since we want to solve for three variables, we need a third equation. This can be taken as

$$A + 2B + C = 2E + D, \qquad (3)$$

where

$$D = \tfrac{1}{4}\sqrt{3} \qquad (4)$$

$$D + E = \tfrac{1}{6}\pi, \qquad (5)$$

leading to

$$A+2B+C = D+2E = 2(D+E)-D = \tfrac{1}{3}\pi - \tfrac{1}{4}\sqrt{3}. \qquad (6)$$

Combining the equations (1), (2), and (6) gives the matrix equation

$$\begin{bmatrix} 1 & 4 & 4 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ \tfrac{1}{4}\pi \\ \tfrac{1}{3}\pi - \tfrac{1}{4}\sqrt{3} \end{bmatrix}, \qquad (7)$$

which can be inverted to yield

$$A = 1 - \sqrt{3} - \tfrac{1}{3}\pi \qquad (8)$$

$$B = -1 + \tfrac{1}{2}\sqrt{3} + \tfrac{1}{12}\pi \qquad (9)$$

$$C = 1 - \tfrac{1}{4}\sqrt{3} + \tfrac{1}{6}\pi. \qquad (10)$$

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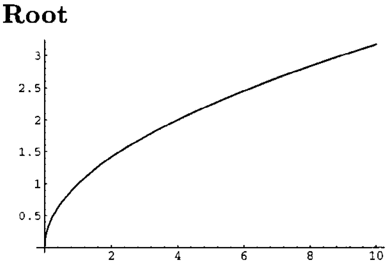
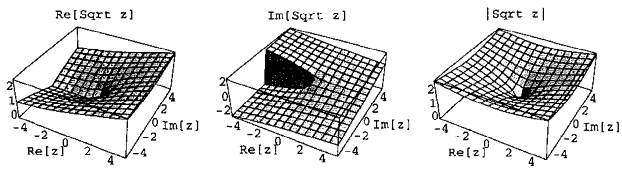
Square Root





A square root of x is a number r such that $r^2 = x$. This is written $r = x^{1/2}$ (x to the 1/2 POWER) or $r = \sqrt{x}$. The square root function $f(x) = \sqrt{x}$ is the INVERSE FUNCTION of $f(x) = x^2$. Square roots are also called RADICALS or SURDS. A general COMPLEX NUMBER z has *two* square roots. For example, for the real POSITIVE number $x = 9$, the two square roots are $\sqrt{9} = \pm 3$, since $3^2 = (-3)^2 = 9$. Similarly, for the real NEGATIVE number $x = -9$, the two square roots are $\sqrt{-9} = \pm 3i$, where i is the IMAGINARY NUMBER defined by $i^2 = -1$. In common usage, unless otherwise specified, "the" square root is generally taken to mean the POSITIVE square root.

The square root of 2 is the IRRATIONAL NUMBER $\sqrt{2} \approx 1.41421356$ (Sloane's A002193), which has the simple periodic CONTINUED FRACTION 1, 2, 2, 2, 2, 2, ... The square root of 3 is the IRRATIONAL NUMBER $\sqrt{3} \approx 1.73205081$ (Sloane's A002194), which has the simple periodic CONTINUED FRACTION 1, 1, 2, 1, 2, 1, 2, ... In general, the CONTINUED FRACTIONS of the square roots of all POSITIVE integers are periodic.

The square roots of a COMPLEX NUMBER are given by

$$\sqrt{x+iy} = \pm\sqrt{x^2+y^2}\left\{\cos\left[\frac{1}{2}\tan^{-1}\left(\frac{y}{x}\right)\right] + i\sin\left[\frac{1}{2}\tan^{-1}\left(\frac{y}{x}\right)\right]\right\}. \quad (1)$$

As can be seen in the above figure, the IMAGINARY PART of the complex square root function has a BRANCH CUT along the NEGATIVE real axis.

A NESTED RADICAL of the form $\sqrt{a \pm b\sqrt{c}}$ can sometimes be simplified into a simple square root by equating

$$\sqrt{a \pm b\sqrt{c}} = \sqrt{d} \pm \sqrt{e}. \quad (2)$$

Squaring gives

$$a \pm b\sqrt{c} = d + e \pm 2\sqrt{de}, \quad (3)$$

so

$$a = d + e \quad (4)$$
$$b^2 c = 4de. \quad (5)$$

Solving for d and e gives

$$d, e = \frac{a \pm \sqrt{a^2 - b^2 c}}{2}. \quad (6)$$

A sequence of approximations a/b to $\sqrt{n}$ can be derived by factoring

$$a^2 - nb^2 = \pm 1 \quad (7)$$

(where -1 is possible only if -1 is a QUADRATIC RESIDUE of n). Then

$$(a + b\sqrt{n})(a - b\sqrt{n}) = \pm 1 \quad (8)$$

$$(a + b\sqrt{n})^k (a - b\sqrt{n})^k = (\pm 1)^k = \pm 1, \quad (9)$$

and

$$(1 + \sqrt{n})^1 = 1 + \sqrt{n} \quad (10)$$
$$(1 + \sqrt{n})^2 = (1 + n) + 2\sqrt{n} \quad (11)$$
$$(1 + \sqrt{n})(a + b\sqrt{n}) = (a + bn) + \sqrt{n}(a + b). \quad (12)$$

Therefore, a and b are given by the RECURRENCE RELATIONS

$$a_i = a_{i-1} + b_{i-1} n \quad (13)$$
$$b_i = a_{i-1} + b_{i-1} \quad (14)$$

with $a_1 = b_1 = 1$. The error obtained using this method is

$$\left|\frac{a}{b} - \sqrt{n}\right| = \frac{1}{b(a + b\sqrt{n})} < \frac{1}{2b^2}. \quad (15)$$

The first few approximants to $\sqrt{n}$ are therefore given by

$$1, \tfrac{1}{2}(1+n), \frac{1+3n}{3+n}, \frac{1+6n+n^2}{4(n+1)}, \frac{1+10n+5n^2}{5+10n+n^2}, \ldots. \quad (16)$$

This ALGORITHM is sometimes known as the BHASKARA-BROUCKNER ALGORITHM. For the case $n = 2$, this gives the convergents to $\sqrt{2}$ as 1, 3/2, 7/5, 17/12, 41/29, 99/70, ...

Another general technique for deriving this sequence, known as NEWTON'S ITERATION, is obtained by letting $x = \sqrt{n}$. Then $x = n/x$, so the SEQUENCE

$$x_k = \frac{1}{2}\left(x_{k-1} + \frac{n}{x_{k-1}}\right) \quad (17)$$

converges quadratically to the root. The first few approximants to $\sqrt{n}$ are therefore given by

$$1, \tfrac{1}{2}(1+n), \frac{1+6n+n^2}{4(n+1)}, \frac{1+26n+70n^2+28n^3+n^4}{8(1+n)(1+6n+n^2)}, \ldots. \quad (18)$$

For $\sqrt{2}$, this gives the convergents 1, 3/2, 17/12, 577/408, 665857/470832, ...

see also CONTINUED SQUARE ROOT, CUBE ROOT, NESTED RADICAL, NEWTON'S ITERATION, QUADRATIC SURD, ROOT OF UNITY, SQUARE NUMBER, SQUARE TRIANGULAR NUMBER, SURD

References
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Square Root Inequality

$$2\sqrt{n+1}-2\sqrt{n}<\frac{1}{\sqrt{n}}<2\sqrt{n}-2\sqrt{n-1}\,.$$

Square Root Method

The square root method is an algorithm which solves the MATRIX EQUATION

$$\mathsf{A}\mathbf{u}=\mathbf{g} \tag{1}$$

for $\mathbf{u}$, with A a $p\times p$ SYMMETRIC MATRIX and $\mathbf{g}$ a given VECTOR. Convert A to a TRIANGULAR MATRIX such that

$$\mathsf{T}^{\mathrm{T}}\mathsf{T}=\mathsf{A}, \tag{2}$$

where T^{T} is the MATRIX TRANSPOSE. Then

$$\mathsf{T}^{\mathrm{T}}\mathbf{k}=\mathbf{g} \tag{3}$$

$$\mathsf{T}\mathbf{u}=\mathbf{k}, \tag{4}$$

so

$$\mathsf{T}=\begin{bmatrix} s_{11} & s_{12} & \cdots & \cdots \\ 0 & s_{22} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{pp}\end{bmatrix}, \tag{5}$$

giving the equations

$$\begin{aligned} {s_{11}}^2&=a_{11}\\ s_{11}s_{12}&=a_{12}\\ {s_{12}}^2+{s_{22}}^2&=a_{22}\\ {s_{1j}}^2+{s_{2j}}^2+\ldots+{s_{jj}}^2&=a_{jj}\\ s_{1j}+s_{2j}s_{2k}+\ldots+s_{jj}s_{jk}&=a_{jk}. \end{aligned} \tag{6}$$

These give

$$\begin{aligned} s_{11}&=\sqrt{a_{11}}\\ s_{12}&=\frac{a_{12}}{s_{11}}\\ s_{22}&=\sqrt{a_{22}-{s_{12}}^2}\\ s_{jj}&=\sqrt{a_{jj}-{s_{ij}}^2-{s_{2j}}^2-\ldots-{s_{j-1,j}}^2}\\ s_{jk}&=\frac{a_{jk}-s_{1j}s_{1k}-s_{2j}s_{2k}-\ldots-s_{j-1,j}s_{j-1,k}}{s_{jj}}, \end{aligned} \tag{7}$$

giving T from A. Now solve for $\mathbf{k}$ in terms of the s_{ij}s and $\mathbf{g}$,

$$\begin{aligned} s_{11}k_1&=g_1\\ s_{12}k_1+s_{22}k_2&=g_2\\ s_{1j}k_1+s_{2j}k_2+\ldots+s_{jj}k_j&=g_j, \end{aligned} \tag{8}$$

which gives

$$\begin{aligned} k_1&=\frac{g_1}{s_{11}}\\ k_2&=\frac{g_2-s_{12}k_1}{s_{22}}\\ k_j&=\frac{g_j-s_{1j}k_1-s_{2j}k_2-\ldots-s_{j-1,j}k_{j-1}}{s_{jj}}. \end{aligned} \tag{9}$$

Finally, find $\mathbf{u}$ from the s_{ij}s and $\mathbf{k}$,

$$\begin{aligned} s_{11}u_1+s_{12}u_2\ldots+s_{1p}u_p&=k_1\\ s_{22}u_2+\ldots+s_{2p}u_p&=k_2\\ s_{pp}u_p&=k_p, \end{aligned} \tag{10}$$

giving the desired solution,

$$\begin{aligned} u_p&=\frac{k_p}{s_{pp}}\\ u_{p-1}&=\frac{k_{p-1}-s_{p-1,p}u_p}{s_{p-1,p-1}}\\ u_j&=\frac{k_j-s_{j,j+1}u_{j+1}-s_{j,j+2}u_{j+2}-\ldots-s_{jp}u_p}{s_{jj}}. \end{aligned} \tag{11}$$

see also LU DECOMPOSITION

References
Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics, Pt. 2, 2nd ed.* Princeton, NJ: Van Nostrand, pp. 298–300, 1951.

Square Triangular Number

A number which is simultaneously SQUARE and TRIANGULAR. The first few are 1, 36, 1225, 41616, 1413721, 48024900, ... (Sloane's A001110), corresponding to $T_1=S_1$, $T_8=S_6$, $T_{49}=S_{35}$, $T_{288}=S_{204}$, $T_{1681}=S_{1189}$, ... (Pietenpol 1962), but there are an infinite number, as first shown by Euler in 1730 (Dickson 1952).

The general FORMULA for a square triangular number ST_n is b^2c^2, where b/c is the nth convergent to the CONTINUED FRACTION of $\sqrt{2}$ (Ball and Coxeter 1987, p. 59; Conway and Guy 1996). The first few are

$$\frac{1}{1},\frac{3}{2},\frac{7}{5},\frac{17}{12},\frac{41}{29},\frac{99}{70},\frac{239}{169},\ldots. \tag{1}$$

The NUMERATORS and DENOMINATORS give solutions to the PELL EQUATION

$$x^2-2y^2=\pm1, \tag{2}$$

but can also be obtained by doubling the previous FRACTION and adding to the FRACTION before that. The connection with the PELL EQUATION can be seen by letting N denote the Nth TRIANGULAR NUMBER and M the Mth SQUARE NUMBER, then

$$\tfrac{1}{2}N(N+1) = M^2. \tag{3}$$

Defining

$$x \equiv 2N+1 \tag{4}$$
$$y \equiv 2M \tag{5}$$

then gives the equation

$$x^2 - 2y^2 = 1 \tag{6}$$

(Conway and Guy 1996). Numbers which are simultaneously TRIANGULAR and SQUARE PYRAMIDAL also satisfy the DIOPHANTINE EQUATION

$$3(2y+1)^2 = 8x^3 + 12x^2 + 4x + 3. \tag{7}$$

The only solutions are $x = -1$, 0, 1, 5, 6, and 85 (Guy 1994, p. 147).

A general FORMULA for square triangular numbers is

$$ST_n = \left[\frac{(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}}{4\sqrt{2}}\right]^2 \tag{8}$$
$$= \tfrac{1}{32}[(17+12\sqrt{2})^n + (17-12\sqrt{2})^n - 2]. \tag{9}$$

The square triangular numbers also satisfy the RECURRENCE RELATION

$$ST_n = 34ST_{n-1} - ST_{n-2} + 2 \tag{10}$$
$$u_{n+2} = 6u_{n+1} - u_n, \tag{11}$$

with $u_0 = 0$, $u_1 = 1$, where $ST_n \equiv {u_n}^2$. A curious product formula for ST_n is given by

$$ST_n = 2^{2n-5}\prod_{k=1}^{2n}\left[3+\cos\left(\frac{k\pi}{n}\right)\right]. \tag{12}$$

An amazing GENERATING FUNCTION is

$$f(x) = \frac{1+x}{(1-x)(1-34x+x^2)} = 1 + 36x + 1225x^2 + \ldots \tag{13}$$

(Sloane and Plouffe 1995).

see also SQUARE NUMBER, SQUARE ROOT, TRIANGULAR NUMBER

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Guy, R. K. "Sums of Squares" and "Figurate Numbers." §C20 and §D3 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 136–138 and 147–150, 1994.

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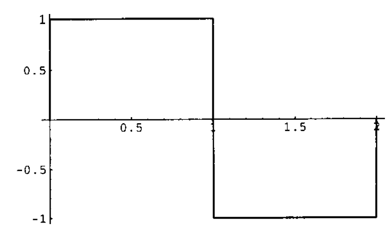
Square Wave



The square wave is a periodic waveform consisting of instantaneous transitions between two levels which can be denoted ± 1. The square wave is sometimes also called the RADEMACHER FUNCTION. Let the square wave have period $2L$. The square wave function is ODD, so the FOURIER SERIES has $a_0 = a_n = 0$ and

$$b_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx$$
$$= \frac{4}{n\pi}\sin^2(\tfrac{1}{2}n\pi) = \frac{4}{n\pi}\begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd.}\end{cases}$$

The FOURIER SERIES for the square wave is therefore

$$f(x) = \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right).$$

see also HADAMARD MATRIX, WALSH FUNCTION

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Squared

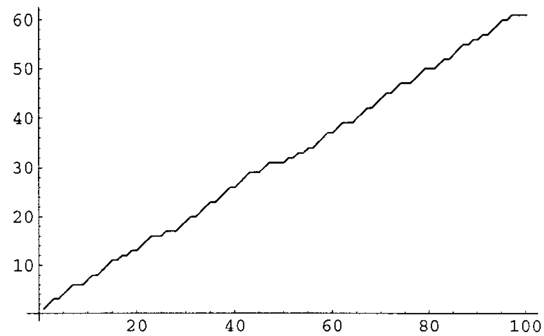
A number to the POWER 2 is said to be squared, so that x^2 is called "x squared."

see also CUBED, SQUARE ROOT

Squared Square

see PERFECT SQUARE DISSECTION

Squarefree



A number is said to be squarefree (or sometimes QUADRATFREI; Shanks 1993) if its PRIME decomposition contains no repeated factors. All PRIMES are therefore trivially squarefree. The squarefree numbers are 1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, ... (Sloane's A005117). The SQUAREFUL numbers (i.e., those that contain at least one square) are 4, 8, 9, 12, 16, 18, 20, 24, 25, ... (Sloane's A013929).

The asymptotic number $Q(n)$ of squarefree numbers $\leq n$ is given by

$$Q(n) = \frac{6n}{\pi^2} + \mathcal{O}(\sqrt{n}) \tag{1}$$

(Hardy and Wright 1979, pp. 269–270). $Q(n)$ for $n = 10, 100, 1000, \ldots$ are 7, 61, 608, 6083, 60794, 607926, ..., while the asymptotic density is $1/\zeta(2) = 6/\pi^2 \approx 0.607927$, where $\zeta(n)$ is the RIEMANN ZETA FUNCTION.

The MÖBIUS FUNCTION is given by

$$\mu(n) \equiv \begin{cases} 0 & \text{if } n \text{ has one or more repeated prime factors} \\ 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is product of } k \text{ distinct primes,} \end{cases} \tag{2}$$

so $\mu(n) \neq 0$ indicates that n is squarefree. The asymptotic formula for $Q(x)$ is equivalent to the formula

$$\sum_{n=1}^{x} |\mu(n)| = \frac{6x}{\pi^2} + \mathcal{O}(\sqrt{x}) \tag{3}$$

(Hardy and Wright 1979, p. 270)

There is no known polynomial-time algorithm for recognizing squarefree INTEGERS or for computing the squarefree part of an INTEGER. In fact, this problem may be no easier than the general problem of integer factorization (obviously, if an integer n can be factored completely, n is squarefree IFF it contains no duplicated factors). This problem is an important unsolved problem in NUMBER THEORY because computing the RING of integers of an algebraic number field is reducible to computing the squarefree part of an INTEGER (Lenstra 1992, Pohst and Zassenhaus 1997). The *Mathematica*® (Wolfram Research, Champaign, IL) function `NumberTheory`NumberTheoryFunctions`` `SquareFreeQ[n]` determines whether a number is squarefree.

The largest known SQUAREFUL FIBONACCI NUMBER is F_{336}, and no SQUAREFUL FIBONACCI NUMBERS F_p are known with p PRIME. All numbers less than 2.5×10^{15} in SYLVESTER'S SEQUENCE are squarefree, and no SQUAREFUL numbers in this sequence are known (Vardi 1991). Every CARMICHAEL NUMBER is squarefree. The BINOMIAL COEFFICIENTS $\binom{2n-1}{n}$ are squarefree only for $n = 2, 3, 4, 6, 9, 10, 12, 36, \ldots$, with no others less than $n = 1500$. The CENTRAL BINOMIAL COEFFICIENTS are SQUAREFREE only for $n = 1, 2, 3, 4, 5, 7, 8, 11, 17, 19, 23, 71, \ldots$ (Sloane's A046098), with no others less than 1500.

see also BINOMIAL COEFFICIENT, BIQUADRATEFREE, COMPOSITE NUMBER, CUBEFREE, ERDŐS SQUAREFREE CONJECTURE, FIBONACCI NUMBER, KORSELT'S CRITERION, MÖBIUS FUNCTION, PRIME NUMBER, RIEMANN ZETA FUNCTION, SÁRKÖZY'S THEOREM, SQUARE NUMBER, SQUAREFUL, SYLVESTER'S SEQUENCE

References

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Squareful

A number is squareful, also called NONSQUAREFREE, if it contains at least one SQUARE in its prime factorization. Such a number is also called SQUAREFUL. The first few are 4, 8, 9, 12, 16, 18, 20, 24, 25, ... (Sloane's A013929). The greatest multiple prime factors for the squareful integers are 2, 2, 3, 2, 2, 3, 2, 2, 5, 3, 2, 2, 3, ... (Sloane's A046028). The least multiple prime factors for squareful integers are 2, 2, 3, 2, 2, 3, 2, 2, 5, 3, 2, 2, 2, ... (Sloane's A046027).

see also GREATEST PRIME FACTOR, LEAST PRIME FACTOR, SMARANDACHE NEAR-TO-PRIMORIAL FUNCTION, SQUAREFREE

References

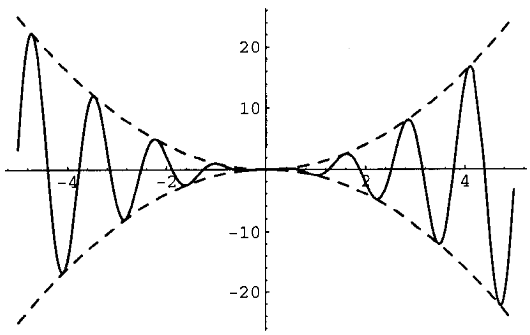
Sloane, N. J. A. Sequences A013929, A046027, and A046028 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Squaring

Squaring is the GEOMETRIC CONSTRUCTION, using only COMPASS and STRAIGHTEDGE, of a SQUARE which has the same area as a given geometric figure. Squaring is also called QUADRATURE. An object which can be constructed by squaring is called SQUARABLE.

see also CIRCLE SQUARING, COMPASS, CONSTRUCTIBLE NUMBER, GEOMETRIC CONSTRUCTION, RECTANGLE SQUARING, STRAIGHTEDGE, TRIANGLE SQUARING

Squeezing Theorem



Let there be two functions $f_-(x)$ and $f_+(x)$ such that $f(x)$ is "squeezed" between the two,

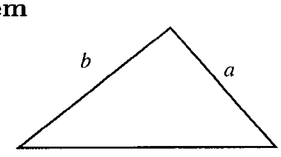
$$f_-(x) \le f(x) \le f_+(x).$$

If

$$r = \lim_{x \to a} f_-(x) = \lim_{x \to a} f_+(x),$$

then $\lim_{x \to a} f(x) = r$. In the above diagram the functions $f_-(x) = -x^2$ and $f_+(x) = x^2$ "squeeze" $x^2 \sin(cx)$ at 0, so $\lim_{x \to 0} x^2 \sin(cx) = 0$. The squeezing theorem is also called the SANDWICH THEOREM.

SSS Theorem



Specifying three sides uniquely determines a TRIANGLE whose AREA is given by HERON'S FORMULA,

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \tag{1}$$

where

$$s \equiv \tfrac{1}{2}(a+b+c) \tag{2}$$

is the SEMIPERIMETER of the TRIANGLE. Let R be the CIRCUMRADIUS, then

$$A = \frac{abc}{4R}. \tag{3}$$

Using the LAW OF COSINES

$$a^2 = b^2 + c^2 - 2bc\cos A \tag{4}$$

$$b^2 = a^2 + c^2 - 2ac\cos B \tag{5}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \tag{6}$$

gives the three ANGLES as

$$A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) \tag{7}$$

$$B = \cos^{-1}\left(\frac{b^2 + c^2 - b^2}{2ac}\right) \tag{8}$$

$$C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right). \tag{9}$$

see also AAA THEOREM, AAS THEOREM, ASA THEOREM, ASS THEOREM, HERON'S FORMULA, SAS THEOREM, SEMIPERIMETER, TRIANGLE

Stability

The robustness of a given outcome to small changes in initial conditions or small random fluctuations. CHAOS is an example of a process which is not stable.

see also STABILITY MATRIX

Stability Matrix

Given a system of two ordinary differential equations

$$\dot{x} = f(x, y) \tag{1}$$

$$\dot{y} = g(x, y), \tag{2}$$

let x_0 and y_0 denote FIXED POINTS with $\dot{x} = \dot{y} = 0$, so

$$f(x_0, y_0) = 0 \tag{3}$$

$$g(x_0, y_0) = 0. \tag{4}$$

Then expand about (x_0, y_0) so

$$\delta\dot{x} = f_x(x_0, y_0)\delta x + f_y(x_0, y_0)\delta y + f_{xy}(x_0, y_0)\delta x\delta y + \ldots \tag{5}$$

$$\delta\dot{y} = g_x(x_0, y_0)\delta x + g_y(x_0, y_0)\delta y + g_{xy}(x_0, y_0)\delta x\delta y + \ldots. \tag{6}$$

To first-order, this gives

$$\frac{d}{dt}\begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}, \tag{7}$$

where the 2×2 MATRIX, or its generalization to higher dimension, is called the stability matrix. Analysis of the EIGENVALUES (and EIGENVECTORS) of the stability matrix characterizes the type of FIXED POINT.

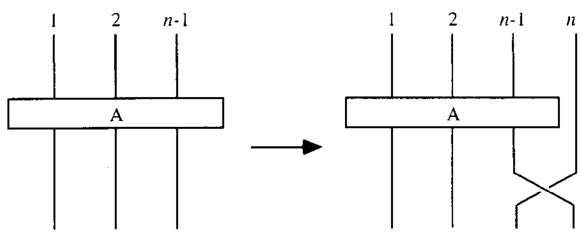
see also ELLIPTIC FIXED POINT (DIFFERENTIAL EQUATIONS), FIXED POINT, HYPERBOLIC FIXED POINT

(DIFFERENTIAL EQUATIONS), LINEAR STABILITY, STABLE IMPROPER NODE, STABLE NODE, STABLE SPIRAL POINT, STABLE STAR, UNSTABLE IMPROPER NODE, UNSTABLE NODE, UNSTABLE SPIRAL POINT, UNSTABLE STAR

References

Tabor, M. "Linear Stability Analysis." §1.4 in *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, pp. 20–31, 1989.

Stabilization



A type II MARKOV MOVE.

see also MARKOV MOVES

Stable Equivalence

Two VECTOR BUNDLES are stably equivalent IFF ISOMORPHIC VECTOR BUNDLES are obtained upon WHITNEY SUMMING each VECTOR BUNDLE with a trivial VECTOR BUNDLE.

see also VECTOR BUNDLE, WHITNEY SUM

Stable Improper Node

A FIXED POINT for which the STABILITY MATRIX has equal NEGATIVE EIGENVALUES.

see also ELLIPTIC FIXED POINT (DIFFERENTIAL EQUATIONS), FIXED POINT, HYPERBOLIC FIXED POINT (DIFFERENTIAL EQUATIONS), STABLE NODE, STABLE SPIRAL POINT, UNSTABLE IMPROPER NODE, UNSTABLE NODE, UNSTABLE SPIRAL POINT, UNSTABLE STAR

References

Tabor, M. "Classification of Fixed Points." §1.4.b in *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, pp. 22–25, 1989.

Stable Node

A FIXED POINT for which the STABILITY MATRIX has both EIGENVALUES NEGATIVE, so $\lambda_1 < \lambda_2 < 0$.

see also ELLIPTIC FIXED POINT (DIFFERENTIAL EQUATIONS), FIXED POINT, HYPERBOLIC FIXED POINT (DIFFERENTIAL EQUATIONS), STABLE IMPROPER NODE, STABLE SPIRAL POINT, STABLE STAR, UNSTABLE IMPROPER NODE, UNSTABLE NODE, UNSTABLE SPIRAL POINT, UNSTABLE STAR

References

Tabor, M. "Classification of Fixed Points." §1.4.b in *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, pp. 22–25, 1989.

Stable Spiral Point

A FIXED POINT for which the STABILITY MATRIX has EIGENVALUES of the form $\lambda_\pm = -\alpha \pm i\beta$ (with $\alpha, \beta > 0$).

see also ELLIPTIC FIXED POINT (DIFFERENTIAL EQUATIONS), FIXED POINT, HYPERBOLIC FIXED POINT (DIFFERENTIAL EQUATIONS), STABLE IMPROPER NODE, STABLE NODE, STABLE STAR, UNSTABLE IMPROPER NODE, UNSTABLE NODE, UNSTABLE SPIRAL POINT, UNSTABLE STAR

References

Tabor, M. "Classification of Fixed Points." §1.4.b in *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, pp. 22–25, 1989.

Stable Star

A FIXED POINT for which the STABILITY MATRIX has one zero EIGENVECTOR with NEGATIVE EIGENVALUE $\lambda < 0$.

see also ELLIPTIC FIXED POINT (DIFFERENTIAL EQUATIONS), FIXED POINT, HYPERBOLIC FIXED POINT (DIFFERENTIAL EQUATIONS), STABLE IMPROPER NODE, STABLE NODE, STABLE SPIRAL POINT, UNSTABLE IMPROPER NODE, UNSTABLE NODE, UNSTABLE SPIRAL POINT, UNSTABLE STAR

References

Tabor, M. "Classification of Fixed Points." §1.4.b in *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, pp. 22–25, 1989.

Stable Type

A POLYNOMIAL equation whose ROOTS all have NEGATIVE REAL PARTS. For a REAL QUADRATIC EQUATION

$$z^2 + Bz + C = 0,$$

the stability conditions are $B, C > 0$. For a REAL CUBIC EQUATION

$$z^3 + Az^2 + Bz + C = 0,$$

the stability conditions are $A, B, C > 0$ and $AB > C$.

References

Birkhoff, G. and Mac Lane, S. *A Survey of Modern Algebra, 3rd ed.* New York: Macmillan, pp. 108–109, 1965.

Stack

A DATA STRUCTURE which is a special kind of LIST in which elements may be added to or removed from the top only. These actions are called a PUSH or a POP, respectively. Actions may be taken by popping one or more values, operating on them, and then pushing the result back onto the stack.

Stacks are used as the basis for computer languages such as FORTH, PostScript® (Adobe Systems), and the RPN language used in Hewlett-Packard® programmable calculators.

see also LIST, POP, PUSH, QUEUE

Stäckel Determinant

A DETERMINANT used to determine in which coordinate systems the HELMHOLTZ DIFFERENTIAL EQUATION is separable (Morse and Feshbach 1953). A determinant

$$S = |\Phi_{mn}| = \begin{vmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{vmatrix} \tag{1}$$

in which Φ_{ni} are functions of u_i alone is called a Stäckel determinant. A coordinate system is separable if it obeys the ROBERTSON CONDITION, namely that the SCALE FACTORS h_i in the LAPLACIAN

$$\nabla^2 = \sum_{i=1}^{3} \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_i} \left(\frac{h_1 h_2 h_3}{{h_i}^2} \frac{\partial}{\partial u_i} \right) \tag{2}$$

can be rewritten in terms of functions $f_i(u_i)$ defined by

$$\begin{aligned} &\frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_i} \left(\frac{h_1 h_2 h_3}{{h_i}^2} \frac{\partial}{\partial u_i} \right) \\ &\quad = \frac{g(u_{i+1}, u_{i+2})}{h_1 h_2 h_3} \frac{\partial}{\partial u_i} \left[f_i(u_i) \frac{\partial}{\partial u_i} \right] \\ &\quad = \frac{1}{{h_i}^2 f_i} \frac{\partial}{\partial u_i} \left(f_i \frac{\partial}{\partial u_i} \right) \end{aligned} \tag{3}$$

such that S can be written

$$S = \frac{h_1 h_2 h_3}{f_1(u_1) f_2(u_2) f_3(u_3)}. \tag{4}$$

When this is true, the separated equations are of the form

$$\frac{1}{f_n} \frac{\partial}{\partial u_n} \left(f_n \frac{\partial X_n}{\partial u_n} \right) + ({k_1}^2 \Phi_{n1} + {k_2}^2 \Phi_{n2} + {k_3}^2 \Phi_{n3}) X_n = 0 \tag{5}$$

The Φ_{ij}s obey the minor equations

$$M_1 = \Phi_{22}\Phi_{33} - \Phi_{23}\Phi_{32} = \frac{S}{h_1^2} \tag{6}$$

$$M_2 = \Phi_{13}\Phi_{31} - \Phi_{12}\Phi_{33} = \frac{S}{h_2^2} \tag{7}$$

$$M_3 = \Phi_{12}\Phi_{23} - \Phi_{13}\Phi_{22} = \frac{S}{h_3^2}, \tag{8}$$

which are equivalent to

$$M_1\Phi_{11} + M_2\Phi_{21} + M_3\Phi_{31} = S \tag{9}$$

$$M_1\Phi_{12} + M_2\Phi_{22} + M_3\Phi_{32} = 0 \tag{10}$$

$$M_1\Phi_{13} + M_2\Phi_{23} + M_3\Phi_{33} = 0. \tag{11}$$

This gives a total of four equations in nine unknowns. Morse and Feshbach (1953, pp. 655–666) give not only the Stäckel determinants for common coordinate systems, but also the elements of the determinant (although it is not clear how these are derived).

see also HELMHOLTZ DIFFERENTIAL EQUATION, LAPLACE'S EQUATION, POISSON'S EQUATION, ROBERTSON CONDITION, SEPARATION OF VARIABLES

References
Morse, P. M. and Feshbach, H. "Tables of Separable Coordinates in Three Dimensions." *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 509–511 and 655–666, 1953.

Stamp Folding

The number of ways of folding a strip of stamps has several possible variants. Considering only positions of the hinges for unlabeled stamps without regard to orientation of the stamps, the number of foldings is denoted $U(n)$. If the stamps are labelled and orientation is taken into account, the number of foldings is denoted $N(n)$. Finally, the number of symmetric foldings is denoted $S(n)$. The following table summarizes these values for the first n.

n	$S(n)$	$U(n)$	$N(n)$
1	1	1	1
2	1	1	1
3	2	2	6
4	4	5	16
5	6	14	50
6	8	39	144
7	18	120	462
8	20	358	1392
9	56	1176	4536
10		3572	

see also MAP FOLDING

References
Gardner, M. "The Combinatorics of Paper-Folding." In *Wheels, Life, and Other Mathematical Amusements.* New York: W. H. Freeman, pp. 60–73, 1983.
Ruskey, F. "Information of Stamp Folding." `http://sue.csc.uvic.ca/~cos/inf/perm/StampFolding.html`.
Sloane, N. J. A. *A Handbook of Integer Sequences.* Boston, MA: Academic Press, p. 22, 1973.

Standard Deviation

The standard deviation is defined as the SQUARE ROOT of the VARIANCE,

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\mu_2' - \mu^2}, \tag{1}$$

where $\mu = \langle x \rangle$ is the MEAN and $\mu_2' = \langle x^2 \rangle$ is the second MOMENT about 0. The variance σ^2 is equal to the second MOMENT about the MEAN,

$$\sigma^2 = \mu_2. \tag{2}$$

The square root of the SAMPLE VARIANCE is the "sample" standard deviation,

$$s_N = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2}. \tag{3}$$

It is a BIASED ESTIMATOR of the population standard deviation. As unbiased ESTIMATOR is given by

$$s_{N-1} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}. \tag{4}$$

Physical scientists often use the term ROOT-MEAN-SQUARE as a synonym for standard deviation when they refer to the SQUARE ROOT of the mean squared deviation of a signal from a given baseline or fit.

see also MEAN, MOMENT, ROOT-MEAN-SQUARE, SAMPLE VARIANCE, STANDARD ERROR, VARIANCE

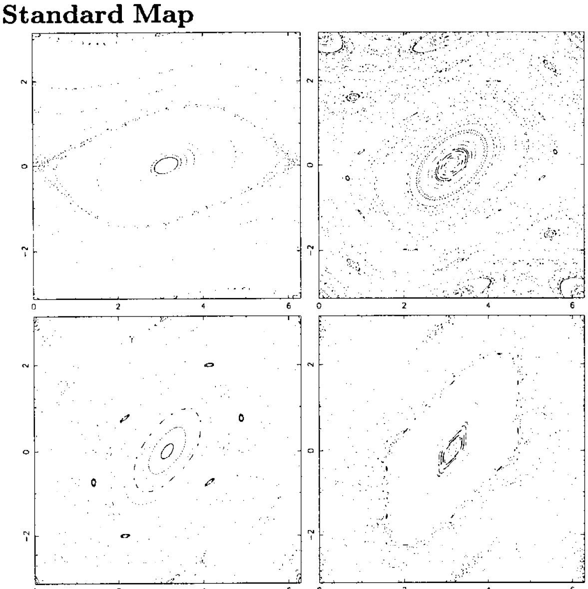
Standard Error

The square root of the ESTIMATED VARIANCE of a quantity. The standard error is also sometimes used to mean

$$\mathrm{var}(\bar{x}) = \sum_{i=1}^{n}\left(\frac{1}{n}\right)^2 {\sigma_i}^2 = \sum_{i=1}^{n}\left(\frac{1}{n}\right)^2 \sigma^2 = \frac{\sigma^2}{n}.$$

see also STANDARD DEVIATION

Standard Map



A 2-D MAP, also called the TAYLOR-GREENE-CHIRIKOV MAP in some of the older literature.

$$I_{n+1} = I_n + K\sin\theta_n \tag{1}$$

$$\theta_{n+1} = \theta_n + I_{n+1} = I_n + \theta_n + K\sin\theta_n, \tag{2}$$

where I and θ are computed mod 2π and K is a POSITIVE constant. An analytic estimate of the width of the CHAOTIC zone (Chirikov 1979) finds

$$\delta I = Be^{-AK^{-1/2}}. \tag{3}$$

Numerical experiments give $A \approx 5.26$ and $B \approx 240$. The value of K at which global CHAOS occurs has been bounded by various authors. GREENE'S METHOD is the most accurate method so far devised.

Author	Bound	Fraction	Decimal
Hermann	$>$	$\frac{1}{34}$	0.029411764
Italians	$>$	–	0.65
Greene	$\approx$	–	0.971635406
MacKay and Pearson	$<$	$\frac{63}{64}$	0.984375000
Mather	$<$	$\frac{4}{3}$	1.333333333

FIXED POINTS are found by requiring that

$$I_{n+1} = I_n \tag{4}$$

$$\theta_{n+1} = \theta_n. \tag{5}$$

The first gives $K\sin\theta_n = 0$, so $\sin\theta_n = 0$ and

$$\theta_n = 0, \pi. \tag{6}$$

The second requirement gives

$$I_n + K\sin\theta_n = I_n = 0. \tag{7}$$

The FIXED POINTS are therefore $(I,\theta) = (0,0)$ and $(0,\pi)$. In order to perform a LINEAR STABILITY analysis, take differentials of the variables

$$dI_{n+1} = dI_n + K\cos\theta_n\, d\theta_n \tag{8}$$

$$d\theta_{n+1} = dI_n + (1 + K\cos\theta_n)\, d\theta_n. \tag{9}$$

In MATRIX form,

$$\begin{bmatrix} \delta I_{n+1} \\ \delta\theta_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & K\cos\theta_n \\ 1 & 1 + K\cos\theta_n \end{bmatrix} \begin{bmatrix} \delta I_n \\ \delta\theta_n \end{bmatrix}. \tag{10}$$

The EIGENVALUES are found by solving the CHARACTERISTIC EQUATION

$$\begin{vmatrix} 1-\lambda & K\cos\theta_n \\ 1 & 1 + K\cos\theta_n - \lambda \end{vmatrix} = 0, \tag{11}$$

so

$$\lambda^2 - \lambda(K\cos\theta_n + 2) + 1 = 0 \tag{12}$$

$$\lambda_\pm = \tfrac{1}{2}[K\cos\theta_n + 2 \pm \sqrt{(K\cos\theta_n + 2)^2 - 4}\,]. \tag{13}$$

For the FIXED POINT $(0,\pi)$,

$$\begin{aligned} \lambda_\pm^{(0,\pi)} &= \tfrac{1}{2}[2 - K \pm \sqrt{(2-K)^2 - 4}\,] \\ &= \tfrac{1}{2}(2 - K \pm \sqrt{K^2 - 4K}\,). \end{aligned} \tag{14}$$

The FIXED POINT will be stable if $|\Re(\lambda^{(0,\pi)})| < 2$. Here, that means

$$\tfrac{1}{2}|2-K| < 1 \tag{15}$$

$$|2-K| < 2 \tag{16}$$

$$-2 < 2-K < 2 \tag{17}$$

$$-4 < -K < 0 \tag{18}$$

so $K \in [0,4)$. For the FIXED POINT (0, 0), the EIGENVALUES are

$$\lambda_{\pm}^{(0,0)} = \tfrac{1}{2}[2+K \pm \sqrt{(K+2)^2-4}\,] = \tfrac{1}{2}(2+K \pm \sqrt{K^2+4K}\,). \quad (19)$$

If the map is unstable for the larger EIGENVALUE, it is unstable. Therefore, examine $\lambda_{+}^{(0,0)}$. We have

$$\frac{1}{2}\left|2+K+\sqrt{K^2+4K}\right| < 1, \quad (20)$$

so

$$-2 < 2+K+\sqrt{K^2+4K} < 2 \quad (21)$$

$$-4-K < \sqrt{K^2+4K} < -K. \quad (22)$$

But $K > 0$, so the second part of the inequality cannot be true. Therefore, the map is unstable at the FIXED POINT (0, 0).

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Chirikov, B. V. "A Universal Instability of Many-Dimensional Oscillator Systems." *Phys. Rep.* **52**, 264–379, 1979.

Standard Normal Distribution

A NORMAL DISTRIBUTION with zero MEAN ($\mu = 0$) and unity STANDARD DEVIATION ($\sigma^2 = 1$).

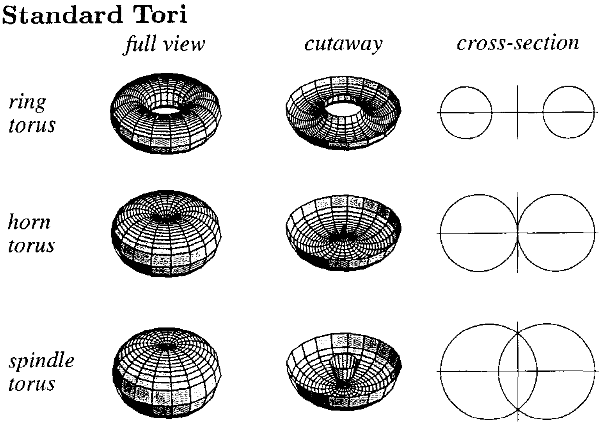
see also NORMAL DISTRIBUTION

Standard Space

A SPACE which is ISOMORPHIC to a BOREL SUBSET B of a POLISH SPACE equipped with its SIGMA ALGEBRA of BOREL SETS.

see also BOREL SET, POLISH SPACE, SIGMA ALGEBRA

Standard Tori



One of the three classes of TORI illustrated above and given by the parametric equations

$$x = (c + a\cos v)\cos u \quad (1)$$

$$y = (c + a\cos v)\sin u \quad (2)$$

$$z = a\sin v. \quad (3)$$

The three different classes of standard tori arise from the three possible relative sizes of a and c. $c > a$ corresponds to the RING TORUS shown above, $c = a$ corresponds to a HORN TORUS which touches itself at the point (0, 0, 0), and $c < a$ corresponds to a self-intersecting SPINDLE TORUS (Pinkall 1986). If no specification is made, "torus" is taken to mean RING TORUS.

The standard tori and their inversions are CYCLIDES.

see also APPLE, CYCLIDE, HORN TORUS, LEMON, RING TORUS, SPINDLE TORUS, TORUS

References
Pinkall, U. "Cyclides of Dupin." §3.3 in *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, pp. 28–30, 1986.

Standardized Moment

Defined for samples x_i, $i = 1, \ldots, N$ by

$$\alpha_r \equiv \frac{1}{N}\sum_{i=1}^{N} {z_i}^r = \frac{\mu_r}{\sigma^r}, \quad (1)$$

where

$$z_i \equiv \frac{x_i - \bar{x}}{s_x}. \quad (2)$$

The first few are

$$\alpha_1 = 0 \quad (3)$$

$$\alpha_2 = 1 \quad (4)$$

$$\alpha_3 = \frac{\mu_3}{s^3} \quad (5)$$

$$\alpha_4 = \frac{\mu_4}{s^4}. \quad (6)$$

see also KURTOSIS, MOMENT, SKEWNESS

Standardized Score

see z-SCORE

Stanley's Theorem

The total number of 1s that occur among all unordered PARTITIONS of a POSITIVE INTEGER is equal to the sum of the numbers of distinct parts of (i.e., numbers in) those PARTITIONS.

see also ELDER'S THEOREM, PARTITION

References
Honsberger, R. *Mathematical Gems III.* Washington, DC: Math. Assoc. Amer., pp. 6–8, 1985.

Star

In formal geometry, a star is a set of $2n$ VECTORS $\pm\mathbf{a}_1$, ..., $\pm\mathbf{a}_n$ which form a fixed center in EUCLIDEAN 3-SPACE. In common usage, a star is a STAR POLYGON (i.e., regular convex polygon) such as the PENTAGRAM or HEXAGRAM

see also CROSS, EUTACTIC STAR, STAR OF GOLIATH, STAR POLYGON

Star of David

see HEXAGRAM

Star Figure

A STAR POLYGON-like figure $\left\{\frac{p}{q}\right\}$ for which p and q are not RELATIVELY PRIME.

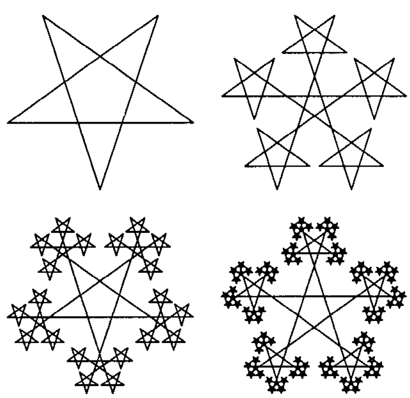
see also STAR POLYGON

Star (Fixed Point)

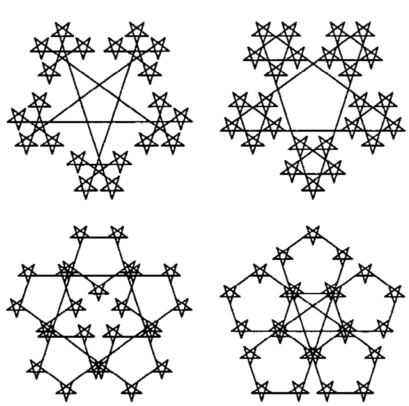
A FIXED POINT which has one zero EIGENVECTOR.

see STABLE STAR, UNSTABLE STAR

Star Fractal



A FRACTAL composed of repeated copies of a PENTAGRAM or other polygon.



The above figure shows a generalization to different offsets from the center.

References

Lauwerier, H. *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, pp. 72–77, 1991.

Weisstein, E. W. "Fractals." http://www.astro.virginia.edu/~eww6n/math/notebooks/Fractal.m.

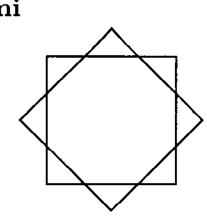
Star of Goliath

see NONAGRAM

Star Graph

The k-star graph is a TREE on $k+1$ nodes with one node having valency k and the others having valency 1. Star graphs S_n are always GRACEFUL.

Star of Lakshmi



The STAR FIGURE $\{8/2\}$, which is used by Hindus to symbolize *Ashtalakshmi,* the eight forms of wealth. This symbol appears prominently in the Lugash national museum portrayed in the fictional film *Return of the Pink Panther.*

see also DISSECTION, HEXAGRAM, PENTAGRAM, STAR FIGURE, STAR POLYGON

References

Savio, D. Y. and Suryanaroyan, E. R. "Chebyshev Polynomials and Regular Polygons." *Amer. Math. Monthly* **100**, 657–661, 1993.

Star Number

The number of cells in a generalized Chinese checkers board (or "centered" HEXAGRAM).

$$S_n = 6n(n+1)+1 = S_{n-1} + 12(n-1). \qquad (1)$$

The first few are 1, 13, 37, 73, 121, ... (Sloane's A003154). Every star number has DIGITAL ROOT 1 or 4, and the final digits must be one of: 01, 21, 41, 61, 81, 13, 33, 53, 73, 93, or 37.

The first TRIANGULAR star numbers are 1, 253, 49141, 9533161, ... (Sloane's A006060), and can be computed using

$$\begin{aligned} TS_n &= \frac{3[(7+4\sqrt{3})^{2n-1} + (7-4\sqrt{3})^{2n-1}] - 10}{32} \\ &= 194\,TS_{n-1} + 60 - TS_{n-2}. \qquad (2) \end{aligned}$$

The first few SQUARE star numbers are 1, 121, 11881, 1164241, 114083761, ... (Sloane's A006061). SQUARE star numbers are obtained by solving the DIOPHANTINE EQUATION

$$2x^2 + 1 = 3y^2 \qquad (3)$$

and can be computed using

$$SS_n = \frac{[(5+2\sqrt{6})^n(\sqrt{6}-2)-(5-2\sqrt{6})^n(\sqrt{6}+2)]^2}{4}. \tag{4}$$

see also HEX NUMBER, SQUARE NUMBER, TRIANGULAR NUMBER

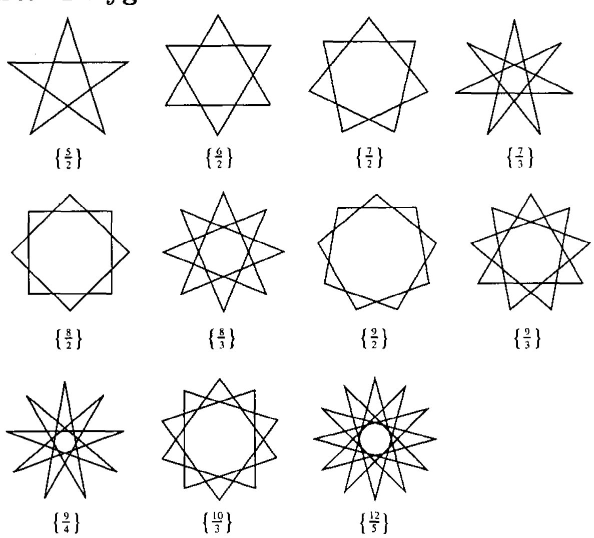
References

Gardner, M. "Hexes and Stars." Ch. 2 in *Time Travel and Other Mathematical Bewilderments.* New York: W. H. Freeman, 1988.

Hindin, H. "Stars, Hexes, Triangular Numbers, and Pythagorean Triples." *J. Recr. Math.* **16**, 191–193, 1983–1984.

Sloane, N. J. A. Sequences A003154/M4893, A006060/M5425, and A006061/M5385 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Star Polygon



A star polygon $\{p/q\}$, with p, q POSITIVE INTEGERS, is a figure formed by connecting with straight lines every qth point out of p regularly spaced points lying on a CIRCUMFERENCE. The number q is called the DENSITY of the star polygon. Without loss of generality, take $q < p/2$.

The usual definition (Coxeter 1969) requires p and q to be RELATIVELY PRIME. However, the star polygon can also be generalized to the STAR FIGURE (or "improper" star polygon) when p and q share a common divisor (Savio and Suryanaroyan 1993). For such a figure, if all points are not connected after the first pass, i.e., if $(p, q) \neq 1$, then start with the first unconnected point and repeat the procedure. Repeat until all points are connected. For $(p, q) \neq 1$, the $\{p/q\}$ symbol can be factored as

$$\left\{\frac{p}{q}\right\} = n\left\{\frac{p'}{q'}\right\}, \tag{1}$$

where

$$p' \equiv \frac{p}{n} \tag{2}$$

$$q' \equiv \frac{q}{n}, \tag{3}$$

to give n $\{p'/q'\}$ figures, each rotated by $2\pi/p$ radians, or $360°/p$.

If $q = 1$, a REGULAR POLYGON $\{p\}$ is obtained. Special cases of $\{p/q\}$ include $\{5/2\}$ (the PENTAGRAM), $\{6/2\}$ (the HEXAGRAM, or STAR OF DAVID), $\{8/2\}$ (the STAR OF LAKSHMI), $\{8/3\}$ (the OCTAGRAM), $\{10/3\}$ (the DECAGRAM), and $\{12/5\}$ (the DODECAGRAM).

The star polygons were first systematically studied by Thomas Bradwardine.

see also DECAGRAM, HEXAGRAM, NONAGRAM, OCTAGRAM, PENTAGRAM, REGULAR POLYGON, STAR OF LAKSHMI, STELLATED POLYHEDRON

References

Coxeter, H. S. M. "Star Polygons." §2.8 in *Introduction to Geometry, 2nd ed.* New York: Wiley, pp. 36–38, 1969.

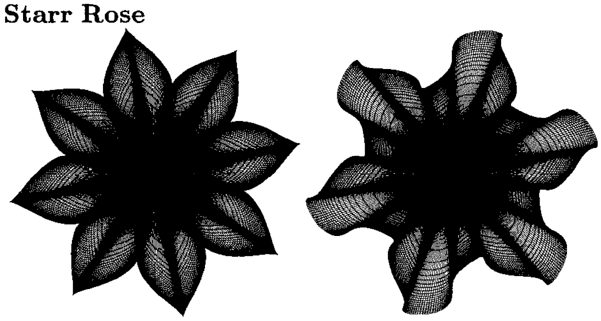
Frederickson, G. "Stardom." Ch. 16 in *Dissections: Plane and Fancy.* New York: Cambridge University Press, pp. 172–186, 1997.

Savio, D. Y. and Suryanaroyan, E. R. "Chebyshev Polynomials and Regular Polygons." *Amer. Math. Monthly* **100**, 657–661, 1993.

Star Polyhedron

see KEPLER-POINSOT SOLID

Starr Rose



$a = 8, b = 16, c = 16$ $\quad$ $a = 6, b = 18, c = 18$

see also MAURER ROSE

References

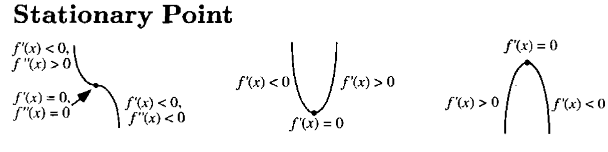
Wagon, S. "Variations of Circular Motion." §4.5 in *Mathematica in Action.* New York: W. H. Freeman, pp. 137–140, 1991.

State Space

The measurable space $(S', \mathbb{S}')$ into which a RANDOM VARIABLE from a PROBABILITY SPACE is a measurable function.

see also PROBABILITY SPACE, RANDOM VARIABLE

Stationary Point



A point x_0 at which the DERIVATIVE of a FUNCTION $f(x)$ vanishes,

$$f'(x_0) = 0.$$

A stationary point may be a MINIMUM, MAXIMUM, or INFLECTION POINT.

see also CRITICAL POINT, DERIVATIVE, EXTREMUM, FIRST DERIVATIVE TEST, INFLECTION POINT, MAXIMUM, MINIMUM, SECOND DERIVATIVE TEST

Stationary Tangent

see INFLECTION POINT

Stationary Value

The value at a STATIONARY POINT.

Statistic

A function of one or more random variables.

see also ANDERSON-DARLING STATISTIC, KUIPER STATISTIC, VARIATE

Statistical Test

A test used to determine the statistical SIGNIFICANCE of an observation. Two main types of error can occur:

1. A TYPE I ERROR occurs when a false negative result is obtained in terms of the NULL HYPOTHESIS by obtaining a *false positive measurement.*
2. A TYPE II ERROR occurs when a false positive result is obtained in terms of the NULL HYPOTHESIS by obtaining a *false negative measurement.*

The probability that a statistical test will be positive for a true statistic is sometimes called the test's SENSITIVITY, and the probability that a test will be negative for a negative statistic is sometimes called the SPECIFICITY. The following table summarizes the names given to the various combinations of the actual state of affairs and observed test results.

result	name
true positive result	sensitivity
false negative result	1 − sensitivity
true negative result	specificity
false positive result	1 − specificity

Multiple-comparison corrections to statistical tests are used when several statistical tests are being performed simultaneously. For example, let's suppose you were measuring leg length in eight different lizard species and wanted to see whether the MEANS of any pair were different. Now, there are $8!/2!6! = 28$ pairwise comparisons possible, so even if all of the *population* means are equal, it's quite likely that at least one pair of sample means would differ significantly at the 5% level. An ALPHA VALUE of 0.05 is therefore appropriate for each individual comparison, but not for the set of *all* comparisons.

In order to avoid a lot of spurious positives, the ALPHA VALUE therefore needs to be lowered to account for the number of comparisons being performed. This is a correction for multiple comparisons. There are *many* different ways to do this. The simplest, and the most conservative, is the BONFERRONI CORRECTION. In practice, more people are more willing to accept false positives (false rejection of NULL HYPOTHESIS) than false negatives (false acceptance of NULL HYPOTHESIS), so less conservative comparisons are usually used.

see also ANOVA, BONFERRONI CORRECTION, CHI-SQUARED TEST, FISHER'S EXACT TEST, FISHER SIGN TEST, KOLMOGOROV-SMIRNOV TEST, LIKELIHOOD RATIO, LOG LIKELIHOOD PROCEDURE, NEGATIVE LIKELIHOOD RATIO, PAIRED t-TEST, PARAMETRIC TEST, PREDICTIVE VALUE, SENSITIVITY, SIGNIFICANCE TEST, SPECIFICITY, TYPE I ERROR, TYPE II ERROR, WILCOXON RANK SUM TEST, WILCOXON SIGNED RANK TEST

Statistics

The mathematical study of the LIKELIHOOD and PROBABILITY of events occurring based on known information and inferred by taking a limited number of samples. Statistics plays an extremely important role in many aspects of economics and science, allowing educated guesses to be made with a minimum of expensive or difficult-to-obtain data.

see also BOX-AND-WHISKER PLOT, BUFFON-LAPLACE NEEDLE PROBLEM, BUFFON'S NEEDLE PROBLEM, CHERNOFF FACE, COIN FLIPPING, DE MERE'S PROBLEM, DICE, DISTRIBUTION, GAMBLER'S RUIN, INDEX, LIKELIHOOD, MOVING AVERAGE, P-VALUE, POPULATION COMPARISON, POWER (STATISTICS), PROBABILITY, RESIDUAL VS. PREDICTOR PLOT, RUN, SHARING PROBLEM, STATISTICAL TEST, TAIL PROBABILITY

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Staudt-Clausen Theorem

see VON STAUDT-CLAUSEN THEOREM

Steenrod Algebra

The Steenrod algebra has to do with the COHOMOLOGY operations in singular COHOMOLOGY with INTEGER mod 2 COEFFICIENTS. For every $n \in \mathbb{Z}$ and $i \in \{0, 1, 2, 3, \ldots\}$ there are natural transformations of FUNCTORS

$$Sq^i : H^n(\bullet; \mathbb{Z}_2) \to H^{n+i}(\bullet; \mathbb{Z}_2)$$

satisfying:

1. $Sq^i = 0$ for $i > n$.
2. $Sq^n(x) = x \smile x$ for all $x \in H^n(X, A; \mathbb{Z}_2)$ and all pairs (X, A).
3. $Sq^0 = id_{H^n(\bullet;\mathbb{Z}_2)}$.
4. The Sq^i maps commute with the coboundary maps in the long exact sequence of a pair. In other words,

   $$Sq^i : H^*(\bullet; \mathbb{Z}_2) \to H^{*+i}(\bullet; \mathbb{Z}_2)$$

   is a degree i transformation of cohomology theories.
5. (CARTAN RELATION)

   $$Sq^i(x \smile y) = \Sigma_{j+k=i} Sq^j(x) \smile Sq^k(y).$$

6. (ADEM RELATIONS) For $i < 2j$,

   $$Sq^i \circ Sq^j(x) = \Sigma_{k=0}^{\lfloor i \rfloor} \binom{j-k-1}{i-2k} Sq^{i+j-k} \circ Sq^k(x).$$

7. $Sq^i \circ \Sigma = \Sigma \circ Sq^i$ where Σ is the cohomology suspension isomorphism.

The existence of these cohomology operations endows the cohomology ring with the structure of a MODULE over the Steenrod algebra $\mathcal{A}$, defined to be $T(F_{\mathbb{Z}_2}\{Sq^i : i \in \{0, 1, 2, 3, \ldots\}\})/R$, where $F_{\mathbb{Z}_2}(\bullet)$ is the free module functor that takes any set and sends it to the free $\mathbb{Z}_2$ module over that set. We think of $F_{\mathbb{Z}_2}\{Sq^i : i \in \{0, 1, 2, \ldots\}\}$ as being a graded $\mathbb{Z}_2$ module, where the i-th gradation is given by $\mathbb{Z}_2 \cdot Sq^i$. This makes the tensor algebra $T(F_{\mathbb{Z}_2}\{Sq^i : i \in \{0, 1, 2, 3, \ldots\}\})$ into a GRADED ALGEBRA over $\mathbb{Z}_2$. R is the IDEAL generated by the elements $Sq^i Sq^j + \Sigma_{k=0}^{\lfloor i \rfloor} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k$ and

$1 + Sq^0$ for $0 < i < 2j$. This makes $\mathcal{A}$ into a graded $\mathbb{Z}_2$ algebra.

By the definition of the Steenrod algebra, for any SPACE (X, A), $H^*(X, A; \mathbb{Z}_2)$ is a MODULE over the Steenrod algebra $\mathcal{A}$, with multiplication induced by $Sq^i \cdot x \equiv Sq^i(x)$. With the above definitions, cohomology with COEFFICIENTS in the RING $\mathbb{Z}_2$, $H^*(\bullet; \mathbb{Z}_2)$ is a FUNCTOR from the category of pairs of TOPOLOGICAL SPACES to graded modules over $\mathcal{A}$.

see also ADEM RELATIONS, CARTAN RELATION, COHOMOLOGY, GRADED ALGEBRA, IDEAL, MODULE, TOPOLOGICAL SPACE

Steenrod-Eilenberg Axioms

see EILENBERG-STEENROD AXIOMS

Steenrod's Realization Problem

When can homology classes be realized as the image of fundamental classes of MANIFOLDS? The answer is known, and singular BORDISM GROUPS provide insight into this problem.

see also BORDISM GROUP, MANIFOLD

Steepest Descent Method

An ALGORITHM for calculating the GRADIENT $\nabla f(\mathbf{P})$ of a function at an n-D point $\mathbf{P}$. The steepest descent method starts at a point $\mathbf{P}_0$ and, as many times as needed, moves from $\mathbf{P}_i$ to $\mathbf{P}_{i+1}$ by minimizing along the line extending from $\mathbf{P}_i$ in the direction of $-\nabla f(\mathbf{P}_i)$, the local downhill gradient. This method has the severe drawback of requiring a great many iterations for functions which have long, narrow valley structures. In such cases, a CONJUGATE GRADIENT METHOD is preferable.

see also CONJUGATE GRADIENT METHOD, GRADIENT

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Steffenson's Formula

$$f_p = f_0 + \tfrac{1}{2}p(p+1)\delta_{1/2} - \tfrac{1}{2}(p-1)p\delta_{-1/2} + (S_3 + S_4)\delta_{1/2}^3 + (S_3 - S_4)\delta_{-1/2}^3 + \ldots, \quad (1)$$

for $p \in [-\frac{1}{2}, \frac{1}{2}]$, where δ is the CENTRAL DIFFERENCE and

$$S_{2n+1} = \frac{1}{2}\binom{p+n}{2n+1} \quad (2)$$

$$S_{2n+2} = \frac{p}{2n+2}\binom{p+n}{2n+1} \quad (3)$$

$$S_{2n+1} - S_{2n+2} = \binom{p+n+1}{2n+2} \quad (4)$$

$$S_{2n+1} - S_{2n+2} = -\binom{p+n}{2n+2}, \quad (5)$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT.

see also CENTRAL DIFFERENCE, STIRLING'S FINITE DIFFERENCE FORMULA

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Steffensen's Inequality

Let $f(x)$ be a NONNEGATIVE and monotonic decreasing function in $[a, b]$ and $g(x)$ satisfy such that $0 \leq g(x) \leq 1$ in $[a, b]$, then

$$\int_{b-k}^{b} f(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq \int_a^{a+k} f(x)\,dx,$$

where

$$k = \int_a^b g(x)\,dx.$$

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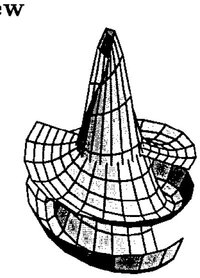
Steinbach Screw



A SURFACE generated by the parametric equations

$$x(u, v) = u \cos v$$
$$y(u, v) = u \sin v$$
$$z(u, v) = v \cos u.$$

The above image uses $u \in [-4, 4]$ and $v \in [0, 6.25]$.

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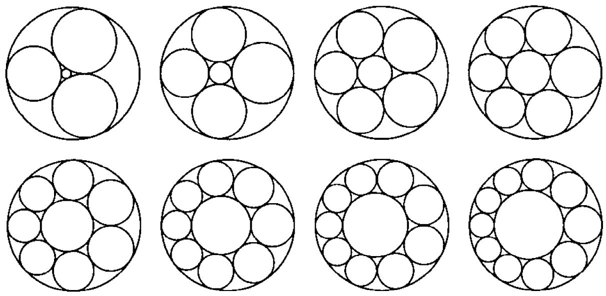
Steiner Chain



Given two nonconcentric CIRCLES with one interior to the other, if small TANGENT CIRCLES can be inscribed around the region between the two CIRCLES such that the final CIRCLE is TANGENT to the first, the CIRCLES form a Steiner chain.

The simplest way to construct a Steiner chain is to perform an INVERSION on a symmetrical arrangement on n circles packed between a central circle of radius b and an outer concentric circle of radius a. In this arrangement,

$$\sin\left(\frac{\pi}{n}\right) = \frac{a-b}{a+b}, \quad (1)$$

so the ratio of the radii for the small and large circles is

$$\frac{b}{a} = \frac{1-\sin\left(\frac{\pi}{n}\right)}{1+\sin\left(\frac{\pi}{n}\right)}. \quad (2)$$

To transform the symmetrical arrangement into a Steiner chain, find an INVERSION CENTER which transforms two centers initially offset by a fixed distance c to the same point. This can be done by equating

$$\frac{k^2 x}{x^2-a^2} = \frac{k^2(x-c)}{(x-c)^2-b^2}, \quad (3)$$

giving the offset of the inversion center from the large circle's center as

$$x = \frac{a^2-b^2+c^2 \pm \sqrt{(a^2-b^2+c^2)^2-4a^2c^2}}{2c}. \quad (4)$$

Plugging in a fixed value of a fixes b, which therefore determines x for a given c. Equivalently, a Steiner chain results whenever the INVERSIVE DISTANCE between the two original circles is given by

$$\delta = 2\ln\left[\sec\left(\frac{\pi}{n}\right)+\tan\left(\frac{\pi}{n}\right)\right] \quad (5)$$

$$= 2\ln\left[\tan\left(\frac{\pi}{4}+\frac{\pi}{2n}\right)\right] \quad (6)$$

(Coxeter and Greitzer 1967). The centers of the circles in a Steiner chain lie on an ELLIPSE (Ogilvy 1990, p. 57).

STEINER'S PORISM states that if a Steiner chain is formed from one starting circle, then a Steiner chain is also formed from any other starting circle.

see also ARBELOS, COXETER'S LOXODROMIC SEQUENCE OF TANGENT CIRCLES, HEXLET, PAPPUS CHAIN, STEINER'S PORISM

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Steiner Construction

A construction done using only a STRAIGHTEDGE. The PONCELET-STEINER THEOREM proves that all constructions possible using a COMPASS and STRAIGHTEDGE are possible using a STRAIGHTEDGE alone, as long as a fixed CIRCLE and its center, two intersecting CIRCLES without their centers, or three nonintersecting CIRCLES are drawn beforehand.

see also GEOMETRIC CONSTRUCTION, MASCHERONI CONSTRUCTION, PONCELET-STEINER THEOREM, STRAIGHTEDGE

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Steiner, J. *Geometric Constructions with a Ruler, Given a Fixed Circle with Its Center.* Translated from the first German ed. (1833). New York: Scripta Mathematica, 1950.

Steiner's Ellipse

Let $\alpha' : \beta' : \gamma'$ be the ISOTOMIC CONJUGATE POINT of a point with TRILINEAR COORDINATES $\alpha : \beta : \gamma$. The isotomic conjugate of the LINE AT INFINITY having trilinear equation

$$a\alpha + b\beta + c\gamma = 0$$

is

$$\frac{\beta'\gamma'}{a} + \frac{\gamma'\alpha'}{b} + \frac{\alpha'\beta'}{c} = 0,$$

known as Steiner's ellipse (Vandeghen 1965).

see also ISOTOMIC CONJUGATE POINT, LINE AT INFINITY

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Steiner's Hypocycloid

see DELTOID

Steiner-Lehmus Theorem

Any TRIANGLE that has two equal ANGLE BISECTORS (each measured from a VERTEX to the opposite sides) is an ISOSCELES TRIANGLE. This theorem is also called the INTERNAL BISECTORS PROBLEM and LEHMUS' THEOREM.

see also ISOSCELES TRIANGLE

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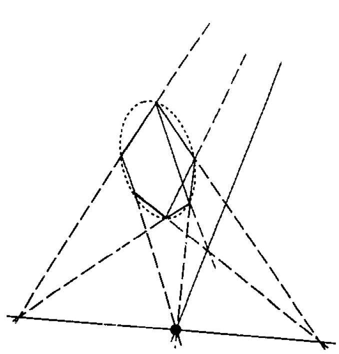
Steiner Points

There are two different types of points known as Steiner points.

The point of CONCURRENCE of the three lines drawn through the VERTICES of a TRIANGLE PARALLEL to the corresponding sides of the first BROCARD TRIANGLE. It lies on the CIRCUMCIRCLE opposite the TARRY POINT and has TRIANGLE CENTER FUNCTION

$$\alpha = bc(a^2 - b^2)(a^2 - c^2).$$

The BRIANCHON POINT for KIEPERT'S PARABOLA is the Steiner point. The LEMOINE POINT K is the Steiner point of the first BROCARD TRIANGLE. The SIMSON LINE of the Steiner point is PARALLEL to the line OK, when O is the CIRCUMCENTER and K is the LEMOINE POINT.



If triplets of opposites sides on a CONIC SECTION in PASCAL'S THEOREM are extended for all permutations of VERTICES, 60 PASCAL LINES are produced. The 20 points of their 3 by 3 intersections are called Steiner points.

see also BRIANCHON POINT, BROCARD TRIANGLES, CIRCUMCIRCLE, CONIC SECTION, KIEPERT'S PARABOLA, LEMOINE POINT, PASCAL LINE, PASCAL'S THEOREM, STEINER SET, STEINER TRIPLE SYSTEM, TARRY POINT

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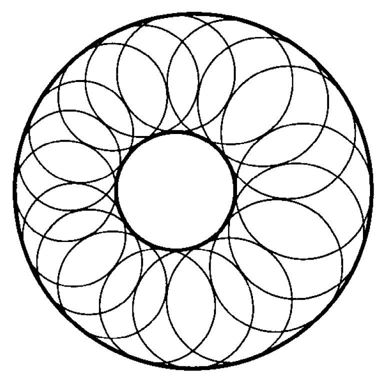
Steiner's Porism



If a STEINER CHAIN is formed from one starting circle, then a STEINER CHAIN is formed from any other starting circle. In other words, given two nonconcentric CIRCLES, draw CIRCLES successively touching them and each other. If the last touches the first, this will also happen for any position of the first CIRCLE.

see also HEXLET, STEINER CHAIN

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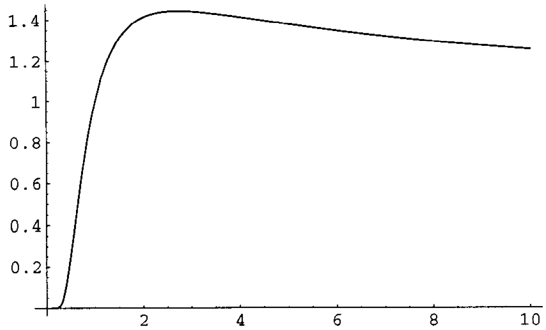
Steiner's Problem



For what value of x is $f(x) = x^{1/x}$ a MAXIMUM? The maximum occurs at $x = e$, where

$$f'(x) = x^{-2+1/x}(1 - \ln x) = 0,$$

which gives a maximum of

$$e^{1/e} = 1.444667861\ldots.$$

The function has an inflection point at $x = 0.581933\ldots$, where

$$f''(x) = x^{-4+1/x}[1 - 3x + (\ln x)(2x - 2 + \ln x)] = 0.$$

see also FERMAT'S PROBLEM

Steiner Quadruple System

A Steiner quadruple system is a STEINER SYSTEM $S(t = 3, k = 4, v)$, where S is a v-set and B is a collection of k-sets of S such that every t-subset of S is contained in exactly one member of B. Barrau (1908) established the uniqueness of $S(3, 4, 8)$,

1	2	4	8	3	5	6	7
2	3	5	8	1	4	6	7
3	4	6	8	1	2	5	7
4	5	7	8	1	2	3	6
1	5	6	8	2	3	4	7
2	6	7	8	1	3	4	5
1	3	7	8	2	4	5	6

and $S(3, 4, 10)$

1	2	4	5	1	2	3	7	1	3	5	8
2	3	5	6	2	3	4	8	2	4	6	9
3	4	6	7	3	4	5	9	3	5	7	0
4	5	7	8	4	5	6	0	1	4	6	8
5	6	8	9	1	5	6	7	2	5	7	9
6	7	9	0	2	6	7	8	3	6	8	0
1	7	8	0	3	7	8	9	1	4	7	9
1	2	8	9	4	8	9	0	2	5	8	0
2	3	9	0	1	5	9	0	1	3	6	9
1	3	4	0	1	2	6	0	2	4	7	0

Fitting (1915) subsequently constructed the cyclic systems $S(3, 4, 26)$ and $S(3, 4, 34)$, and Bays and de Weck (1935) showed the existence of at least one $S(3, 4, 14)$. Hanani (1960) proved that a NECESSARY and SUFFICIENT condition for the existence of an $S(3, 4, v)$ is that $v \equiv 2$ or $4 \pmod 6$.

The number of nonisomorphic steiner quadruple systems of orders 8, 10, 14, and 16 are 1, 1, 4 (Mendelsohn and Hung 1972), and at least 31,021 (Lindner and Rosa 1976).

see also STEINER SYSTEM, STEINER TRIPLE SYSTEM

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Steiner's Segment Problem

Given n points, find the line segments with the shortest possible total length which connect the points. The segments need not necessarily be straight from one point to another.

For three points, if all ANGLES are less than 120°, then the line segments are those connecting the three points to a central point P which makes the ANGLES $\langle A\rangle PB$, $\langle B\rangle PC$, and $\langle C\rangle PA$ all 120°. If one ANGLE is greater that 120°, then P coincides with the offending ANGLE.

For four points, P is the intersection of the two diagonals, but the required minimum segments are not necessarily these diagonals.

A modified version of the problem is, given two points, to find the segments with the shortest total length connecting the points such that each branch point may be connected to only three segments. There is no general solution to this version of the problem.

Steiner Set

Three sets of three LINES such that each line is incident with two from both other sets.

see also SOLOMON'S SEAL LINES, STEINER POINTS, STEINER TRIPLE SYSTEM

Steiner Surface

A projection of the VERONESE SURFACE into 3-D (which must contain singularities) is called a Steiner surface. A classification of Steiner surfaces allowing complex parameters and projective transformations was accomplished in the 19th century. The surfaces obtained by restricting to real parameters and transformations were classified into 10 types by Coffman *et al.* (1996). Examples of Steiner surfaces include the ROMAN SURFACE (Coffman type 1) and CROSS-CAP (type 3).

The Steiner surface of type 2 is given by the implicit equation

$$x^2y^2 - x^2z^2 + y^2z^2 - xyz = 0,$$

and can be transformed into the ROMAN SURFACE or CROSS-CAP by a complex projective change of coordinates (but not by a real transformation). It has two pinch points and three double lines and, unlike the ROMAN SURFACE or CROSS-CAP, is not compact in any affine neighborhood.

The Steiner surface of type 4 has the implicit equation

$$y^2 - 2xy^2 - xz^2 + x^2y^2 + x^2z^2 - z^4 = 0,$$

and two of the three double lines of surface 2 coincide along a line where the two noncompact "components" are tangent.

see also CROSS-CAP, ROMAN SURFACE, VERONESE VARIETY

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Nordstrand, T. "Steiner Relative." http://www.uib.no/people/nfytn/stmtxt.htm.
Nordstrand, T. "Steiner Relative [2]." http://www.uib.no/people/nfytn/stm2txt.htm.

Steiner System

A Steiner system is a set X of v points, and a collection of subsets of X of size k (called blocks), such that any t points of X are in exactly one of the blocks. The special case $t = 2$ and $k = 3$ corresponds to a so-called STEINER TRIPLE SYSTEM. For a PROJECTIVE PLANE, $v = n^2 + n + 1$, $k = n + 1$, $t = 2$, and the blocks are simply lines.

see also STEINER QUADRUPLE SYSTEM, STEINER TRIPLE SYSTEM.

References
Colbourn, C. J. and Dinitz, J. H. (Eds.) *CRC Handbook of Combinatorial Designs.* Boca Raton, FL: CRC Press, 1996.
Woolhouse, W. S. B. "Prize Question 1733." *Lady's and Gentleman's Diary.* 1844.

Steiner's Theorem

Let LINES x and y join a variable point on a CONIC SECTION to two fixed points on the same CONIC SECTION. Then x and y are PROJECTIVELY related.

see also CONIC SECTION, PROJECTION

Steiner Triple System

Let X be a set of $v \geq 3$ elements together with a set B of 3-subset (triples) of X such that every 2-SUBSET of X occurs in exactly one triple of B. Then B is called a Steiner triple system and is a special case of a STEINER SYSTEM with $t = 2$ and $k = 3$. A Steiner triple system $S(v) = S(v, k = 3, \lambda = 1)$ of order v exists IFF $v \equiv 1, 3 \pmod 6$ (Kirkman 1847). In addition, if Steiner triple systems S_1 and S_2 of orders v_1 and v_2 exist, then so does a Steiner triple system S of order v_1v_2 (Ryser 1963, p. 101).

Examples of Steiner triple systems $S(v)$ of small orders v are

$$\begin{aligned} S_3 &= \{\{1,2,3\}\} \\ S_7 &= \{\{1,2,4\},\{2,3,5\},\{3,4,6\},\{4,5,7\}, \\ &\quad \{5,6,1\},\{6,7,2\},\{7,1,3\}\} \\ S_9 &= \{\{1,2,3\},\{4,5,6\},\{7,8,9\},\{1,4,7\}, \\ &\quad \{2,5,8\},\{3,6,9\},\{1,5,9\},\{2,6,7\}\}. \end{aligned}$$

The number of nonisomorphic Steiner triple systems $S(v)$ of orders $v = 7, 9, 13, 15, 19, \ldots$ (i.e., $6k + 1, 3$) are 1, 1, 20, 80, $> 1.1 \times 10^9$, ... (Colbourn and Dinitz 1996, pp. 14–15; Sloane's A030129). $S(7)$ is the same as the finite PROJECTIVE PLANE of order 2. $S(9)$ is a finite AFFINE PLANE which can be constructed from the array

$$\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}.$$

One of the two $S(13)$s is a finite HYPERBOLIC PLANE. The 80 Steiner triple systems $S(15)$ have been studied by Tonchev and Weishaar (1997). There are more than 1.1×10^9 Steiner triple systems of order 19 (Stinson and Ferch 1985; Colbourn and Dinitz 1996, p. 15).

see also HADAMARD MATRIX, KIRKMAN TRIPLE SYSTEM, STEINER QUADRUPLE SYSTEM, STEINER SYSTEM

References
Colbourn, C. J. and Dinitz, J. H. (Eds.) "Steiner Triple Systems." §4.5 in *CRC Handbook of Combinatorial Designs.* Boca Raton, FL: CRC Press, pp.14–15 and 70, 1996.
Kirkman, T. P. "On a Problem in Combinatorics." *Cambridge Dublin Math. J.* **2**, 191–204, 1847.
Lindner, C. C. and Rodger, C. A. *Design Theory.* Boca Raton, FL: CRC Press, 1997.
Ryser, H. J. *Combinatorial Mathematics.* Buffalo, NY: Math. Assoc. Amer., pp. 99–102, 1963.
Sloane, N. J. A. Sequence A030129 in "An On-Line Version of the Encyclopedia of Integer Sequences."
Stinson, D. R. and Ferch, H. "2000000 Steiner Triple Systems of Order 19." *Math. Comput.* **44**, 533–535, 1985.

Tonchev, V. D. and Weishaar, R. S. "Steiner Triple Systems of Order 15 and Their Codes." *J. Stat. Plan. Inference* **58**, 207–216, 1997.

Steinerian Curve

The LOCUS of points whose first POLARS with regard to the curves of a linear net have a common point. It is also the LOCUS of points of CONCURRENCE of line POLARS of points of the JACOBIAN CURVE. It passes through all points common to all curves of the system and is of order $3(n-1)^2$.

see also CAYLEYIAN CURVE, JACOBIAN CURVE

References
Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 150, 1959.

Steinhaus-Moser Notation

A NOTATION for LARGE NUMBERS defined by Steinhaus (1983, pp. 28–29). In this notation, $\triangle_n$ denotes n^n, $\boxed{n}$ denotes "n in n TRIANGLES," and n denotes "n in n SQUARES." A modified version due to Moser eliminates the circle notation, continuing instead with POLYGONS of ever increasing size, so n in a PENTAGON is n with n SQUARES around it, etc.

see also CIRCLE NOTATION, LARGE NUMBER, MEGA, MOSER

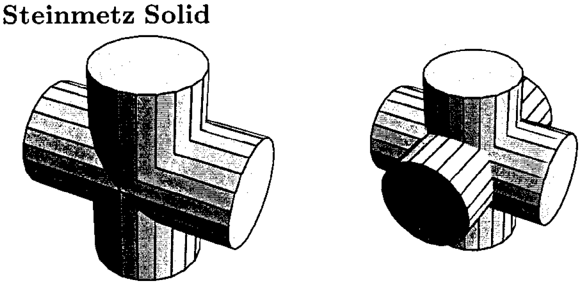
References
Steinhaus, H. *Mathematical Snapshots, 3rd American ed.* New York: Oxford University Press, 1983.

Steinitz's Theorem

A GRAPH G is the edge graph of a POLYHEDRON IFF G is a SIMPLE, PLANAR GRAPH which is 3-connected.

see also PLANAR GRAPH, SIMPLE GRAPH

Steinmetz Solid



The solid common to two (or three) right circular CYLINDERS of equal RADII intersecting at RIGHT ANGLES is called the Steinmetz solid. (Two CYLINDERS intersecting at RIGHT ANGLES are sometimes called a BICYLINDER, and three intersecting CYLINDERS a TRICYLINDER.)

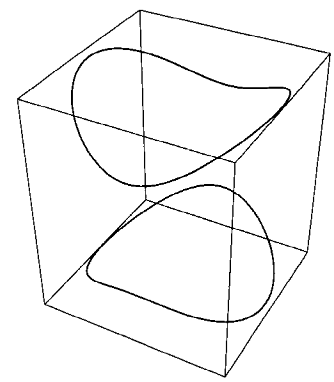
The VOLUME common to two intersecting right CYLINDERS of RADIUS r is

$$V_2(r,r) = \tfrac{16}{3}r^3. \qquad (1)$$

If the two right CYLINDERS are of *different* RADII a and b with $a > b$, then the VOLUME common to them is

$$V_2(a,b) = \tfrac{8}{3}a[(a^2+b^2)E(k) - (a^2-b^2)K(k)], \qquad (2)$$

where $K(k)$ is the complete ELLIPTIC INTEGRAL OF THE FIRST KIND, $E(k)$ is the complete ELLIPTIC INTEGRAL OF THE SECOND KIND, and $k \equiv b/a$ is the MODULUS.



The curves of intersection of two cylinders of RADII a and b, shown above, are given by the parametric equations

$$x(t) = a\cos t \qquad (3)$$

$$y(t) = a\sin t \qquad (4)$$

$$z(t) = \pm\sqrt{b^2 - a^2\sin^2 t} \qquad (5)$$

(Gray 1993).

The VOLUME common to two ELLIPTIC CYLINDERS

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 \qquad \frac{y^2}{b^2} + \frac{z^2}{c'^2} = 1 \qquad (6)$$

with $c < c'$ is

$$V_2(a,c;b,c') = \frac{8ab}{3c}[(c'^2+c^2)E(k) - (c'^2-c^2)K(k)], \qquad (7)$$

where $k = c/c'$ (Bowman 1961, p. 34).

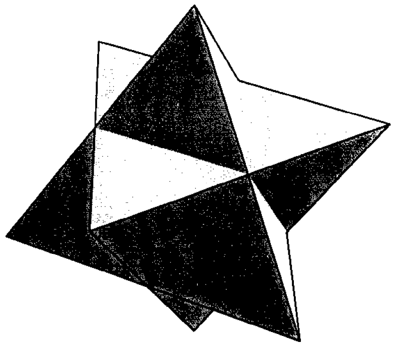
For three CYLINDERS of RADII r intersecting at RIGHT ANGLES, the VOLUME of intersection is

$$V_3(r,r,r) = 8(2-\sqrt{2})r^3. \qquad (8)$$

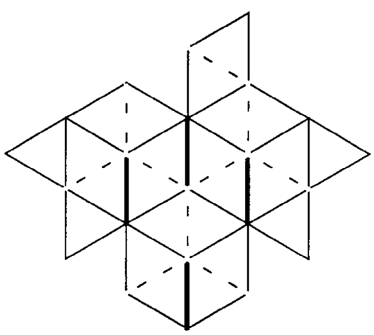
see also BICYLINDER, CYLINDER

References
Bowman, F. *Introduction to Elliptic Functions, with Applications.* New York: Dover, 1961.
Gardner, M. *The Unexpected Hanging and Other Mathematical Diversions.* Chicago, IL: Chicago University Press, pp. 183–185, 1991.
Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 149–150, 1993.
Wells, D. G. #555 in *The Penguin Book of Curious and Interesting Puzzles.* London: Penguin Books, 1992.

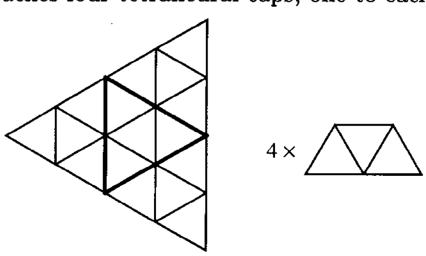
Stella Octangula

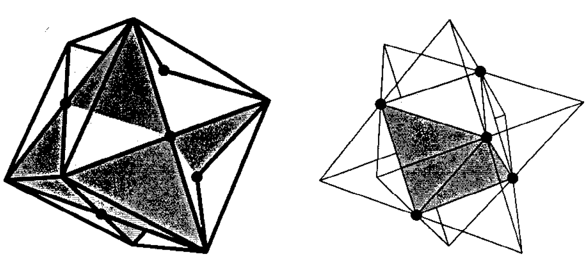


A POLYHEDRON COMPOUND composed of a TETRAHEDRON and its RECIPROCAL (a second TETRAHEDRON rotated 180° with respect to the first). The stella octangula is also called a STELLATED TETRAHEDRON. It can be constructed using the following NET by cutting along the solid lines, folding back along the plain lines, and folding forward along the dotted lines.



Another construction builds a single TETRAHEDRON, then attaches four tetrahedral caps, one to each face.





The edges of the two tetrahedra form the 12 DIAGONALS of a CUBE. The solid common to both tetrahedra is an OCTAHEDRON (Ball and Coxeter 1987).

see also CUBE, OCTAHEDRON, POLYHEDRON COMPOUND, TETRAHEDRON

References
Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 135–137, 1987.
Coxeter, H. S. M. *Introduction to Geometry, 2nd ed.* New York: Wiley, p. 158, 1969.
Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., p. 129, 1989.

Stella Octangula Number

A FIGURATE NUMBER of the form,

$$StOct_n = O_n + 8T_{n-1} = n(2n^2 - 1).$$

The first few are 1, 14, 51, 124, 245, ... (Sloane's A007588). The GENERATING FUNCTION for the stella octangula numbers is

$$\frac{x(x^2 + 10x + 1)}{(x-1)^4} = x + 14x^2 + 51x^3 + 124x^4 + \ldots.$$

References
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, p. 51, 1996.
Sloane, N. J. A. Sequence A007588/M4932 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Stellated Polyhedron

A convex regular POLYHEDRON. Stellated polyhedra include the KEPLER-POINSOT SOLIDS, which consist of three DODECAHEDRON STELLATIONS and one of the ICOSAHEDRON STELLATIONS. Coxeter (1982) shows that 59 ICOSAHEDRON STELLATIONS exist. The CUBE and the TETRAHEDRON cannot be stellated. The OCTAHEDRON has only one stellation, the STELLA OCTANGULA which is a compound of two TETRAHEDRA.

There are therefore a total of $3 + 1 + (59 - 1) + 1 = 63$ stellated POLYHEDRA, although some are COMPOUND POLYHEDRA and therefore not UNIFORM POLYHEDRA. The set of all possible EDGES of the stellations can be obtained by finding all intersections on the facial planes.

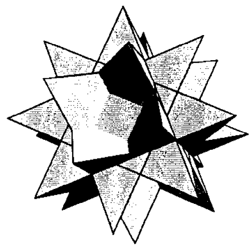
see also ARCHIMEDEAN SOLID STELLATION, DODECAHEDRON STELLATIONS, ICOSAHEDRON STELLATIONS, KEPLER-POINSOT SOLID, POLYHEDRON, STELLA OCTANGULA, STELLATED TRUNCATED HEXAHEDRON, STELLATION, UNIFORM POLYHEDRON

References
Coxeter, H. S. M. *The Fifty-Nine Icosahedra.* New York: Springer-Verlag, 1982.
Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Publications, 1989.
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: University Press, 1974.

Stellated Tetrahedron

see STELLA OCTANGULA

Stellated Truncated Hexahedron



The UNIFORM POLYHEDRON U_{19}, also called the QUASITRUNCATED HEXAHEDRON, whose DUAL POLYHEDRON is the GREAT TRIAKIS OCTAHEDRON. It has SCHLÄFLI SYMBOL $t'\{4,3\}$ and WYTHOFF SYMBOL $2\,3\,|\,\frac{4}{3}$. Its faces are $8\{3\}+6\{\frac{8}{3}\}$. For $a=1$, its CIRCUMRADIUS is

$$R=\tfrac{1}{2}\sqrt{7-4\sqrt{2}}.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, p. 144, 1989.

Stellation

The process of constructing POLYHEDRA by extending the facial PLANES past the EDGES of a given POLYHEDRON.

see also ARCHIMEDEAN SOLID STELLATION, DODECAHEDRON STELLATIONS, FACETING, ICOSAHEDRON STELLATIONS, KEPLER-POINSOT SOLID, POLYHEDRON, STELLA OCTANGULA, STELLATED POLYHEDRON, STELLATED TRUNCATED HEXAHEDRON, STELLATION TRUNCATION, UNIFORM POLYHEDRON

References

Fleurent, G. M. "Symmetry and Polyhedral Stellation Ia and Ib. Symmetry 2: Unifying Human Understanding, Part 1." *Comput. Math. Appl.* **17**, 167–193, 1989.

Messer, P. W. "Les étoilements du rhombitricontaèdre et plus." *Structural Topology* **21**, 25–46, 1995.

Messer, P. W. and Wenninger, M. J. "Symmetry and Polyhedral Stellation. II. Symmetry 2: Unifying Human Understanding, Part 1." *Comput. Math. Appl.* **17**, 195–201, 1989.

Stem-and-Leaf Diagram

The "stem" is a column of the data with the last digit removed. The final digits of each column are placed next to each other in a row next to the appropriate column. Then each row is sorted in numerical order. This diagram was invented by John Tukey.

References

Tukey, J. W. *Explanatory Data Analysis.* Reading, MA: Addison-Wesley, pp. 7–16, 1977.

Step

1.5 times the H-SPREAD.

see also FENCE, H-SPREAD

References

Tukey, J. W. *Explanatory Data Analysis.* Reading, MA: Addison-Wesley, p. 44, 1977.

Step Function

A function on the REALS $\mathbb{R}$ is a step function if it can be written as a finite linear combination of semi-open intervals $[a,b)\subseteq\mathbb{R}$. Therefore, a step function f can be written as

$$f(x)=\alpha_1 f_1(x)+\cdots+\alpha_n f_n(x),$$

where $\alpha_i\in\mathbb{R}$, $f_i(x)=1$ if $x\in[a_i,b_i)$ and 0 otherwise, for $i=1,\ldots,n$.

see also HEAVISIDE STEP FUNCTION

Step Polynomial

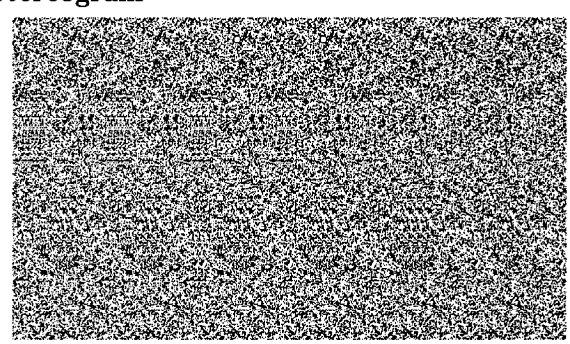
see HERMITE'S INTERPOLATING FUNDAMENTAL POLYNOMIAL

Steradian

The unit of SOLID ANGLE. The SOLID ANGLE corresponding to all of space being subtended is 4π steradian.

see also RADIAN, SOLID ANGLE

Stereogram



A plane image or pair of 2-D images which, when appropriately viewed using both eyes, produces an image which appears to be three-dimensional. By taking a pair of photographs from slightly different angles and then allowing one eye to view each image, a stereogram is not difficult to produce.

Amazingly, it turns out that the 3-D effect can be produced by both eyes looking at a *single* image by defocusing the eyes at a certain distance. Such stereograms are called "random-dot stereograms."

References

Bar-Natan, D. "Random-Dot Stereograms." *Math. J.* **1**, 69–71, 1991.

Fineman, M. *The Nature of Visual Illusion.* New York: Dover, pp. 89–93, 1996.

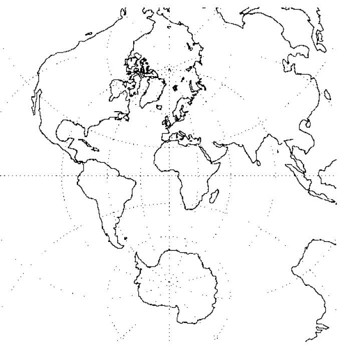
Julesz, B. *Foundations of Cyclopean Perception.* Chicago, IL: University of Chicago Press, 1971.

Julesz, B. "Stereoscopic Vision." *Vision Res.* **26**, 1601–1611, 1986.

Terrell, M. S. and Terrell, R. E. "Behind the Scenes of a Random Dot Stereogram." *Amer. Math. Monthly* **101**, 715–724, 1994.

Tyler, C. "Sensory Processing of Binocular Disparity." In *Vergence Eye Movements: Basic and Clinical Aspects.* Boston, MA: Butterworth, pp. 199–295, 1983.

Stereographic Projection



A MAP PROJECTION in which GREAT CIRCLES are CIRCLES and LOXODROMES are LOGARITHMIC SPIRALS.

$$x = k\cos\phi\sin(\lambda-\lambda_0) \tag{1}$$

$$y = k[\cos\phi_1\sin\phi - \sin\phi_1\cos\phi\cos(\lambda-\lambda_0)], \tag{2}$$

where

$$k = \frac{2}{1+\sin\phi_1\sin\phi+\cos\phi_1\cos\phi\cos(\lambda-\lambda_0)}. \tag{3}$$

The inverse FORMULAS are given by

$$\phi = \sin^{-1}\left(\cos c\sin\phi_1 + \frac{y\sin c\cos\phi_1}{\rho}\right) \tag{4}$$

$$\lambda = \lambda_0 + \tan^{-1}\left(\frac{x\sin c}{\rho\cos\phi_1\cos c - y\sin\phi_1\sin c}\right), \tag{5}$$

where

$$\rho = \sqrt{x^2+y^2} \tag{6}$$

$$c = 2\tan^{-1}(\tfrac{1}{2}\rho). \tag{7}$$

see also GALL'S STEREOGRAPHIC PROJECTION

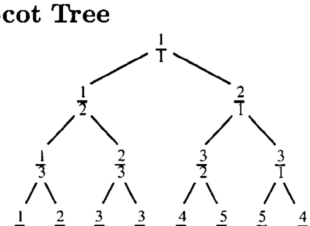
References
Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 150–153, 1967.
Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, pp. 154–163, 1987.

Stereology

The exploration of 3-D space from 2-D sections of PROJECTIONS of solid bodies.

see also AXONOMETRY, CORK PLUG, CROSS-SECTION, PROJECTION, TRIP-LET

Stern-Brocot Tree

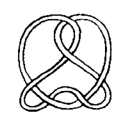


A special type of BINARY TREE obtained by starting with the fractions $\frac{0}{1}$ and $\frac{1}{0}$ and iteratively inserting $(m+m')/(n+n')$ between each two adjacent fractions m/n and m'/n'. The result can be arranged in tree form as illustrated above. The FAREY SEQUENCE F_n defines a subtree of the Stern-Brocot tree obtained by pruning off unwanted branches (Vardi 1991, Graham *et al.* 1994).

see also BINARY TREE, FAREY SEQUENCE, FORD CIRCLE

References
Brocot, A. "Calcul des rouages par approximation, nouvelle méthode." *Revue Chonométrique* **6**, 186–194, 1860.
Graham, R. L.; Knuth, D. E.; and Patashnik, O. *Concrete Mathematics: A Foundation for Computer Science, 2nd ed.* Reading, MA: Addison-Wesley, pp. 116–117, 1994.
Stern, M. A. "Über eine zahlentheoretische Funktion." *J. reine angew. Math.* **55**, 193–220, 1858.
Vardi, I. *Computational Recreations in Mathematica.* Redwood City, CA: Addison-Wesley, p. 253, 1991.

Stevedore's Knot

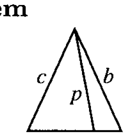


The 6-crossing KNOT 06_{001} having CONWAY-ALEXANDER POLYNOMIAL

$$\Delta(t) = 2t^2 - 5t + 2.$$

References
Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, pp. 225, 1976.

Stewart's Theorem



$$a(p^2+mn) = b^2m + c^2n,$$

where

$$a \equiv m+n.$$

References
Altshiller-Court, N. "Stewart's Theorem." §6B in *College Geometry: A Second Course in Plane Geometry for Colleges and Normal Schools, 2nd ed., rev. enl.* New York: Barnes and Noble, pp. 152–153, 1952.
Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., p. 6, 1967.

Stick Number

Let the stick number $s(K)$ of a KNOT K be the least number of straight sticks needed to make a KNOT K. The smallest stick number of any KNOT is $s(T) = 6$, where T is the TREFOIL KNOT. If J and K are KNOTS, then

$$s(J+K) \leq s(J) + s(K) + 1.$$

For a nontrivial KNOT K, let $c(K)$ be the CROSSING NUMBER (i.e., the least number of crossings in any projection of K). Then

$$\tfrac{1}{2}[5+\sqrt{25+8(c(K)-2)}] \leq s(K) \leq 2c(K).$$

The following table gives the stick number for some common knots.

Knot	s
trefoil knot	6
Whitehead link	8

see also CROSSING NUMBER (LINK), TRIANGLE COUNTING

References

Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, pp. 27–30, 1994.

Stickelberger Relation

Let P be a PRIME IDEAL in D_m not containing m. Then

$$(\Phi(P)) = P^{\sum t\sigma_t^{-1}},$$

where the sum is over all $1 \leq t < m$ which are RELATIVELY PRIME to m. Here D_m is the RING of integers in $\mathbb{Q}(\zeta_m)$, $\Phi(P) = g(P)^m$, and other quantities are defined by Ireland and Rosen (1990).

see also PRIME IDEAL

References

Ireland, K. and Rosen, M. "The Stickelberger Relation and the Eisenstein Reciprocity Law." Ch. 14 in *A Classical Introduction to Modern Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 203–227, 1990.

Stiefel Manifold

The Stiefel manifold of ORTHONORMAL k-frames in $\mathbb{R}^n$ is the collection of vectors $(v_1, \ldots, v_k)$ where v_i is in $\mathbb{R}^n$ for all i, and the k-tuple $(v_1, \ldots, v_k)$ is ORTHONORMAL. This is a submanifold of $\mathbb{R}^{nk}$, having DIMENSION $nk - (k+1)k/2$.

Sometimes the "orthonormal" condition is dropped in favor of the mildly weaker condition that the k-tuple $(v_1, \ldots, v_k)$ is linearly independent. Usually, this does not affect the applications since Stiefel manifolds are usually considered only during HOMOTOPY THEORETIC considerations. With respect to HOMOTOPY THEORY, the two definitions are more or less equivalent since GRAM-SCHMIDT ORTHONORMALIZATION gives rise to a smooth deformation retraction of the second type of Stiefel manifold onto the first.

see also GRASSMANN MANIFOLD

Stiefel-Whitney Class

The ith Stiefel-Whitney class of a REAL VECTOR BUNDLE (or TANGENT BUNDLE or a REAL MANIFOLD) is in the ith cohomology group of the base SPACE involved. It is an OBSTRUCTION to the existence of $(n-i+1)$ REAL linearly independent VECTOR FIELDS on that VECTOR BUNDLE, where n is the dimension of the FIBER. Here, OBSTRUCTION means that the ith Stiefel-Whitney class being NONZERO implies that there do *not* exist $(n-i+1)$ everywhere linearly dependent VECTOR FIELDS (although the Stiefel-Whitney classes are not always the OBSTRUCTION).

In particular, the nth Stiefel-Whitney class is the obstruction to the existence of an everywhere NONZERO VECTOR FIELD, and the first Stiefel-Whitney class of a MANIFOLD is the obstruction to orientability.

see also CHERN CLASS, OBSTRUCTION, PONTRYAGIN CLASS, STIEFEL-WHITNEY NUMBER

Stiefel-Whitney Number

The Stiefel-Whitney number is defined in terms of the STIEFEL-WHITNEY CLASS of a MANIFOLD as follows. For any collection of STIEFEL-WHITNEY CLASSES such that their cup product has the same DIMENSION as the MANIFOLD, this cup product can be evaluated on the MANIFOLD's FUNDAMENTAL CLASS. The resulting number is called the PONTRYAGIN NUMBER for that combination of Pontryagin classes.

The most important aspect of Stiefel-Whitney numbers is that they are COBORDISM invariant. Together, PONTRYAGIN and Stiefel-Whitney numbers determine an oriented MANIFOLD's COBORDISM class.

see also CHERN NUMBER, PONTRYAGIN NUMBER, STIEFEL-WHITNEY CLASS

Stieltjes Constants

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Expanding the RIEMANN ZETA FUNCTION about $z = 1$ gives

$$\zeta(z) = \frac{1}{z-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\gamma_n(z-1)^n, \tag{1}$$

where

$$\gamma_n \equiv \lim_{m\to\infty}\left[\sum_{k=1}^{m}\frac{(\ln k)^n}{k} - \frac{(\ln m)^{n+1}}{n+1}\right]. \tag{2}$$

An alternative definition is given by

$$\gamma_n' \equiv \frac{(-1)^n}{n!}\gamma_n. \tag{3}$$

The case $n = 0$ gives the EULER-MASCHERONI CONSTANT γ. The first few numerical values are given in the following table.

n	γ_n
0	0.5772156649
1	−0.07281584548
2	−0.009690363192
3	0.002053834420
4	0.002325370065
5	0.0007933238173

Briggs (1955–1956) proved that there infinitely many γ_n of each SIGN. Berndt (1972) gave upper bounds of

$$|\gamma_n| < \begin{cases} \frac{4(n-1)!}{\pi^n} & \text{for } n \text{ even} \\ \frac{2(n-1)!}{\pi^n} & \text{for } n \text{ odd.} \end{cases} \tag{4}$$

Vacca (1910) proves that the EULER-MASCHERONI CONSTANT may be expressed as

$$\gamma = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \lfloor \lg k \rfloor, \tag{5}$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION. Hardy (1912) gave the FORMULA

$$\frac{2\gamma_1}{\ln 2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} [2 \lg k - \lfloor \lg(2k) \rfloor] \lfloor \lg k \rfloor. \tag{6}$$

Kluyver (1927) gave similar series for γ_n with $n > 1$.

A set of constants related to γ_n is

$$\delta_n \equiv \lim_{m\to\infty} \left[\sum_{k=1}^{m} (\ln k)^n - \int_1^m (\ln x)^n \, dx - \tfrac{1}{2}(\ln m)^n \right] \tag{7}$$

(Sitaramachandrarao 1986, Lehmer 1988).

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Stieltjes Integral

The Stieltjes integral is a generalization of the RIEMANN INTEGRAL. Let $f(x)$ and $\alpha(x)$ be real-values bounded functions defined on a CLOSED INTERVAL $[a, b]$. Take a partition of the INTERVAL

$$a = x_0 < x_1 < x_2, \ldots < x_{n-1} < x_n = b, \tag{1}$$

and consider the Riemann sum

$$\sum_{i=0}^{n-1} f(\xi_i)[\alpha(x_{i+1}) - \alpha(x_i)] \tag{2}$$

with $\xi_i \in [x_i, x_{i+1}]$. If the sum tends to a fixed number I as $\max(x_{i+1} - x_i) \to 0$, then I is called the Stieltjes integral, or sometimes the RIEMANN-STIELTJES INTEGRAL. The Stieltjes integral of P with respect to F is denoted

$$\int P(x)\, dF(x), \tag{3}$$

where

$$\int P(x)\, dF(x) = \begin{cases} \int f(x)\, dx & \text{for } x \text{ continuous} \\ \sum_x f(x) & \text{for } x \text{ discrete.} \end{cases} \tag{4}$$

If P and F have a common point of discontinuity, then the integral does not exist. However, if the Stieltjes integral exists and F has a derivative F', then

$$\int P(x)\, dF(x) = \int P(x) F'(x)\, dx. \tag{5}$$

For enumeration of many of the integral's properties, see Dresher (1981, p. 105).

see also RIEMANN INTEGRAL

References

Dresher, M. *The Mathematics of Games of Strategy: Theory and Applications.* New York: Dover, 1981.

Hardy, G. H.; Littlewood, J. E.; and Pólya, G. *Inequalities, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 152–155, 1988.

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Stieltjes' Theorem

The $m + 1$ ELLIPSOIDAL HARMONICS when κ_1, κ_2, and κ_3 are given can be arranged in such a way that the rth function has $r - 1$ zeros between $-a^2$ and $-b^2$ and the remaining $m + r - 1$ zeros between $-b^2$ and $-c^2$ (Whittaker and Watson 1990).

see also ELLIPSOIDAL HARMONIC

References

Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis, 4th ed.* Cambridge, England: Cambridge University Press, pp. 560–562, 1990.

Stieltjes-Wigert Polynomial

Orthogonal POLYNOMIALS associated with WEIGHTING FUNCTION

$$w(x) = \pi^{-1/2} k \exp(-k^2 \ln^2 x) = \pi^{-1/2} k x^{-k^2 \ln x} \quad (1)$$

for $x \in (0, \infty)$ and $k > 0$. Using

$$\begin{bmatrix} n \\ \nu \end{bmatrix} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-\nu+1})}{(1-q)(1-q^2)\cdots(1-q^\nu)} \quad (2)$$

where $0 < \nu < n$,

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1, \quad (3)$$

and

$$q = \exp[-(2k^2)^{-1}]. \quad (4)$$

Then

$$p_n(x) = (-1)^n q^{n/2+1/4}[(1-q)(1-q^2)\cdots(1-q^n)]^{-1/2} \times \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix} q^{\nu^2}(-q^{1/2}x)^\nu \quad (5)$$

for $n > 0$ and

$$p_0(x) = q^{1/4}. \quad (6)$$

References

Szegő, G. *Orthogonal Polynomials, 4th ed.* Providence, RI: Amer. Math. Soc., p. 33, 1975.

Stirling's Approximation

Stirling's approximation gives an approximate value for the FACTORIAL function $n!$ or the GAMMA FUNCTION $\Gamma(n)$ for $n \gg 1$. The approximation can most simply be derived for n an INTEGER by approximating the sum over the terms of the FACTORIAL with an INTEGRAL, so that

$$\ln n! = \ln 1 + \ln 2 + \ldots + \ln n = \sum_{k=1}^{n} \ln k \approx \int_1^n \ln x\, dx$$
$$= [x \ln x - x]_1^n = n \ln n - n + 1 \approx n \ln n - n. \quad (1)$$

The equation can also be derived using the integral definition of the FACTORIAL,

$$n! = \int_0^\infty e^{-x} x^n\, dx. \quad (2)$$

Note that the derivative of the LOGARITHM of the integrand can be written

$$\frac{d}{dx} \ln(e^{-x} x^n) = \frac{d}{dx}(n \ln x - x) = \frac{n}{x} - 1. \quad (3)$$

The integrand is sharply peaked with the contribution important only near $x = n$. Therefore, let $x \equiv n + \xi$ where $\xi \ll n$, and write

$$\ln(x^n e^{-x}) = n \ln x - x = n \ln(n + \xi) - (n + \xi). \quad (4)$$

Now,

$$\ln(n+\xi) = \ln\left[n\left(1 + \frac{\xi}{n}\right)\right] = \ln n + \ln\left(1 + \frac{\xi}{n}\right)$$
$$= \ln n + \frac{\xi}{n} - \frac{1}{2}\frac{\xi^2}{n^2} + \ldots, \quad (5)$$

so

$$\ln(x^n e^{-x}) = n \ln(n+\xi) - (n+\xi)$$
$$= n \ln n + \xi - \frac{1}{2}\frac{\xi^2}{n} - n - \xi + \ldots$$
$$= n \ln n - n - \frac{\xi^2}{2n} + \ldots. \quad (6)$$

Taking the EXPONENTIAL of each side then gives

$$x^n e^{-x} \approx e^{n \ln n} e^{-n} e^{-\xi^2/2n} = n^n e^{-n} e^{-\xi^2/2n}. \quad (7)$$

Plugging into the integral expression for $n!$ then gives

$$n! \approx \int_{-n}^{\infty} n^n e^{-n} e^{-\xi^2/2n}\, d\xi \approx n^n e^{-n} \int_{-\infty}^{\infty} e^{-\xi^2/2n}\, d\xi$$
$$= n^n. \quad (8)$$

Evaluating the integral gives

$$n! \approx n^n e^{-n} \sqrt{2\pi n}, \quad (9)$$
$$\approx \sqrt{2\pi}\, n^{n+1/2} e^{-n}. \quad (10)$$

Taking the LOGARITHM of both sides then gives

$$\ln n! \approx n \ln n - n + \tfrac{1}{2} \ln(2\pi n) = (n + \tfrac{1}{2}) \ln n - n + \tfrac{1}{2} \ln(2\pi). \quad (11)$$

This is STIRLING'S SERIES with only the first term retained and, for large n, it reduces to Stirling's approximation

$$\ln n! \approx n \ln n - n. \quad (12)$$

Gosper notes that a better approximation to $n!$ (i.e., one which approximates the terms in STIRLING'S SERIES instead of truncating them) is given by

$$n! \approx \sqrt{(2n + \tfrac{1}{3})\pi}\, n^n e^{-n}. \quad (13)$$

This also gives a much closer approximation to the FACTORIAL of 0, $0! = 1$, yielding $\sqrt{\pi/3} \approx 1.02333$ instead of 0 obtained with the conventional Stirling approximation.

see also STIRLING'S SERIES

Stirling Cycle Number

see STIRLING NUMBER OF THE FIRST KIND

Stirling's Finite Difference Formula

$$f_p = f_0 + \tfrac{1}{2}p(\delta_{1/2} + \delta_{-1/2}) + \tfrac{1}{2}p^2\delta_0^2 + S_3(\delta_{1/2}^2 + \delta_{-1/2}^2) + S_4\delta_0^4 + \ldots,$$

for $p \in [-1/2, 1/2]$, where δ is the CENTRAL DIFFERENCE and

$$S_{2n+1} = \frac{1}{2}\binom{p+n}{2n+1}$$

$$S_{2n+2} = \frac{p}{2n+2}\binom{p+n}{2n+1},$$

with $\binom{n}{k}$ a BINOMIAL COEFFICIENT.

see also CENTRAL DIFFERENCE, STEFFENSON'S FORMULA

References

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Stirling's Formula

see STIRLING'S SERIES

Stirling Number of the First Kind

The definition of the (signed) Stirling number of the first kind is a number $S_n^{(m)}$ such that the number of permutations of n elements which contain exactly m CYCLES is

$$(-1)^{n-m}S_n^{(m)}. \tag{1}$$

This means that $S_n^{(m)} = 0$ for $m > n$ and $S_n^{(n)} = 1$. The GENERATING FUNCTION is

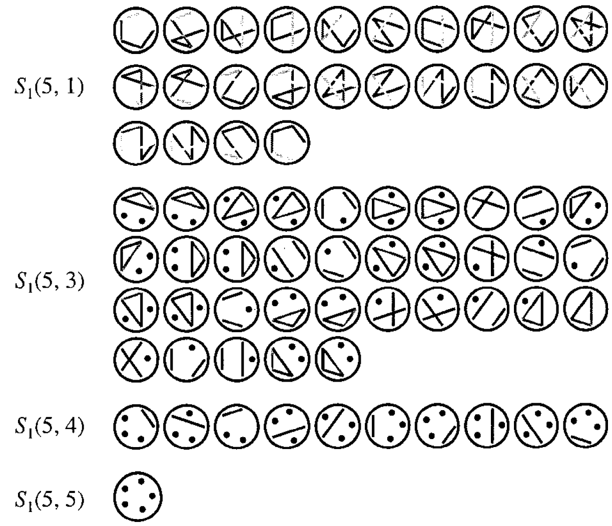
$$x(x-1)\cdots(x-n+1) = \sum_{m=0}^{n} S_n^{(m)}x^m. \tag{2}$$

This is the Stirling number of the first kind returned by the *Mathematica*® (Wolfram Research, Champaign, IL) command `StirlingS1[n,m]`. The triangle of signed Stirling numbers of the first kind is

$$\begin{array}{c} 1 \\ -1 \quad 1 \\ 2 \quad -3 \quad 1 \\ -6 \quad 11 \quad 6 \quad 1 \\ 24 \quad -50 \quad 35 \quad -10 \quad 1 \end{array}$$

(Sloane's A008275).

The NONNEGATIVE version simply gives the number of PERMUTATIONS of n objects having m CYCLES (with cycles in opposite directions counted as distinct) and is obtained by taking the ABSOLUTE VALUE of the signed version. The nonnegative Stirling number of the first kind is denoted $S_1(n,m) = |S_n^{(m)}|$ or $\left[\begin{matrix} n \\ m \end{matrix}\right]$. Diagrams illustrating $S_1(5,1) = 24$, $S_1(5,3) = 35$, $S_1(5,4) = 10$, and $S_1(5,5) = 1$ (Dickau) are shown below.



The nonnegative Stirling numbers of the first kind satisfy the curious identity

$$\sum_{n=1}^{\infty}\left[\sum_{k=0}^{n-2}\frac{(e^x - x - 1)^{k+1}S_1(n, n-k)}{(k+1)!}\right]e^{-xn} = \ln(x+1) \tag{3}$$

(Gosper) and have the GENERATING FUNCTION

$$(1+x)(1+2x)\cdots(1+nx) = \sum_{k=1}^{n} S_1(n,m)x^k \tag{4}$$

and satisfy

$${}_1(n+1,k) = nS_1(n,k) + S_1(n,k-1). \tag{5}$$

The Stirling numbers can be generalized to nonintegral arguments (a sort of "Stirling polynomial") using the identity

$$\frac{\Gamma(j+h)}{j^h\Gamma(j)} = \sum_{k=0}^{\infty}\frac{S_1(h,h-k)}{j^k} = 1 + \frac{(h-1)h}{2j} + \frac{(h-2)(3h-1)(h-1)h}{24j^2} + \frac{(h-3)(h-2)(h-1)^2h^2}{48j^3} + \ldots, \tag{6}$$

which is a generalization of an ASYMPTOTIC SERIES for a ratio of GAMMA FUNCTIONS $\Gamma(j+1/2)/\Gamma(j)$ (Gosper).

see also CYCLE (PERMUTATION), HARMONIC NUMBER, PERMUTATION, STIRLING NUMBER OF THE SECOND KIND

References

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Stirling Number of the Second Kind

The number of ways of partitioning a set of n elements into m nonempty SETS (i.e., m BLOCKS), also called a STIRLING SET NUMBER. For example, the SET $\{1,2,3\}$ can be partitioned into three SUBSETS in one way: $\{\{1\},\{2\},\{3\}\}$; into two SUBSETS in three ways: $\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$, and $\{\{1\},\{2,3\}\}$; and into one SUBSET in one way: $\{\{1,2,3\}\}$.

The Stirling numbers of the second kind are denoted $s_n^{(m)}$, $S_2(n,m)$, $s(n,m)$, or $\left\{ n \atop m \right\}$, so the Stirling numbers of the second kind for three elements are

$$s(3,1) = 1 \tag{1}$$

$$s(3,2) = 3 \tag{2}$$

$$s(3,3) = 1. \tag{3}$$

Since a set of n elements can only be partitioned in a single way into 1 or n SUBSETS,

$$s(n,1) = s(n,n) = 1. \tag{4}$$

The triangle of Stirling numbers of the second kind is

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 3 & & 1 & & & \\
 & & 1 & & 7 & & 6 & & 1 & & \\
 & 1 & & 15 & & 25 & & 10 & & 1 & \\
1 & & 31 & & 90 & & 65 & & 15 & & 1
\end{array}$$

(Sloane's A008277).

The Stirling numbers of the second kind can be computed from the sum

$$s(n,k) = \frac{1}{k!}\sum_{i=0}^{k-1}(-1)^i \binom{k}{i}(k-i)^n, \tag{5}$$

with $\binom{n}{k}$ a BINOMIAL COEFFICIENT, or the GENERATING FUNCTIONS

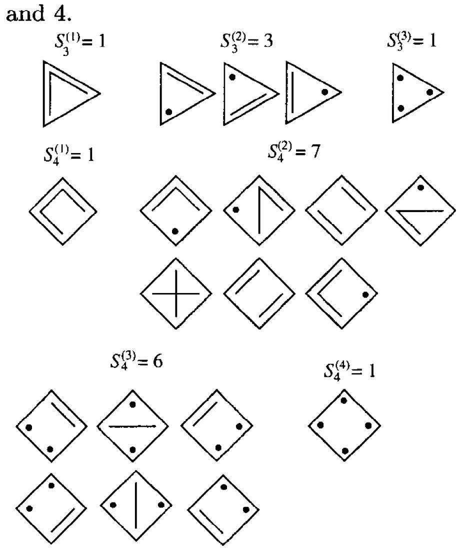
$$x^n = \sum_{m=0}^{n} s(n,m)x(x-1)\cdots(x-m+1), \tag{6}$$

$$\sum_{n\geq k} s(n,k)\frac{x^n}{n!} = \frac{1}{k!}(e^x-1)^k, \tag{7}$$

and

$$\frac{1}{(1-x)(1-2x)\cdots(1-kx)} = \sum_{n=1}^{k} s(n,k)x^n. \tag{8}$$

The following diagrams (Dickau) illustrate the definition of the Stirling numbers of the second kind $s(n,m)$ for $n=3$ and 4.



Stirling numbers of the second kind obey the RECURRENCE RELATIONS

$$s(n,k) = s(n-1,k-1) + ks(n-1,k). \tag{9}$$

An identity involving Stirling numbers of the second kind is

$$f(m,n) \equiv \sum_{k=1}^{\infty} k^n \left(\frac{m}{m+1}\right)^l = (m+1)\sum_{k=1}^{m} k!s(n,k)m^k. \tag{10}$$

It turns out that $f(1,n)$ can have only 0, 2, or 6 as a last DIGIT (Riskin 1995).

see also BELL NUMBER, COMBINATION LOCK, LENGYEL'S CONSTANT, MINIMAL COVER, STIRLING NUMBER OF THE FIRST KIND

References

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Stirling's Series

The ASYMPTOTIC SERIES for the GAMMA FUNCTION is given by

$$\Gamma(z) = e^{-z} z^{z-1/2}\sqrt{2\pi}\left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + \ldots\right) \quad (1)$$

(Sloane's A001163 and A001164). The series for $z!$ is obtained by adding an additional factor of z,

$$z! = \Gamma(z+1) = e^{-z} z^{z+1/2}\sqrt{2\pi}\left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + \ldots\right). \quad (2)$$

The expansion of $\ln\Gamma(z)$ is what is usually called Stirling's series. It is given by the simple analytic expression

$$\ln\Gamma(z) = \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)z^{2n-1}} \quad (3)$$

$$= \tfrac{1}{2}\ln(2\pi) + (z+\tfrac{1}{2})\ln z - z + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \ldots, \quad (4)$$

where B_n is a BERNOULLI NUMBER.

see also BERNOULLI NUMBER, K-FUNCTION, STIRLING'S APPROXIMATION

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 257, 1972.
Arfken, G. "Stirling's Series." §10.3 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 555–559, 1985.
Conway, J. H. and Guy, R. K. "Stirling's Formula." In *The Book of Numbers.* New York: Springer-Verlag, pp. 260–261, 1996.
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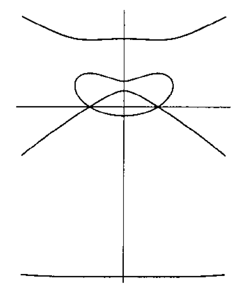
Stirling Set Number

see STIRLING NUMBER OF THE SECOND KIND

Stirrup Curve



A plane curve given by the equation

$$(x^2-1)^2 = y^2(y-1)(y-2)(y+5).$$

References

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Stochastic

see RANDOM VARIABLE

Stochastic Calculus of Variations

see MALLIAVIN CALCULUS

Stochastic Group

The GROUP of all nonsingular $n \times n$ STOCHASTIC MATRICES over a FIELD F. It is denoted $S(n, F)$. If p is PRIME and F is the GALOIS FIELD of ORDER $q = p^m$, $S(n, q)$ is written instead of $S(n, F)$. Particular examples include

$$\begin{aligned} S(2,2) &= \mathbb{Z}_2 \\ S(2,3) &= S_3 \\ S(2,4) &= A_4 \\ S(3,2) &= S_4 \\ S(2,5) &= \mathbb{Z}_4 \times_\theta \mathbb{Z}_5, \end{aligned}$$

where $\mathbb{Z}_2$ is an ABELIAN GROUP, S_n are SYMMETRIC GROUPS on n elements, and $\times_\theta$ denotes the semidirect product with $\theta : \mathbb{Z}_4 \to \mathrm{Aut}(\mathbb{Z}_5)$ (Poole 1995).

see also STOCHASTIC MATRIX

References

Poole, D. G. "The Stochastic Group." *Amer. Math. Monthly* **102**, 798–801, 1995.

Stochastic Matrix

A Stochastic matrix is the transition matrix for a finite MARKOV CHAIN, also called a MARKOV MATRIX. Elements of the matrix must be REAL NUMBERS in the CLOSED INTERVAL [0, 1].

A completely independent type of stochastic matrix is defined as a SQUARE MATRIX with entries in a FIELD F such that the sum of elements in each column equals 1. There are two nonsingular 2×2 STOCHASTIC MATRICES over $\mathbb{Z}_2$ (i.e., the integers mod 2),

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

There are six nonsingular stochastic 3×3 MATRICES over $\mathbb{Z}_3$,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}.$$

In fact, the set S of all nonsingular stochastic $n \times n$ matrices over a FIELD F forms a GROUP under MATRIX MULTIPLICATION. This GROUP is called the STOCHASTIC GROUP.

see also MARKOV CHAIN, STOCHASTIC GROUP

References
Poole, D. G. "The Stochastic Group." *Amer. Math. Monthly* **102**, 798–801, 1995.

Stochastic Process

A stochastic process is a family of RANDOM VARIABLES $\{x(t,\bullet), t \in \mathcal{J}\}$ from some PROBABILITY SPACE $(S, \mathbb{S}, P)$ into a STATE SPACE $(S', \mathbb{S}')$. Here, $\mathcal{J}$ is the INDEX SET of the process.

see also INDEX SET, PROBABILITY SPACE, RANDOM VARIABLE, STATE SPACE

References
Doob, J. L. "The Development of Rigor in Mathematical Probability (1900–1950)." *Amer. Math. Monthly* **103**, 586–595, 1996.

Stochastic Resonance

A stochastic resonance is a phenomenon in which a nonlinear system is subjected to a periodic modulated signal so weak as to be normally undetectable, but it becomes detectable due to resonance between the weak deterministic signal and stochastic NOISE. The earliest definition of stochastic resonance was the maximum of the output signal strength as a function of NOISE (Bulsara and Gammaitoni 1996).

see also KRAMERS RATE, NOISE

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Bulsara, A. R. and Gammaitoni, L. "Tuning in to Noise." *Phys. Today* **49**, 39–45, March 1996.

Stöhr Sequence

Let $a_1 = 1$ and define a_{n+1} to be the least INTEGER greater than a_n for $n \geq k$ which cannot be written as the SUM of at most h addends among the terms a_1, a_2, $\ldots$, a_n.

see also GREEDY ALGORITHM, s-ADDITIVE SEQUENCE, ULAM SEQUENCE

References
Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 233, 1994.

Stokes Phenomenon

The asymptotic expansion of the AIRY FUNCTION $\mathrm{Ai}(z)$ (and other similar functions) has a different form in different sectors of the COMPLEX PLANE.

see also AIRY FUNCTIONS

References
Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 609–611, 1953.

Stokes' Theorem

For w a DIFFERENTIAL $(n-1)$-FORM with compact support on an oriented n-dimensional MANIFOLD M,

$$\int_M dw = \int_{\partial M} w, \qquad (1)$$

where dw is the EXTERIOR DERIVATIVE of the differential form w. This connects to the "standard" GRADIENT, CURL, and DIVERGENCE THEOREMS by the following relations. If f is a function on $\mathbb{R}^3$,

$$\mathrm{grad}(f) = c^{-1}\, df, \qquad (2)$$

where $c : \mathbb{R}^3 \to \mathbb{R}^{3*}$ (the dual space) is the duality isomorphism between a VECTOR SPACE and its dual, given by the Euclidean INNER PRODUCT on $\mathbb{R}^3$. If f is a VECTOR FIELD on a $\mathbb{R}^3$,

$$\mathrm{div}(f) = {}^*d^*c(f), \qquad (3)$$

where $*$ is the HODGE STAR operator. If f is a VECTOR FIELD on $\mathbb{R}^3$,

$$\mathrm{curl}(f) = c^{-1*}dc(f). \qquad (4)$$

With these three identities in mind, the above Stokes' theorem in the three instances is transformed into the GRADIENT, CURL, and DIVERGENCE THEOREMS respectively as follows. If f is a function on $\mathbb{R}^3$ and γ is a curve in $\mathbb{R}^3$, then

$$\int_\gamma \mathrm{grad}(f)\cdot d\mathbf{l} = \int_\gamma df = f(\gamma(1)) - f(\gamma(0)), \qquad (5)$$

which is the GRADIENT THEOREM. If $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a VECTOR FIELD and M an embedded compact 3-manifold with boundary in $\mathbb{R}^3$, then

$$\int_{\partial M} f \cdot dA = \int_{\partial M} {}^*cf = \int_M d{*}cf = \int_M \operatorname{div}(f)\,dV, \quad (6)$$

which is the DIVERGENCE THEOREM. If f is a VECTOR FIELD and M is an oriented, embedded, compact 2-MANIFOLD with boundary in $\mathbb{R}^3$, then

$$\int_{\partial M} f\,dl = \int_{\partial M} cf = \int_M dc(f) = \int_M \operatorname{curl}(f) \cdot dA, \quad (7)$$

which is the CURL THEOREM.

Physicists generally refer to the CURL THEOREM

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s} \quad (8)$$

as Stokes' theorem.

see also CURL THEOREM, DIVERGENCE THEOREM, GRADIENT THEOREM

Stolarsky Array

A INTERSPERSION array given by

1	2	3	5	8	13	21	34	55	⋯
4	6	10	16	26	42	68	110	178	⋯
7	11	18	29	47	76	123	199	322	⋯
9	15	24	39	63	102	165	267	432	⋯
12	19	31	50	81	131	212	343	555	⋯
14	23	37	60	97	157	254	411	665	⋯
17	28	45	73	118	191	309	500	809	⋯
20	32	52	84	136	220	356	576	932	⋯
22	36	58	94	152	246	398	644	1042	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

the first row of which is the FIBONACCI NUMBERS.

see also INTERSPERSION, WYTHOFF ARRAY

References

Kimberling, C. "Interspersions and Dispersions." *Proc. Amer. Math. Soc.* **117**, 313–321, 1993.

Morrison, D. R. "A Stolarsky Array and Wythoff Pairs." In *A Collection of Manuscripts Related to the Fibonacci Sequence.* Santa Clara, CA: Fibonacci Assoc., pp. 134–136, 1980.

Stolarsky-Harborth Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let $b(k)$ be the number of 1s in the BINARY expression of k. Then the number of ODD BINOMIAL COEFFICIENTS $\binom{k}{j}$ where $0 \le j \le k$ is $2^{b(k)}$ (Glaisher 1899, Fine 1947). The number of ODD elements in the first n rows of PASCAL'S TRIANGLE is

$$f(n) = \sum_{k=0}^{n-1} 2^{b(k)}. \quad (1)$$

This function is well approximated by n^θ, where

$$\theta \equiv \frac{\ln 3}{\ln 2} = 1.58496\ldots. \quad (2)$$

Stolarsky and Harborth showed that

$$0.812556 \le \liminf_{n\to\infty} \frac{f(n)}{n^\theta} < 0.812557 < \limsup_{n\to\infty} \frac{f(n)}{n^\theta} = 1. \quad (3)$$

The value

$$SH = \liminf_{n\to\infty} \frac{f(n)}{n^\theta} \quad (4)$$

is called the Stolarsky-Harborth constant.

References

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/stlrsky/stlrsky.html`.

Fine, N. J. "Binomial Coefficients Modulo a Prime." *Amer. Math. Monthly* **54**, 589–592, 1947.

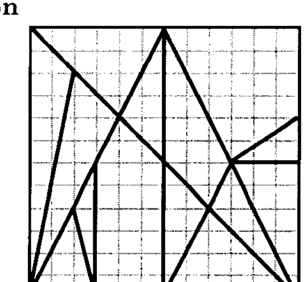
Wolfram, S. "Geometry of Binomial Coefficients." *Amer. Math. Monthly* **91**, 566–571, 1984.

Stolarsky's Inequality

If $0 \le g(x) \le 1$ and g is nonincreasing on the INTERVAL [0,1], then for all possible values of a and b,

$$\int_0^1 g(x^{1/(a+b)})\,dx \ge \int_0^1 g(x^{1/a})\,dx \int_0^1 g(x^{1/b})\,dx.$$

Stomachion



A DISSECTION game similar to TANGRAMS described in fragmentary manuscripts attributed to Archimedes and was referred to as the LOCULUS OF ARCHIMEDES (Archimedes' box) in Latin texts. The word Stomachion has as its root the Greek word for stomach. The game consisted of 14 flat pieces of various shapes arranged in the shape of a square. Like TANGRAMS, the object is to rearrange the pieces to form interesting shapes.

see also DISSECTION, TANGRAM

References

Rorres, C. "Stomachion Introduction." `http://www.mcs.drexel.edu/~crorres/Archimedes/Stomachion/intro.html`.

Rorres, C. "Stomachion Construction." `http://www.mcs.drexel.edu/~crorres/Archimedes/Stomachion/construction.html`.

Stone Space

Let $P(L)$ be the set of all PRIME IDEALS of L, and define $r(a) = \{P|a \notin P\}$. Then the Stone space of L is the TOPOLOGICAL SPACE defined on $P(L)$ by postulating that the sets of the form $r(a)$ are a subbase for the open sets.

see also PRIME IDEAL, TOPOLOGICAL SPACE

References

Grätzer, G. *Lattice Theory: First Concepts and Distributive Lattices.* San Francisco, CA: W. H. Freeman, p. 119, 1971.

Stone-von Neumann Theorem

A theorem which specifies the structure of the generic unitary representation of the Weyl relations and thus establishes the equivalence of Heisenberg's matrix mechanics and Schrödinger's wave mechanics formulations of quantum mechanics.

References

Neumann, J. von. "Die Eindeutigkeit der Schrödingerschen Operationen." *Math. Ann.* **104**, 570–578, 1931.

Stopper Knot

A KNOT used to prevent the end of a string from slipping through a hole.

References

Owen, P. *Knots.* Philadelphia, PA: Courage, p. 11, 1993.

Størmer Number

A Størmer number is a POSITIVE INTEGER n for which the largest PRIME factor p of n^2+1 is at least $2n$. Every GREGORY NUMBER t_x can be expressed uniquely as a sum of t_ns where the ns are Størmer numbers. Conway and Guy (1996) give a table of Størmer numbers reproduced below (Sloane's A005529). In a paper on INVERSE TANGENT relations, Todd (1949) gives a similar compilation.

n	p	n	p	n	p	n	p	n	p
1	2	10	101	19	181	26	617	35	613
2	5	11	61	20	401	27	73	36	1297
4	17	12	29	22	97	28	157	37	137
5	13	14	197	23	53	29	421	39	761
6	37	15	113	24	577	33	109	40	1601
9	41	16	257	25	313	34	89	42	353

see also GREGORY NUMBER, INVERSE TANGENT

References

Conway, J. H. and Guy, R. K. "Størmer's Numbers." *The Book of Numbers.* New York: Springer-Verlag, pp. 245–248, 1996.

Sloane, N. J. A. Sequence A005529/M1505 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Todd, J. "A Problem on Arc Tangent Relations." *Amer. Math. Monthly* **56**, 517–528, 1949.

Straight Angle

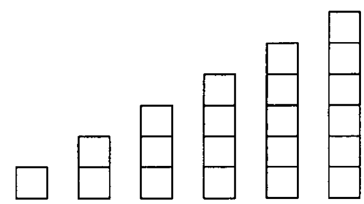
An ANGLE of $180° = \pi$ RADIANS.

see also DIGON, RIGHT ANGLE

Straight Line

see LINE

Straight Polyomino



The straight polyomino of order n is the n-POLYOMINO in which all squares are placed along a line.

see also L-POLYOMINO, SKEW POLYOMINO, SQUARE POLYOMINO, T-POLYOMINO

Straightedge

An idealized mathematical object having a rigorously straight edge which can be used to draw a LINE SEGMENT. Although GEOMETRIC CONSTRUCTIONS are sometimes said to be performed with a RULER and COMPASS, the term straightedge is preferable to RULER since markings on the straightedge (usually assumed to be present on a RULER) are not allowed by the classical Greek rules.

see also COMPASS, GEOMETRIC CONSTRUCTION, GEOMETROGRAPHY, MASCHERONI CONSTANT, POLYGON, PONCELET-STEINER THEOREM, RULER, SIMPLICITY, STEINER CONSTRUCTION

Strange Attractor

An attracting set that has zero MEASURE in the embedding PHASE SPACE and has FRACTAL dimension. Trajectories within a strange attractor appear to skip around randomly.

see also CORRELATION EXPONENT, FRACTAL

References

Benmizrachi, A.; Procaccia, I.; and Grassberger, P. "Characterization of Experimental (Noisy) Strange Attractors." *Phys. Rev. A* **29**, 975–977, 1984.

Grassberger, P. "On the Hausdorff Dimension of Fractal Attractors." *J. Stat. Phys.* **26**, 173–179, 1981.

Grassberger, P. and Procaccia, I. "Measuring the Strangeness of Strange Attractors." *Physica D* **9**, 189–208, 1983a.

Grassberger, P. and Procaccia, I. "Characterization of Strange Attractors." *Phys. Rev. Let.* **50**, 346–349, 1983b.

Lauwerier, H. *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, pp. 137–138, 1991.
Sprott, J. C. *Strange Attractors: Creating Patterns in Chaos.* New York: Henry Holt, 1993.

Strange Loop

A phenomenon in which, whenever movement is made upwards or downwards through the levels of some heirarchial system, the system unexpectedly arrives back where it started. Hofstadter (1987) uses the strange loop as a paradigm in which to interpret paradoxes in logic (such as GRELLING'S PARADOX and RUSSELL'S PARADOX) and calls a system in which a strange loop appears a TANGLED HIERARCHY.

see also GRELLING'S PARADOX, RUSSELL'S PARADOX, TANGLED HIERARCHY

References
Hofstadter, D. R. *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Vintage Books, p. 10, 1989.

Strangers

Two numbers which are RELATIVELY PRIME.

References
Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 145, 1983.

Strassen Formulas

The usual number of scalar operations (i.e., the total number of additions and multiplications) required to perform $n \times n$ MATRIX MULTIPLICATION is

$$M(n) = 2n^3 - n^2 \tag{1}$$

(i.e., n^3 multiplications and $n^3 - n^2$ additions). However, Strassen (1969) discovered how to multiply two MATRICES in

$$S(n) = 7 \cdot 7^{\lg n} - 6 \cdot 4^{\lg n} \tag{2}$$

scalar operations, where lg is the LOGARITHM to base 2, which is less than $M(n)$ for $n > 654$. For n a power of two ($n = 2^k$), the two parts of (2) can be written

$$7 \cdot 7^{\lg n} = 7 \cdot 7^{\lg 2^k} = 7 \cdot 7^k = 7 \cdot 2^{k \lg 7} = 7(2^k)^{\lg 7} = 7n^{\lg 7} \tag{3}$$

$$6 \cdot 4^{\lg n} = 6 \cdot 4^{\lg 2^k} = 6 \cdot 4^{k \lg 2} = 6 \cdot 4^k = 6(2^k)^2 = 6n^2, \tag{4}$$

so (2) becomes

$$S(2^k) = 7n^{\lg 7} - 6n^2. \tag{5}$$

Two 2×2 matrices can therefore be multiplied

$$\mathsf{C} = \mathsf{AB} \tag{6}$$

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{7}$$

with only

$$S(2) = 7 \cdot 2^{\lg 7} - 6 \cdot 2^2 = 49 - 24 = 25 \tag{8}$$

scalar operations (as it turns out, seven of them are multiplications and 18 are additions). Define the seven products (involving a total of 10 additions) as

$$Q_1 \equiv (a_{11} + a_{22})(b_{11} + b_{22}) \tag{9}$$
$$Q_2 \equiv (a_{21} + a_{22})b_{11} \tag{10}$$
$$Q_3 \equiv a_{11}(b_{12} - b_{22}) \tag{11}$$
$$Q_4 \equiv a_{22}(-b_{11} + b_{21}) \tag{12}$$
$$Q_5 \equiv (a_{11} + a_{12})b_{22} \tag{13}$$
$$Q_6 \equiv (-a_{11} + a_{12})(b_{11} + b_{12}) \tag{14}$$
$$Q_7 \equiv (a_{12} - a_{22})(b_{21} + b_{22}). \tag{15}$$

Then the matrix product is given using the remaining eight additions as

$$c_{11} = Q_1 + Q_4 - Q_5 + Q_7 \tag{16}$$
$$c_{21} = Q_2 + Q_4 \tag{17}$$
$$c_{12} = Q_3 + Q_5 \tag{18}$$
$$c_{22} = Q_1 + Q_3 - Q_2 + Q_6 \tag{19}$$

(Strassen 1969, Press *et al.* 1989).

Matrix inversion of a 2×2 matrix A to yield $\mathsf{C} = \mathsf{A}^{-1}$ can also be done in fewer operations than expected using the formulas

$$R_1 \equiv {a_{11}}^{-1} \tag{20}$$
$$R_2 \equiv a_{21} R_1 \tag{21}$$
$$R_3 \equiv R_1 a_{12} \tag{22}$$
$$R_4 \equiv a_{21} R_3 \tag{23}$$
$$R_5 \equiv R_4 - a_{22} \tag{24}$$
$$R_6 \equiv {R_5}^{-1} \tag{25}$$
$$c_{12} = R_3 R_6 \tag{26}$$
$$c_{21} = R_6 R_2 \tag{27}$$
$$R_7 = R_3 c_{21} \tag{28}$$
$$c_{11} = R_1 - R_7 \tag{29}$$
$$c_{22} = -R_6 \tag{30}$$

(Strassen 1969, Press *et al.* 1989). The leading exponent for Strassen's algorithm for a POWER of 2 is $\lg 7 \approx 2.808$. The best leading exponent currently known is 2.376 (Coppersmith and Winograd 1990). It has been shown that the exponent must be at least 2.

see also COMPLEX MULTIPLICATION, KARATSUBA MULTIPLICATION

References

Coppersmith, D. and Winograd, S. "Matrix Multiplication via Arithmetic Programming." *J. Symb. Comput.* **9**, 251–280, 1990.

Pan, V. *How to Multiply Matrices Faster.* New York: Springer-Verlag, 1982.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Is Matrix Inversion an N^3 Process?" §2.11 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 95–98, 1989.

Strassen, V. "Gaussian Elimination is Not Optimal." *Numerische Mathematik* **13**, 354–356, 1969.

Strassman's Theorem

Let $(K, |\cdot|)$ be a complete non-ARCHIMEDEAN VALUATED FIELD, with VALUATION RING R, and let $f(X)$ be a POWER series with COEFFICIENTS in R. Suppose at least one of the COEFFICIENTS is NONZERO (so that f is not identically zero) and the sequence of COEFFICIENTS converges to 0 with respect to $|\cdot|$. Then $f(X)$ has only finitely many zeros in R.

see also ARCHIMEDEAN VALUATION, MAHLER-LECH THEOREM, VALUATION, VALUATION RING

Strassnitzky's Formula

The MACHIN-LIKE FORMULA

$$\tfrac{1}{4}\pi = \cot^{-1} 2 + \cot^{-1} 5 + \cot^{-1} 8.$$

see also MACHIN'S FORMULA, MACHIN-LIKE FORMULAS

Strategy

A set of moves which a player plans to follow while playing a GAME.

see also GAME, MIXED STRATEGY

Stratified Manifold

A set that is a smooth embedded 2-D MANIFOLD except for a subset that consists of smooth embedded curves, except for a set of ISOLATED POINTS.

References

Morgan, F. "What is a Surface?" *Amer. Math. Monthly* **103**, 369–376, 1996.

Strehl Identity

The sum identity

$$\sum_{j=0}^{\infty} \binom{n}{j}^3 = \sum_{k=0}^{\infty} \binom{n}{k}^2 \binom{2(n-k)}{n},$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT.

see also BINOMIAL COEFFICIENT

Striction Curve

A NONCYLINDRICAL RULED SURFACE always has a parameterization of the form

$$\mathbf{x}(u,v) = \boldsymbol{\sigma}(u) + v\boldsymbol{\delta}(u), \tag{1}$$

where $|\boldsymbol{\delta}| = 1$, $\boldsymbol{\sigma}' \cdot \boldsymbol{\delta}' = 0$, and $\boldsymbol{\sigma}$ is called the striction curve of $\mathbf{x}$. Furthermore, the striction curve does not depend on the choice of the base curve. The striction and DIRECTOR CURVES of the HELICOID

$$\mathbf{x}(u,v) = \begin{bmatrix} 0 \\ 0 \\ bu \end{bmatrix} + av \begin{bmatrix} \cos u \\ \sin u \\ 0 \end{bmatrix} \tag{2}$$

are

$$\boldsymbol{\sigma}(u) = \begin{bmatrix} 0 \\ 0 \\ bu \end{bmatrix} \tag{3}$$

$$\boldsymbol{\delta}(u) = \begin{bmatrix} a\cos u \\ a\sin u \\ 0 \end{bmatrix}. \tag{4}$$

For the HYPERBOLIC PARABOLOID

$$\mathbf{x}(u,v) = \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} 0 \\ 1 \\ u \end{bmatrix}, \tag{5}$$

the striction and DIRECTOR CURVES are

$$\boldsymbol{\sigma}(u) = \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

$$\boldsymbol{\delta}(u) = \begin{bmatrix} 0 \\ 1 \\ u \end{bmatrix}. \tag{7}$$

see also DIRECTOR CURVE, DISTRIBUTION PARAMETER, NONCYLINDRICAL RULED SURFACE, RULED SURFACE,

References

Gray, A. "Noncylindrical Ruled Surfaces" and "Examples of Striction Curves of Noncylindrical Ruled Surfaces." §17.3 and 17.4 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 345–350, 1993.

String Rewriting

A SUBSTITUTION MAP in which rules are used to operate on a string consisting of letters of a certain alphabet. String rewriting is a particularly useful technique for generating successive iterations of certain types of FRACTALS, such as the BOX FRACTAL, CANTOR DUST, CANTOR SQUARE FRACTAL, and SIERPIŃSKI CARPET.

see also RABBIT SEQUENCE, SUBSTITUTION MAP

References

Peitgen, H.-O. and Saupe, D. (Eds.). "String Rewriting Systems." §C.1 in *The Science of Fractal Images.* New York: Springer-Verlag, pp. 273–275, 1988.

Wagon, S. "Recursion via String Rewriting." §6.2 in *Mathematica in Action.* New York: W. H. Freeman, pp. 190–196, 1991.

Strip

see CRITICAL STRIP, MÖBIUS STRIP

Strong Convergence

Strong convergence is the type of convergence usually associated with convergence of a SEQUENCE. More formally, a SEQUENCE $\{x_n\}$ of VECTORS in an INNER PRODUCT SPACE E is called convergent to a VECTOR x in E if

$$\|x_n - x\| \to 0 \quad \text{as } n \to \infty.$$

see also CONVERGENT SEQUENCE, INNER PRODUCT SPACE, WEAK CONVERGENCE

Strong Elliptic Pseudoprime

Let n be an ELLIPTIC PSEUDOPRIME associated with (E, P), and let $n+1 = 2^s k$ with k ODD and $s \geq 0$. Then n is a strong elliptic pseudoprime when either $kP \equiv 0 \pmod{n}$ or $2^r kP \equiv 0 \pmod{n}$ for some r with $1 \leq r < s$.

see also ELLIPTIC PSEUDOPRIME

References

Ribenboim, P. *The New Book of Prime Number Records, 3rd ed.* New York: Springer-Verlag, pp. 132–134, 1996.

Strong Frobenius Pseudoprime

A PSEUDOPRIME which obeys an additional restriction beyond that required for a FROBENIUS PSEUDOPRIME. A number n with $(n, 2a) = 1$ is a strong Frobenius pseudoprime with respect to $x - a$ IFF n is a STRONG PSEUDOPRIME with respect to $f(x)$. Every strong Frobenius pseudoprime with respect to $x - a$ is an EULER PSEUDOPRIME to the base a.

Every strong Frobenius pseudoprime with respect to $f(x) = x^2 - bx - c$ such that $((b^2 + 4c)/n) = -1$ is a STRONG LUCAS PSEUDOPRIME with parameters (b, c). Every strong Frobenius pseudoprime n with respect to $x^2 - bx + 1$ is an EXTRA STRONG LUCAS PSEUDOPRIME to the base b.

see also FROBENIUS PSEUDOPRIME

References

Grantham, J. "Frobenius Pseudoprimes." 1996. `http://www.clark.net/pub/grantham/pseudo/pseudo.ps`

Strong Law of Large Numbers

For a set of random variates x_i from a distribution having unit MEAN,

$$P\left(\lim_{n\to\infty} \frac{x_1 + \ldots + x_n}{n}\right) = P\left(\lim_{n\to\infty} \langle x \rangle\right) = 1.$$

This result is due to Kolmogorov.

see also LAW OF TRULY LARGE NUMBERS, STRONG LAW OF SMALL NUMBERS, WEAK LAW OF LARGE NUMBERS

Strong Law of Small Numbers

There aren't enough small numbers to meet the many demands made of them.

References

Gardner, M. "Patterns in Primes are a Clue to the Strong Law of Small Numbers." *Sci. Amer.* **243**, 18–28, Dec. 1980.

Guy, R. K. "The Strong Law of Small Numbers." *Amer. Math. Monthly* **95**, 697–712, 1988.

Strong Lucas Pseudoprime

Let $U(P, Q)$ and $V(P, Q)$ be LUCAS SEQUENCES generated by P and Q, and define

$$D \equiv P^2 - 4Q.$$

Let n be an ODD COMPOSITE NUMBER with $(n, D) = 1$, and $n - (D/n) = 2^s d$ with d ODD and $s \geq 0$, where (a/b) is the LEGENDRE SYMBOL. If

$$U_d \equiv 0 \pmod{n}$$

or

$$V_{2^r d} \equiv 0 \pmod{n}$$

for some r with $0 \leq r < s$, then n is called a strong Lucas pseudoprime with parameters (P, Q).

A strong Lucas pseudoprime is a LUCAS PSEUDOPRIME to the same base. Arnault (1997) showed that any COMPOSITE NUMBER n is a strong Lucas pseudoprime for at most 4/15 of possible bases (unless n is the PRODUCT of TWIN PRIMES having certain properties).

see also EXTRA STRONG LUCAS PSEUDOPRIME, LUCAS PSEUDOPRIME

References

Arnault, F. "The Rabin-Monier Theorem for Lucas Pseudoprimes." *Math. Comput.* **66**, 869–881, 1997.

Ribenboim, P. "Euler-Lucas Pseudoprimes (elpsp(P, Q)) and Strong Lucas Pseudoprimes (slpsp(P, Q))." §2.X.C in *The New Book of Prime Number Records, 3rd ed.* New York: Springer-Verlag, pp. 130–131, 1996.

Strong Pseudoprime

A strong pseudoprime to a base a is an ODD COMPOSITE NUMBER n with $n - 1 = d \cdot 2^s$ (for d ODD) for which either

$$a^d \equiv 1 \pmod{n} \tag{1}$$

or

$$a^{d \cdot 2^r} \equiv -1 \pmod{n} \tag{2}$$

for some $r \in [0, s)$.

The definition is motivated by the fact that a FERMAT PSEUDOPRIME n to the base b satisfies

$$b^{n-1} - 1 \equiv 0 \pmod{n}. \tag{3}$$

But since n is ODD, it can be written $n = 2m + 1$, and

$$b^{2m} - 1 = (b^m - 1)(b^m + 1) \equiv 0 \pmod{n}. \tag{4}$$

If n is PRIME, it must DIVIDE at least one of the FACTORS, but can't DIVIDE both because it would then DIVIDE their difference

$$(b^m + 1) - (b^m - 1) = 2. \tag{5}$$

Therefore,

$$b^m \equiv \pm 1 \pmod{n}, \tag{6}$$

so write $n = 2^a t + 1$ to obtain

$$b^{n-1} - 1 = (b^t - 1)(b^t + 1)(b^{2t} + 1) \cdots (b^{2^{a-1}t} + 1). \tag{7}$$

If n DIVIDES exactly one of these FACTORS but is COMPOSITE, it is a strong pseudoprime. A COMPOSITE number is a strong pseudoprime to at most 1/4 of all bases less than itself (Monier 1980, Rabin 1980). The strong pseudoprimes provide the basis for MILLER'S PRIMALITY TEST and RABIN-MILLER STRONG PSEUDOPRIME TEST.

A strong pseudoprime to the base a is also an EULER PSEUDOPRIME to the base a (Pomerance *et al.* 1980). The strong pseudoprimes include some EULER PSEUDOPRIMES, FERMAT PSEUDOPRIMES, and CARMICHAEL NUMBERS.

There are 4842 strong psp(2) less than 2.5×10^{10}, where a psp(2) is also known as a POULET NUMBER. The strong k-pseudoprime test for $k = 2$, 3, 5 correctly identifies all PRIMES below 2.5×10^{10} with only 13 exceptions, and if 7 is added, then the only exception less than 2.5×10^{10} is 315031751. Jaeschke (1993) showed that there are only 101 strong pseudoprimes for the bases 2, 3, and 5 less than 10^{12}, nine if 7 is added, and none if 11 is added. Also, the bases 2, 13, 23, and 1662803 have no exceptions up to 10^{12}.

If n is COMPOSITE, then there is a base for which n is not a strong pseudoprime. There are therefore no "strong CARMICHAEL NUMBERS." Let ψ_k denote the smallest strong pseudoprime to all of the first k PRIMES taken as bases (i.e, the smallest ODD NUMBER for which the RABIN-MILLER STRONG PSEUDOPRIME TEST on bases less than or equal to k fails). Jaeschke (1993) computed ψ_k from $k = 5$ to 8 and gave upper bounds for $k = 9$ to 11.

$$\begin{aligned}
\psi_1 &= 2047 \\
\psi_2 &= 1373653 \\
\psi_3 &= 25326001 \\
\psi_4 &= 3215031751 \\
\psi_5 &= 2152302898747 \\
\psi_6 &= 3474749660383 \\
\psi_7 &= 34155071728321 \\
\psi_8 &= 34155071728321 \\
\psi_9 &\leq 41234316135705689041 \\
\psi_{10} &\leq 1553360566073143205541002401 \\
\psi_{11} &\leq 56897193526942024370326972321
\end{aligned}$$

(Sloane's A014233). A seven-step test utilizing these results (Riesel 1994) allows all numbers less than 3.4×10^{14} to be tested.

Pomerance *et al.* (1980) have proposed a test based on a combination of STRONG PSEUDOPRIMES and LUCAS PSEUDOPRIMES. They offer a \$620 reward for discovery of a COMPOSITE NUMBER which passes their test (Guy 1994, p. 28).

see also CARMICHAEL NUMBER, MILLER'S PRIMALITY TEST, POULET NUMBER, RABIN-MILLER STRONG PSEUDOPRIME TEST, ROTKIEWICZ THEOREM, STRONG ELLIPTIC PSEUDOPRIME, STRONG LUCAS PSEUDOPRIME

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Strong Pseudoprime Test

see RABIN-MILLER STRONG PSEUDOPRIME TEST

Strong Subadditivity Inequality

$$\phi(A) + \phi(B) - \phi(A \cup B) \geq \phi(A \cap B).$$

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Strong Triangle Inequality

$$|x + y|_p \leq \max(|x|_p, |y|_p)$$

for all x and y.

see also p-ADIC NUMBER, TRIANGLE INEQUALITY

Strongly Connected Component

A maximal subgraph of a DIRECTED GRAPH such that for every pair of vertices u, v in the SUBGRAPH, there is a directed path from u to v and a directed path from v to u.

see also BI-CONNECTED COMPONENT

Strongly Embedded Theorem

The strongly embedded theorem identifies all SIMPLE GROUPS with a strongly 2-embedded SUBGROUP. In particular, it asserts that no SIMPLE GROUP has a strongly 2-embedded 2'-local SUBGROUP.

see also SIMPLE GROUP, SUBGROUP

Strongly Independent

An infinite sequence $\{a_i\}$ of POSITIVE INTEGERS is called strongly independent if any relation $\sum \epsilon_i a_i$, with $\epsilon_i = 0$, ± 1, or ± 2 and $\epsilon_i = 0$ except finitely often, IMPLIES $\epsilon_i = 0$ for all i.

see also WEAKLY INDEPENDENT

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Strongly Triple-Free Set

see TRIPLE-FREE SET

Strophoid

Let C be a curve, let O be a fixed point (the POLE), and let O' be a second fixed point. Let P and P' be points on a line through O meeting C at Q such that $P'Q = QP = QO'$. The LOCUS of P and P' is called the strophoid of C with respect to the POLE O and fixed point O'. Let C be represented parametrically by $(f(t), g(t))$, and let $O = (x_0, y_0)$ and $O' = (x_1, y_1)$. Then the equation of the strophoid is

$$x = f \pm \sqrt{\frac{(x_1 - f)^2 + (y_1 - g)^2}{1 + m^2}} \tag{1}$$

$$y = g \pm \sqrt{\frac{(x_1 - f)^2 + (y_1 - g)^2}{1 + m^2}}, \tag{2}$$

where

$$m \equiv \frac{g - y_0}{f - x_0}. \tag{3}$$

The name strophoid means "belt with a twist," and was proposed by Montucci in 1846 (MacTutor Archive). The polar form for a general strophoid is

$$r = \frac{b \sin(a - 2\theta)}{\sin(a - \theta)}. \tag{4}$$

If $a = \pi/2$, the curve is a RIGHT STROPHOID. The following table gives the strophoids of some common curves.

Curve	Pole	Fixed Point	Strophoid
line	not on line	on line	oblique strophoid
line	not on line	foot of $\perp$ origin to line	right strophoid
circle	center	on circumf.	Freeth's nephroid

see also RIGHT STROPHOID

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MacTutor History of Mathematics Archive. "Right." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Right.html`.
Yates, R. C. "Strophoid." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 217–220, 1952.

Structurally Stable

A MAP $\phi : M \to M$ where M is a MANIFOLD is C^r structurally stable if any C^r perturbation is TOPOLOGICALLY CONJUGATE to ϕ. Here, C^r perturbation means a FUNCTION ψ such that ψ is close to ϕ and the first r derivatives of ψ are close to those of ϕ.

see also TOPOLOGICALLY CONJUGATE

Structure

see LATTICE

Structure Constant

The structure constant is defined as $i\epsilon_{ijk}$, where ϵ_{ijk} is the PERMUTATION SYMBOL. The structure constant forms the starting point for the development of LIE ALGEBRA.

see also LIE ALGEBRA, PERMUTATION SYMBOL

Structure Factor

The structure factor S_Γ of a discrete set Γ is the FOURIER TRANSFORM of δ-scatterers of equal strengths on all points of Γ,

$$S_\Gamma(k) = \int \sum_{x \in \Gamma} \delta(x' - x) e^{-2\pi i k x'}\, dx' = \sum_{x \in \Gamma} e^{-2\pi i k x}.$$

References

Baake, M.; Grimm, U.; and Warrington, D. H. "Some Remarks on the Visible Points of a Lattice." *J. Phys. A: Math. General* **27**, 2669–2674, 1994.

Struve Differential Equation

The ordinary differential equation

$$z^2 y'' + z y' + (z^2 - \nu^2) y = \frac{4(\frac{1}{2}z)^{\nu+1}}{\sqrt{\pi}\,\Gamma(\nu + \frac{1}{2})},$$

where $\Gamma(z)$ is the GAMMA FUNCTION. The solution is

$$y = a J_\nu(z) + b Y_\nu(z) + \mathcal{H}_\nu(z),$$

where $J_\nu(z)$ and $Y_\nu(z)$ are BESSEL FUNCTIONS OF THE FIRST and SECOND KINDS, and $\mathcal{H}_\nu(z)$ is a STRUVE FUNCTION (Abramowitz and Stegun 1972).

see also BESSEL FUNCTION OF THE FIRST KIND, BESSEL FUNCTION OF THE SECOND KIND, STRUVE FUNCTION

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 496, 1972.

Struve Function

Abramowitz and Stegun (1972, pp. 496–499) define the Struve function as

$$\mathcal{H}_\nu(z) = (\tfrac{1}{2}z)^{\nu+1} \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2}z)^{2k}}{\Gamma(k+\frac{3}{2})\Gamma(k+\nu+\frac{3}{2})}, \qquad (1)$$

where $\Gamma(z)$ is the GAMMA FUNCTION. Watson (1966, p. 338) defines the Struve function as

$$\mathcal{H}_\nu(z) \equiv \frac{2(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^1 (1-t^2)^{\nu-1/2} \sin(zt)\,dt. \quad (2)$$

The series expansion is

$$\mathcal{H}_\nu(z) = \sum_{m=0}^{\infty} (-1)^m \frac{(\frac{1}{2}z)^{2m+\nu+1}}{\Gamma(m+\frac{3}{2})\Gamma(\nu+m+\frac{3}{2})}. \qquad (3)$$

For half integral orders,

$$\begin{aligned}\mathcal{H}_{n+1/2}(z) &\\ &= Y_{n+1/2}(z) + \frac{1}{\pi} \sum_{m=0}^{n} \frac{\Gamma(m+\frac{1}{2})(\frac{1}{2}z)^{-2m+n-1/2}}{\Gamma(n+1-m)} \qquad (4)\\ \mathcal{H}_{-(n+1/2)}(z) &= (-1)^n J_{n+1/2}(z). \qquad (5)\end{aligned}$$

The Struve function and its derivatives satisfy

$$\mathcal{H}_{\nu-1}(z) - \mathcal{H}_{\nu+1}(z) = 2\mathcal{H}_\nu'(z) - \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}. \qquad (6)$$

see also ANGER FUNCTION, BESSEL FUNCTION, MODIFIED STRUVE FUNCTION, WEBER FUNCTIONS

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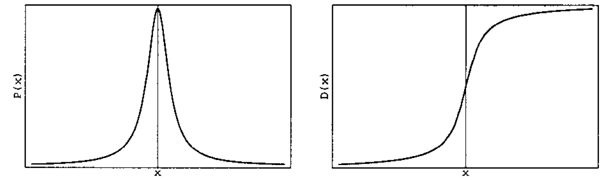
Student's t-Distribution



A DISTRIBUTION published by William Gosset in 1908. His employer, Guinness Breweries, required him to publish under a pseudonym, so he chose "Student." Given n independent measurements x_i, let

$$t \equiv \frac{\bar{x}-\mu}{s/\sqrt{n}}, \qquad (1)$$

where μ is the population MEAN, $\bar{x}$ is the sample MEAN, and s is the ESTIMATOR for population STANDARD DEVIATION (i.e., the SAMPLE VARIANCE) defined by

$$s^2 \equiv \frac{1}{N-1} \sum_{i=1}^{n} (x_i - \bar{x})^2. \qquad (2)$$

Student's t-distribution is defined as the distribution of the random variable t which is (very loosely) the "best" that we can do not knowing σ. If $\sigma = s$, $t = z$ and the distribution becomes the NORMAL DISTRIBUTION. As N increases, Student's t-distribution approaches the NORMAL DISTRIBUTION.

Student's t-distribution is arrived at by transforming to STUDENT'S z-DISTRIBUTION with

$$z \equiv \frac{\bar{x}-\mu}{s}. \qquad (3)$$

Then define

$$t \equiv z\sqrt{n-1}. \qquad (4)$$

The resulting probability and cumulative distribution functions are

$$\begin{aligned} f_r(t) &= \frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{r\pi}\,\Gamma\left(\frac{r}{2}\right)\left(1+\frac{t^2}{r}\right)^{(r+1)/2}} \\ &= \frac{\left(\frac{r}{r+t^2}\right)^{(1+r)/2}}{\sqrt{r}\,B(\frac{1}{2}r,\frac{1}{2})} \qquad (5)\end{aligned}$$

$$\begin{aligned} F_r(t) &= \int_{-\infty}^{t} \frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{r\pi}\,\Gamma\left(\frac{r}{2}\right)\left(1+\frac{t^2}{r}\right)^{(r+1)/2}}\,dt \\ &= \frac{1}{\sqrt{n}B\left(\frac{r}{2},\frac{1}{2}\right)\left(1+\frac{t^2}{r}\right)^{(r+1)/2}} \\ &= \frac{1}{2} + \frac{1}{2}\left[I(1;\tfrac{1}{2}n,\tfrac{1}{2}) - I\left(\frac{r}{r+t^2},\tfrac{1}{2}n,\tfrac{1}{2}\right)\right], \end{aligned} \qquad (6)$$

where

$$r \equiv n-1 \qquad (7)$$

is the number of DEGREES OF FREEDOM, $-\infty < t < \infty$, $\Gamma(z)$ is the GAMMA FUNCTION, $B(a,b)$ is the BETA FUNCTION, and $I(z;a,b)$ is the REGULARIZED BETA FUNCTION defined by

$$I(z;a,b) = \frac{B(z;a,b)}{B(a,b)}. \tag{8}$$

The MEAN, VARIANCE, SKEWNESS, and KURTOSIS of Student's t-distribution are

$$\mu = 0 \tag{9}$$

$$\sigma^2 = \frac{r}{r-2} \tag{10}$$

$$\gamma_1 = 0 \tag{11}$$

$$\gamma_2 = \frac{6}{r-4}. \tag{12}$$

Beyer (1987, p. 514) gives 60%, 70%, 90%, 95%, 97.5%, 99%, 99.5%, and 99.95% confidence intervals, and Goulden (1956) gives 50%, 90%, 95%, 98%, 99%, and 99.9% confidence intervals. A partial table is given below for small r and several common confidence intervals.

r	80%	90%	95%	99%
1	3.08	6.31	12.71	63.66
2	1.89	2.92	4.30	9.92
3	1.64	2.35	3.18	5.84
4	1.53	2.13	2.78	4.60
5	1.48	2.01	2.57	4.03
10	1.37	1.81	2.23	4.14
30	1.31	1.70	2.04	2.75
100	1.29	1.66	1.98	2.63
∞	1.28	1.65	1.96	2.58

The so-called $A(t|n)$ distribution is useful for testing if two observed distributions have the same MEAN. $A(t|n)$ gives the probability that the difference in two observed MEANS for a certain statistic t with n DEGREES OF FREEDOM would be smaller than the observed value purely by chance:

$$A(t|n) = \frac{1}{\sqrt{n}\,B(\frac{1}{2},\frac{1}{2}n)} \int_{-t}^{t} \left(1+\frac{x^2}{n}\right)^{-(1+n)/2} dx. \tag{13}$$

Let X be a NORMALLY DISTRIBUTED random variable with MEAN 0 and VARIANCE σ^2, let Y^2/σ^2 have a CHI-SQUARED DISTRIBUTION with n DEGREES OF FREEDOM, and let X and Y be independent. Then

$$t \equiv \frac{X\sqrt{n}}{Y} \tag{14}$$

is distributed as Student's t with n DEGREES OF FREEDOM.

The noncentral Student's t-distribution is given by

$$P(x) = \frac{n^{n/2}n!}{2^n e^{\lambda^2/2}\Gamma(\frac{1}{2}n)} \times \left\{ \frac{\sqrt{2}\,\lambda x(n+x^2)^{-(1+n/2)}\,{}_1F_1\left(1+\frac{1}{2}n;\frac{3}{2};\frac{\lambda^2x^2}{2(n+x^2)}\right)}{\Gamma[\frac{1}{2}(1+n)]} + \frac{e^{(\lambda^2x^2)/[2(n+x^2)]}\sqrt{\pi}(n+x^2)^{-(n+1)/2}L_{n/2}^{-1/2}\left(-\frac{\lambda^2x^2}{2(n+x^2)}\right)}{\Gamma[\frac{1}{2}(1+n)]} \right\}, \tag{15}$$

where $\Gamma(z)$ is the GAMMA FUNCTION, ${}_1F_1(a;b;z)$ is a CONFLUENT HYPERGEOMETRIC FUNCTION, and $L_n^m(x)$ is an associated LAGUERRE POLYNOMIAL.

see also PAIRED t-TEST, STUDENT'S z-DISTRIBUTION

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Student's z-Distribution

The probability density function and cumulative distribution functions for Student's z-distribution are given by

$$f(z) = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}(1+z^2)^{-n/2} \tag{1}$$

$$D(z) = \frac{-z^{1-n}\Gamma(\frac{1}{2}n)\,{}_2F_1(\frac{1}{2}(n-1),\frac{1}{2}n;\frac{1}{2}(n+1);-z^{-2})}{2\sqrt{\pi}\,\Gamma[\frac{1}{2}(n+1)]}. \tag{2}$$

The MEAN is 0, so the MOMENTS are

$$\mu_1 = 0 \tag{3}$$

$$\mu_2 = \frac{1}{n-3} \tag{4}$$

$$\mu_3 = 0 \tag{5}$$

$$\mu_4 = \frac{3}{(n-3)(n-5)}. \tag{6}$$

The MEAN, VARIANCE, SKEWNESS, and KURTOSIS are

$$\mu = 0 \tag{7}$$

$$\sigma^2 = \frac{1}{n-3} \tag{8}$$

$$\gamma_1 = 0 \tag{9}$$

$$\gamma_2 = \frac{6}{n-5}. \tag{10}$$

Letting

$$z \equiv \frac{(\bar{x} - \mu)}{s}, \tag{11}$$

where x is the sample MEAN and μ is the population MEAN gives STUDENT'S t-DISTRIBUTION.

see also STUDENT'S t-DISTRIBUTION

Study's Theorem

Given three curves ϕ_1, ϕ_2, ϕ_3 with the common group of ordinary points G (which may be empty), let their remaining groups of intersections g_{23}, g_{31}, and g_{12} also be ordinary points. If ϕ_1' is any other curve through g_{23}, then there exist two other curves ϕ_2', ϕ_3' such that the three combined curves $\phi_i\phi_i'$ are of the same order and LINEARLY DEPENDENT, each curve ϕ_k' contains the corresponding group g_{ij}, and every intersection of ϕ_i or ϕ_i' with ϕ_j or ϕ_j' lies on ϕ_k or ϕ_k'.

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Sturm Chain

The series of STURM FUNCTIONS arising in application of the STURM THEOREM.

see also STURM FUNCTION, STURM THEOREM

Sturm Function

Given a function $f(x) \equiv f_0(x)$, write $f_1 \equiv f'(x)$ and define the Sturm functions by

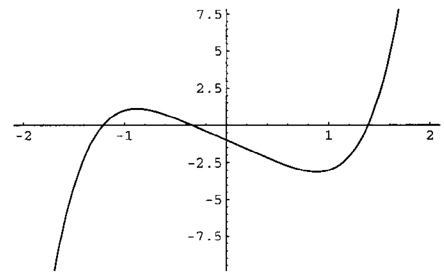
$$f_n(x) = -\left\{f_{n-2}(x) - f_{n-1}(x)\left[\frac{f_{n-2}(x)}{f_{n-1}(x)}\right]\right\}, \tag{1}$$

where $[P(x)/Q(x)]$ is a polynomial quotient. Then construct the following chain of Sturm functions,

$$\begin{aligned} f_0 &= q_0 f_1 - f_2 \\ f_1 &= q_1 f_2 - f_3 \\ f_2 &= q_2 f_3 - f_4 \\ &\vdots \\ f_{s-2} &= q_{s-2} f_{s-1} - f_s, \end{aligned} \tag{2}$$

known as a STURM CHAIN. The chain is terminated when a constant $-f_s(x)$ is obtained.

Sturm functions provide a convenient way for finding the number of real roots of an algebraic equation with real coefficients over a given interval. Specifically, the difference in the number of sign changes between the Sturm functions evaluated at two points $x = a$ and $x = b$ gives the number of real roots in the interval (a, b). This powerful result is known as the STURM THEOREM.



As a specific application of Sturm functions toward finding POLYNOMIAL ROOTS, consider the function $f_0(x) = x^5 - 3x - 1$, plotted above, which has roots -1.21465, -0.334734, $0.0802951 \pm 1.32836i$, and 1.38879 (three of which are real). The DERIVATIVE is given by $f'(x) = 5x^4 - 3$, and the STURM CHAIN is then given by

$$f_0 = x^5 - 3x - 1 \tag{3}$$

$$f_1 = 5x^4 - 3 \tag{4}$$

$$f_2 = \tfrac{1}{5}(12x + 5) \tag{5}$$

$$f_3 = \tfrac{59083}{20736}. \tag{6}$$

The following table shows the signs of f_i and the number of sign changes Δ obtained for points separated by $\Delta x = 2$.

x	f_0	f_1	f_2	f_3	Δ
-2	-1	1	-1	1	3
0	-1	-1	1	1	1
2	1	1	1	1	0

This shows that $3 - 1 = 2$ real roots lie in $(-2, 0)$, and $1 - 0 = 1$ real root lies in $(0, 2)$. Reducing the spacing to $\Delta x = 0.5$ gives the following table.

x	f_0	f_1	f_2	f_3	Δ
-2.0	-1	1	-1	1	3
-1.5	-1	1	-1	1	3
-1.0	1	1	-1	1	2
-0.5	1	-1	-1	1	2
0.0	-1	-1	1	1	1
0.5	-1	-1	1	1	1
1.0	-1	1	1	1	1
1.5	1	1	1	1	0
2.0	1	1	1	1	0

This table isolates the three real roots and shows that they lie in the intervals $(-1.5, -1.0)$, $(-0.5, 0.0)$, and $(1.0, 1.5)$. If desired, the intervals in which the roots fall could be further reduced.

The Sturm functions satisfy the following conditions:

1. Two neighboring functions do not vanish simultaneously at any point in the interval.
2. At a null point of a Sturm function, its two neighboring functions are of different signs.
3. Within a sufficiently small AREA surrounding a zero point of $f_0(x)$, $f_1(x)$ is everywhere greater than zero or everywhere smaller than zero.

see also DESCARTES' SIGN RULE, STURM CHAIN, STURM THEOREM

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Sturm-Liouville Equation

A second-order ORDINARY DIFFERENTIAL EQUATION

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [\lambda w(x) - q(x)]y = 0,$$

where λ is a constant and $w(x)$ is a known function called either the density or WEIGHTING FUNCTION. The solutions (with appropriate boundary conditions) of λ are called EIGENVALUES and the corresponding $u_\lambda(x)$ EIGENFUNCTIONS. The solutions of this equation satisfy important mathematical properties under appropriate boundary conditions (Arfken 1985).

see also ADJOINT OPERATOR, SELF-ADJOINT OPERATOR

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Sturm-Liouville Theory

see STURM-LIOUVILLE EQUATION

Sturm Theorem

The number of REAL ROOTS of an algebraic equation with REAL COEFFICIENTS whose REAL ROOTS are simple over an interval, the endpoints of which are not ROOTS, is equal to the difference between the number of sign changes of the STURM CHAINS formed for the interval ends.

see also STURM CHAIN, STURM FUNCTION

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Sturmian Separation Theorem

Let $\mathsf{A}_r = a_{ij}$ be a SEQUENCE of N SYMMETRIC MATRICES of increasing order with $i, j = 1, 2, \ldots, r$ and $r = 1, 2, \ldots, N$. Let $\lambda_k(\mathsf{A}_r)$ be the kth EIGENVALUE of A_r for $k = 1, 2, \ldots, r$, where the ordering is given by

$$\lambda_1(\mathsf{A}_r) \geq \lambda_2(\mathsf{A}_r) \geq \ldots \geq \lambda_r(\mathsf{A}_r).$$

Then it follows that

$$\lambda_{k+1}(\mathsf{A}_{i+1}) \leq \lambda_k(\mathsf{A}_i) \leq \lambda_k(\mathsf{A}_{i+1}).$$

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Sturmian Sequence

If a SEQUENCE has the property that the BLOCK GROWTH function $B(n) = n + 1$ for all n, then it is said to have minimal block growth, and the sequence is called a Sturmian sequence. An example of this is the sequence arising from the SUBSTITUTION MAP

$$0 \to 01$$
$$1 \to 0,$$

yielding $0 \to 01 \to 010 \to 01001 \to 01001010 \to \ldots$, which gives us the Sturmian sequence 01001010....

STURM FUNCTIONS are sometimes also said to form a Sturmian sequence.

see also STURM FUNCTION, STURM THEOREM

Subalgebra

An ALGEBRA S' which is part of a large ALGEBRA S and shares its properties.

see also ALGEBRA

Subanalytic

$X \subseteq \mathbb{R}^n$ is subanalytic if, for all $x \in \mathbb{R}^n$, there is an open U and $Y \subset \mathbb{R}^{n+m}$ a bounded SEMIANALYTIC set such that $X \cap U$ is the projection of Y into U.

see also SEMIANALYTIC

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Subfactorial

The number of PERMUTATIONS of n objects in which no object appear in its natural place (i.e., so-called "DERANGEMENTS").

$$!n \equiv n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \tag{1}$$

or

$$!n \equiv \left[\frac{n!}{e}\right], \tag{2}$$

where $k!$ is the usual FACTORIAL and $[x]$ is the NINT function. The first few values are $!1 = 0$, $!2 = 1$, $!3 = 2$, $!4 = 9$, $!5 = 44$, $!6 = 265$, $!7 = 1854$, $!8 = 14833$, ... (Sloane's A000166). For example, the only DERANGEMENTS of $\{1,2,3\}$ are $\{2,3,1\}$ and $\{3,1,2\}$, so $!3 = 2$. Similarly, the DERANGEMENTS of $\{1,2,3,4\}$ are $\{2,1,4,3\}$, $\{2,3,4,1\}$, $\{2,4,1,3\}$, $\{3,1,4,2\}$, $\{3,4,1,2\}$, $\{3,4,2,1\}$, $\{4,1,2,3\}$, $\{4,3,1,2\}$, and $\{4,3,2,1\}$, so $!4 = 9$.

The subfactorials are also called the RENCONTRES NUMBERS and satisfy the RECURRENCE RELATIONS

$$!n = n \cdot !(n-1) + (-1)^n \tag{3}$$

$$!(n+1) = n[!n + !(n-1)]. \tag{4}$$

The subfactorial can be considered a special case of a restricted ROOKS PROBLEM.

The only number equal to the sum of subfactorials of its digits is

$$148,349 = !1 + !4 + !8 + !3 + !4 + !9 \tag{5}$$

(Madachy 1979).

see also DERANGEMENT, FACTORIAL, MARRIED COUPLES PROBLEM, ROOKS PROBLEM, SUPERFACTORIAL

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Subfield

If a subset S of the elements of a FIELD F satisfies the FIELD AXIOMS with the same operations of F, then S is called a subfield of F. Let F be a FINITE FIELD of order p^n, then there exists a subfield of ORDER p^m for PRIME p IFF m DIVIDES n.

see also FIELD, SUBMANIFOLD, SUBSPACE

Subgraph

A GRAPH G' whose VERTICES and EDGES form subsets of the VERTICES and EDGES of a given GRAPH G. If G' is a subgraph of G, then G is said to be a SUPERGRAPH of G'.

see also GRAPH (GRAPH THEORY), SUPERGRAPH

Subgroup

A subset of GROUP elements which satisfies the four GROUP requirements. The ORDER of any subgroup of a GROUP ORDER h must be a DIVISOR of h.

see also CARTAN SUBGROUP, COMPOSITION SERIES, FITTING SUBGROUP, GROUP

Sublime Number

Let $\tau(n)$ and $\sigma(n)$ denote the number and sum of the divisors of n, respectively (i.e., the zeroth- and first-order DIVISOR FUNCTIONS). A number N is called sublime if $\tau(N)$ and $\sigma(N)$ are both PERFECT NUMBERS. The only two known sublime numbers are 12 and

$$6086555670238378989670371734243169\cdots$$
$$\cdots 22657830773351885970528324860512791691264.$$

It is not known if any ODD sublime number exists.

see also DIVISOR FUNCTION, PERFECT NUMBER

Submanifold

A C^∞ (infinitely differentiable) MANIFOLD is said to be a submanifold of a C^∞ MANIFOLD M' if M is a SUBSET of M' and the IDENTITY MAP of M into M' is an embedding.

see also MANIFOLD, SUBFIELD, SUBSPACE

Submatrix

An $p \times q$ submatrix of an $m \times n$ MATRIX (with $p \le m$, $n \le q$) is a $p \times q$ MATRIX formed by taking a block of the entries of this size from the original matrix.

see also MATRIX

Subnormal

L is a subnormal SUBGROUP of H if there is a a "normal series" (in the sense of Jordan-Holder) from L to H.

Subordinate Norm

see NATURAL NORM

Subscript

A quantity displayed below the normal line of text (and generally in a smaller point size), as the "i" in a_i, is called a subscript. Subscripts are commonly used to indicate indices (a_{ij} is the entry in the ith row and jth column of a MATRIX A), partial differentiation (y_x is an abbreviation for $\partial y/\partial x$), and a host of other operations and notations in mathematics.

see also SUPERSCRIPT

Subsequence

A subsequence of a SEQUENCE $S = \{x_i\}_{i=1}^n$ is a derived sequence $\{y_i\}_{i=1}^N = \{x_{i+j}\}$ for some $j \geq 0$ and $N \leq n - j$. More generally, the word subsequence is sometimes used to mean a sequence derived from a sequence S by discarding some of its terms.

see also LOWER-TRIMMED SUBSEQUENCE, UPPER-TRIMMED SUBSEQUENCE

Subset

A portion of a SET. B is a subset of A (written $B \subseteq A$) IFF every member of B is a member of A. If B is a PROPER SUBSET of A (i.e., a subset other than the set itself), this is written $B \subset A$.

A SET of n elements has 2^n subsets (including the set itself and the EMPTY SET). For sets of $n = 1, 2, \ldots$ elements, the numbers of subsets are therefore 2, 4, 8, 16, 32, 64, ... (Sloane's A000079). For example, the set $\{1\}$ has the two subsets $\varnothing$ and $\{1\}$. Similarly, the set $\{1, 2\}$ has subsets $\varnothing$ (the EMPTY SET, $\{1\}$, $\{2\}$, and $\{1, 2\}$.

see also EMPTY SET, IMPLIES, k-SUBSET, PROPER SUBSET, SUPERSET, VENN DIAGRAM

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Subspace

Let $\mathbb{V}$ be a REAL VECTOR SPACE (e.g., the real continuous functions $C(I)$ on a CLOSED INTERVAL I, 2-D EUCLIDEAN SPACE $\mathbb{R}^2$, the twice differentiable real functions $C^{(2)}(I)$ on I, etc.). Then $\mathbb{W}$ is a real SUBSPACE of $\mathbb{V}$ if $\mathbb{W}$ is a SUBSET of $\mathbb{V}$ and, for every $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{W}$ and $t \in \mathbb{R}$ (the REALS), $\mathbf{w}_1 + \mathbf{w}_2 \in \mathbb{W}$ and $t\mathbf{w}_1 \in \mathbb{W}$. Let (H) be a homogeneous system of linear equations in $x_1, \ldots, x_n$. Then the SUBSET S of $\mathbb{R}^n$ which consists of all solutions of the system (H) is a subspace of $\mathbb{R}^n$.

More generally, let F_q be a FIELD with $q = p^\alpha$, where p is PRIME, and let $F_{q,n}$ denote the n-D VECTOR SPACE over F_q. The number of k-D linear subspaces of $F_{q,n}$ is

$$N(F_{q,n}) = \binom{n}{k}_q,$$

where this is the q-BINOMIAL COEFFICIENT (Aigner 1979, Exton 1983). The asymptotic limit is

$$N(F_{q,n}) = \begin{cases} c_e q^{n^2/4}[1 + o(1)] & \text{for } n \text{ even} \\ c_o q^{n^2/4}[1 + o(1)] & \text{for } n \text{ odd,} \end{cases}$$

where

$$c_e = \frac{\sum_{k=-\infty}^{\infty} q^{-k^2}}{\prod_{j=1}^{\infty}(1 - q^{-j})}$$

$$c_o = \frac{\sum_{k=-\infty}^{\infty} q^{-(k+1/2)^2}}{\prod_{j=1}^{\infty}(1 - q^{-j})}$$

(Finch). The case $q = 2$ gives the q-ANALOG of the WALLIS FORMULA.

see also q-BINOMIAL COEFFICIENT, SUBFIELD, SUBMANIFOLD

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Substitution Group

see PERMUTATION GROUP

Substitution Map

A MAP which uses a set of rules to transform elements of a sequence into a new sequence using a set of rules which "translate" from the original sequence to its transformation. For example, the substitution map $\{1 \to 0, 0 \to 11\}$ would take 10 to 011.

see also GOLDEN RATIO, MORSE-THUE SEQUENCE, STRING REWRITING, THUE CONSTANT

Subtend

Given a geometric object O in the PLANE and a point P, let A be the ANGLE from one edge of O to the other with VERTEX at P. Then O is said to subtend an ANGLE A from P.

see also ANGLE, VERTEX ANGLE

Subtraction

Subtraction is the operation of taking the DIFFERENCE $x - y$ of two numbers x and y. Here, the symbol between the x and y is called the MINUS SIGN and $x - y$ is read "x MINUS y."

see also ADDITION, DIVISION, MINUS, MINUS SIGN, MULTIPLICATION

Succeeds

The relationship x succeeds (or FOLLOWS) y is written $x \succ y$. The relation x succeeds or is equal to y is written $x \succeq y$.

see also PRECEDES

Successes

see DIFFERENCE OF SUCCESSES

Sufficient

A CONDITION which, if true, guarantees that a result is also true. (However, the result may also be true if the CONDITION is not met.) If a CONDITION is both NECESSARY and SUFFICIENT, then the result is said to be true IFF ("if and only if") the CONDITION holds.

For example, the condition that a decimal number n end in the DIGIT 2 is a sufficient but not NECESSARY condition that n be EVEN.

see also IFF, IMPLIES, NECESSARY

Suitable Number

see IDONEAL NUMBER

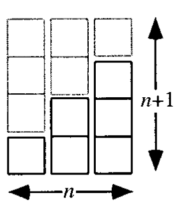
Sum

A sum is the result of an ADDITION. For example, adding 1, 2, 3, and 4 gives the sum 10, written

$$1+2+3+4=10. \tag{1}$$

The numbers being summed are called ADDENDS, or sometimes SUMMANDS. The summation operation can also be indicated using a capital sigma with upper and lower limits written above and below, and the index indicated below. For example, the above sum could be written

$$\sum_{k=1}^{4} k = 10. \tag{2}$$



A simple graphical proof of the sum $\sum_{k=1}^{n} k = n(n+1)/2$ can also be given. Construct a sequence of stacks of boxes, each 1 unit across and k units high, where $k = 1, 2, \ldots, n$. Now add a rotated copy on top, as in the above figure. Note that the resulting figure has WIDTH n and HEIGHT $n+1$, and so has AREA $n(n+1)$. The desired sum is half this, so the AREA of the boxes in the sum is $n(n+1)/2$. Since the boxes are of unit width, this is also the value of the sum.

The sum can also be computed using the first EULER-MACLAURIN INTEGRATION FORMULA

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \tfrac{1}{2}f(1) + \tfrac{1}{2}f(n) + \tfrac{1}{2!}B_2[f'(n) - f'(1)] + \ldots \tag{3}$$

with $f(k) = k$. Then

$$\sum_{k=1}^{n} k = \int_1^n x\,dx + \tfrac{1}{2}\cdot 1 + \tfrac{1}{2}\cdot n + \tfrac{1}{6}(1-1) + \ldots = \tfrac{1}{2}(n^2-1) - \tfrac{1}{2} + h + \tfrac{1}{2}n = \tfrac{1}{2}n(n+1). \tag{4}$$

The general finite sum of integral POWERS can be given by the expression

$$\sum_{k=1}^{n} k^p = \frac{(B+n+1)^{[p+1]} - B^{[p+1]}}{p+1}, \tag{5}$$

where the NOTATION $B^{[k]}$ means the quantity in question is raised to the appropriate POWER k and all terms of the form B^m are replaced with the corresponding BERNOULLI NUMBERS B_m. It is also true that the COEFFICIENTS of the terms in such an expansion sum to 1, as stated by Bernoulli without proof (Boyer 1943).

An analytic solution for a sum of POWERS of integers is

$$\sum_{k=1}^{n} k^p = \zeta(-p) - \zeta(-p, 1+n), \tag{6}$$

where $\zeta(z)$ is the RIEMANN ZETA FUNCTION and $\zeta(z;a)$ is the HURWITZ ZETA FUNCTION. For the special case of p a POSITIVE integer, FAULHABER'S FORMULA gives the SUM explicitly as

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{k=1}^{p+1}(-1)^{\delta_{kp}}\binom{p+1}{k}B_{p+1-k}n^k, \tag{7}$$

where δ_{kp} is the KRONECKER DELTA, $\binom{n}{k}$ is a BINOMIAL COEFFICIENT, and B_k is a BERNOULLI NUMBER. Written explicitly in terms of a sum of POWERS,

$$\sum_{k=1}^{n} k^p = \frac{B_k p!}{k!(p-k+1)!}n^{p-k+1}. \tag{8}$$

Computing the sums for $p = 1, \ldots, 10$ gives

$$\sum_{k=1}^{n} k = \tfrac{1}{2}(n^2+n) \tag{9}$$

$$\sum_{k=1}^{n} k^2 = \tfrac{1}{6}(2n^3+3n^2+n) \tag{10}$$

$$\sum_{k=1}^{n} k^3 = \tfrac{1}{4}(n^4+2n^3+n^2) \tag{11}$$

$$\sum_{k=1}^{n} k^4 = \tfrac{1}{30}(6n^5+15n^4+10n^3-n) \tag{12}$$

$$\sum_{k=1}^{n} k^5 = \tfrac{1}{12}(2n^6+6n^5+5n^4-n^2) \tag{13}$$

$$\sum_{k=1}^{n} k^6 = \tfrac{1}{42}(6n^7 + 21n^6 + 21n^5 - 7n^3 + n) \quad (14)$$

$$\sum_{k=1}^{n} k^7 = \tfrac{1}{24}(3n^8 + 12n^7 + 14n^6 - 7n^4 + 2n^2) \quad (15)$$

$$\sum_{k=1}^{n} k^8 = \tfrac{1}{90}(10n^9 + 45n^8 + 60n^7 - 42n^5 + 20n^3 - 3n) \quad (16)$$

$$\sum_{k=1}^{n} k^9 = \tfrac{1}{20}(2n^{10} + 10n^9 + 15n^8 - 14n^6 + 10n^4 - 3n^2) \quad (17)$$

$$\sum_{k=1}^{n} k^{10} = \tfrac{1}{66}(6n^{11} + 33n^{10} + 55n^9 - 66n^7 + 66n^5 - 33n^3 + 5n). \quad (18)$$

Factoring the above equations results in

$$\sum_{k=1}^{n} k = \tfrac{1}{2}n(n+1) \quad (19)$$

$$\sum_{k=1}^{n} k^2 = \tfrac{1}{6}n(n+1)(2n+1) \quad (20)$$

$$\sum_{k=1}^{n} k^3 = \tfrac{1}{4}n^2(n+1)^2 \quad (21)$$

$$\sum_{k=1}^{n} k^4 = \tfrac{1}{30}n(n+1)(2n+1)(3n^2 + 3n - 1) \quad (22)$$

$$\sum_{k=1}^{n} k^5 = \tfrac{1}{12}n^2(n+1)^2(2n^2 + 2n - 1) \quad (23)$$

$$\sum_{k=1}^{n} k^6 = \tfrac{1}{42}n(n+1)(2n+1)(3n^4 + 6n^3 - 3n + 1) \quad (24)$$

$$\sum_{k=1}^{n} k^7 = \tfrac{1}{24}n^2(n+1)^2(3n^4 + 6n^3 - n^2 - 4n + 2) \quad (25)$$

$$\sum_{k=1}^{n} k^8 = \tfrac{1}{90}n(n+1)(2n+1)(5n^6 + 15n^5 + 5n^4 - 15n^3 - n^2 + 9n - 3) \quad (26)$$

$$\sum_{k=1}^{n} k^9 = \tfrac{1}{20}n^2(n+1)^2(n^2 + n - 1) \times (2n^4 + 4n^3 - n^2 - 3n + 3) \quad (27)$$

$$\sum_{k=1}^{n} k^{10} = \tfrac{1}{66}n(n+1)(2n+1)(n^2 + n - 1) \times (3n^6 + 9n^5 + 2n^4 - 11n^3 + 3n^2 + 10n - 5). \quad (28)$$

From the above, note the interesting identity

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2. \quad (29)$$

Sums of the following type can also be done analytically.

$$\left(\sum_{k=0}^{\infty} x^k\right)^2 = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} 1\right) x^n = \sum_{n=0}^{\infty}(n+1)x^n \quad (30)$$

$$\left(\sum_{k=0}^{\infty} x^k\right)^3 = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} k\right) x^n = \frac{1}{2}\sum_{n=0}^{\infty}(n+1)(n+2)x^n \quad (31)$$

$$\begin{aligned}\left(\sum_{k=0}^{\infty} x^k\right)^4 &= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} \tfrac{1}{2}(k+1)(k+2)\right] x^n \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} k^2 + 3k + 2\right) x^n \\ &= \frac{1}{2}\sum_{n=0}^{\infty}[\tfrac{1}{6}n(n+1)(2n+1) + 3\tfrac{1}{2}n(n+1) + 2(n+1)]x^n \\ &= \frac{1}{12}\sum_{n=0}^{\infty}(n+1)[n(2n+1) + 9n + 12]x^n \\ &= \frac{1}{12}\sum_{n=0}^{\infty}(n+1)(2n^2 + 10n + 12)x^n \\ &= \frac{1}{6}\sum_{n=0}^{\infty}(n+1)(n+2)(n+3)x^n. \end{aligned} \quad (32)$$

By INDUCTION, the sum for an arbitrary POWER p is

$$\left(\sum_{k=0}^{\infty} x^k\right)^p = \frac{1}{(p-1)!}\sum_{n=0}^{\infty}\frac{(n+p-1)!}{n!}x^n. \quad (33)$$

Other analytic sums include

$$\left(\sum_{k=0}^{n} x^k\right)^2 = \frac{1}{(p-1)!}\sum_{k=0}^{2n}\frac{(n-|n-k|+p-1)!}{(n-|n-k|)!}x^k \quad (34)$$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)^2 = \sum_{n=0}^{\infty} {a_n}^2 x^{2n} + 2\sum_{\substack{n=1\\ i+j=n\\ i<j}}^{\infty} a_i a_j x^n. \quad (35)$$

$$\begin{aligned}\sum xy &= x_1y_1 + x_1y_2 + \ldots + x_2y_1 + x_2y_2 + \ldots \\ &= (x_1 + x_2 + \ldots)y_1 + (x_1 + x_2 + \ldots)y_2 \\ &= \left(\sum x\right)(y_1 + y_2 + \ldots) = \sum x \sum y, \end{aligned} \quad (36)$$

so

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j = \left(\sum_{i=1}^{m} x_i\right)\left(\sum_{j=1}^{n} y_j\right). \quad (37)$$

$$\sum_{j=0}^{n} j x^j = \frac{nx^{n+2}-(n+1)x^{n+1}+x}{(x-1)^2} \quad (38)$$

$$\sum_{j=1}^{n} \frac{{x_j}^r}{\prod_{\substack{k=1\\k\neq j}}^{n}(x_j-x_k)} = \begin{cases} 0 & \text{for } 0 \leq r < n-1 \\ 1 & \text{for } r=n-1 \\ \sum_{j=1}^{n} x_j & \text{for } r=n \end{cases} \quad (39)$$

$$\sum_{k=1}^{n} \frac{\prod_{\substack{r=1\\r\neq k}}^{n}(x+k-r)}{\prod_{\substack{r=1\\r\neq k}}^{n}(k-r)} = 1 \quad (40)$$

$$(n+1)\sum_{m=1}^{n} m^k = \sum_{m=1}^{n}\left[m^{k+1} + \sum_{p=1}^{n}\left(\sum_{m=1}^{p} m^k\right)\right]. \quad (41)$$

To minimize the sum of a set of squares of numbers $\{x_i\}$ about a given number x_0

$$S \equiv \sum_i (x_i - x_0)^2 = \sum_i {x_i}^2 - 2x_0 \sum x_i + N{x_0}^2, \quad (42)$$

take the DERIVATIVE.

$$\frac{d}{dx_0} S = -2\sum_i x_i + 2Nx_0 = 0. \quad (43)$$

Solving for x_0 gives

$$x_0 \equiv \bar{x} = \frac{1}{N}\sum_i x_i, \quad (44)$$

so S is maximized when x_0 is set to the MEAN.

see also ARITHMETIC SERIES, BERNOULLI NUMBER, CLARK'S TRIANGLE, CONVERGENCE IMPROVEMENT, DEDEKIND SUM, DOUBLE SUM, EULER SUM, FACTORIAL SUM, FAULHABER'S FORMULA, GABRIEL'S STAIRCASE, GAUSSIAN SUM, GEOMETRIC SERIES, GOSPER'S METHOD, HURWITZ ZETA FUNCTION, INFINITE PRODUCT, KLOOSTERMAN'S SUM, LEGENDRE SUM, LERCH TRANSCENDENT, PASCAL'S TRIANGLE, PRODUCT, RAMANUJAN'S SUM, RIEMANN ZETA FUNCTION, WHITNEY SUM

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Sum-Product Number

A sum-product number is a number n such that the sum of n's digits times the product of n's digit is n itself, for example

$$135 = (1+3+5)(1\cdot 3\cdot 5).$$

The only sum-product numbers less than 10^7 are 1, 135, and 144.

see also AMENABLE NUMBER

Sum Rule

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x),$$

where d/dx denotes a derivative and $f'(x)$ and $g'(x)$ are the derivatives of f and g, respectively.

see also DERIVATIVE

Summand

see ADDEND

Summatory Function

For an discrete function $f(n)$, the summatory function is defined by

$$F(n) \equiv \sum_{k\in D}^{n} f(k),$$

where D is the DOMAIN of the function.

see also DIVISOR FUNCTION, MANGOLDT FUNCTION, MERTENS FUNCTION, RUDIN-SHAPIRO SEQUENCE, TAU FUNCTION, TOTIENT FUNCTION

Sup

see SUPREMUM, SUPREMUM LIMIT

Super-3 Number

An INTEGER n such that $3n^3$ contains three consecutive 3s in its DECIMAL representation. The first few super-3 numbers are 261, 462, 471, 481, 558, 753, 1036, ... (Sloane's A014569). A. Anderson has conjectured that all numbers ending in 471, 4710, or 47100 are super-3 (Pickover 1995).

For a digit d, super-3 numbers can be generalized to super-d numbers n such that dn^d contains d ds in its DECIMAL representation. The following table gives the first few super-d numbers for small d.

d	Sloane	Super-d numbers
2	032743	19, 31, 69, 81, 105, 106, 107, 119, ...
3	014569	261, 462, 471, 481, 558, 753, 1036, ...
4	032744	1168, 4972, 7423, 7752, 8431, 10267, ...
5	032745	4602, 5517, 7539, 12955, 14555, 20137, ...
6	032746	27257, 272570, 302693, 323576, ...
7	032747	140997, 490996, 1184321, 1259609, ...
8	032748	185423, 641519, 1551728, 1854230, ...
9	032749	17546133, 32613656, 93568867, ...

References
Pickover, C. A. *Keys to Infinity.* New York: Wiley, p. 7, 1995.
Sloane, N. J. A. Sequence A014569 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Super Catalan Number

While the CATALAN NUMBERS are the number of p-GOOD PATHS from (n,n) to (0,0) which do not cross the diagonal line, the super Catalan numbers count the number of LATTICE PATHS with diagonal steps from (n,n) to (0,0) which do not touch the diagonal line $x=y$.

The super Catalan numbers are given by the RECURRENCE RELATION

$$S(n) = \frac{3(2n-3)S(n-1)-(n-3)S(n-2)}{n}$$

(Comtet 1974), with $S(1)=S(2)=1$. (Note that the expression in Vardi (1991, p. 198) contains *two* errors.) A closed form expression in terms of LEGENDRE POLYNOMIALS $P_n(x)$ is

$$S(n) = \frac{3P_{n-1}(3)-P_{n-2}(3)}{4n}$$

(Vardi 1991, p. 199). The first few super Catalan numbers are 1, 1, 3, 11, 45, 197, ... (Sloane's A001003).

see also CATALAN NUMBER

References
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Schröder, E. "Vier combinatorische Probleme." *Z. Math. Phys.* **15**, 361–376, 1870.
Sloane, N. J. A. Sequence A001003/M2898 in "An On-Line Version of the Encyclopedia of Integer Sequences."
Vardi, I. *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, pp. 198–199, 1991.

Super-Poulet Number

A POULET NUMBER whose DIVISORS d all satisfy $d|2^d-2$.

see also POULET NUMBER

Superabundant Number

see HIGHLY COMPOSITE NUMBER

Superegg

A superegg is a solid described by the equation

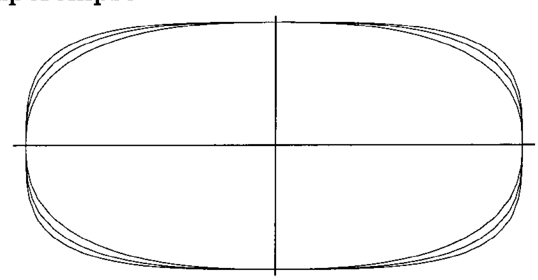
$$\left|\sqrt{\frac{x^2+y^2}{a^2}}\right|^n + \left|\frac{z}{b}\right|^n = 1.$$

Supereggs will balance on either end for any a, b, and n.

see also EGG, SUPERELLIPSE

References
Gardner, M. "Pier Hein's Superellipse." Ch. 18 in *Mathematical Carnival: A New Round-Up of Tantalizers and Puzzles from Scientific American.* New York: Vintage, 1977.

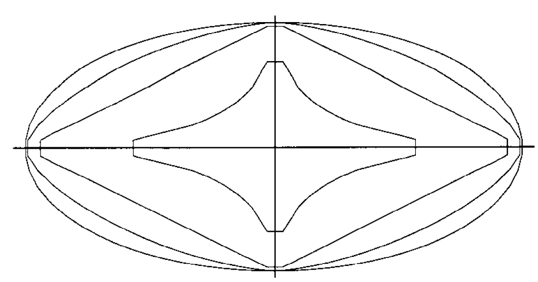
Superellipse



A curve of the form

$$\left|\frac{x}{a}\right|^r + \left|\frac{y}{b}\right|^r = 1.$$

where $r>2$. "The" superellipse is sometimes taken as the curve of the above form with $r=5/2$. Superellipses with $a=b$ are also known as LAMÉ CURVES. The above curves are for $a=1$, $b=2$, and $r=2.5$, 3.0, and 3.5.



A degenerate superellipse is a superellipse with $r\leq 2$. The above curves are for $a=1$, $b=2$, and $r=0.5$, 1.0, 1.5, and 2.0.

see also ELLIPSE, LAMÉ CURVE, SUPEREGG

References
Gardner, M. "Piet Hein's Superellipse." Ch. 18 in *Mathematical Carnival: A New Round-Up of Tantalizers and Puzzles from Scientific American.* New York: Vintage, 1977.

Superfactorial

The superfactorial of n is defined by Pickover (1995) as

$$n\$ \equiv \underbrace{n!^{n!^{\cdot^{\cdot^{\cdot^{n!}}}}}}_{n!}.$$

The first two values are 1 and 4, but subsequently grow so rapidly that 3$ already has a huge number of digits.

Sloane and Plouffe (1995) define the superfactorial by

$$n\$ \equiv \prod_{i=1}^{n} i!,$$

which is equivalent to the integral values of the G-FUNCTION. The first few values are 1, 1, 2, 12, 288, 34560, ... (Sloane's A000178).

see also FACTORIAL, G-FUNCTION, LARGE NUMBER, SUBFACTORIAL

References

Pickover, C. A. *Keys to Infinity.* New York: Wiley, p. 102, 1995.

Sloane, N. J. A. Sequence A000178/M2049 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Supergraph

If G' is a SUBGRAPH of G, then G is said to be a supergraph of G'.

see also GRAPH (GRAPH THEORY), SUBGRAPH

Supernormal

Trials for which the LEXIS RATIO

$$L \equiv \frac{\sigma}{\sigma_B},$$

satisfies $L > 1$, where σ is the VARIANCE in a set of s LEXIS TRIALS and σ_B is the VARIANCE assuming BERNOULLI TRIALS.

see also BERNOULLI TRIAL, LEXIS TRIALS, SUBNORMAL

Superperfect Number

A number n such that

$$\sigma^2(n) = \sigma(\sigma(n)) = 2n,$$

where $\sigma(n)$ is the DIVISOR FUNCTION. EVEN superperfect numbers are just 2^{p-1}, where $M_p = 2^p - 1$ is a MERSENNE PRIME. If any ODD superperfect numbers exist, they are SQUARE NUMBERS and either n or $\sigma(n)$ is DIVISIBLE by at least three distinct PRIMES.

More generally, an m-superperfect number is a number for which $\sigma^m(n) = 2n$. For $m \geq 3$, there are no EVEN m-superperfect numbers.

see also MERSENNE NUMBER

References

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Superposition Principle

For a linear homogeneous ORDINARY DIFFERENTIAL EQUATION, if $y_1(x)$ and $y_2(x)$ are solutions, then so is $y_1(x) + y_2(x)$.

Superregular Graph

For a VERTEX x of a GRAPH, let Γ_x and Δ_x denote the SUBGRAPHS of $\Gamma - x$ induced by the VERTICES adjacent to and nonadjacent to x, respectively. The empty graph is defined to be superregular, and Γ is said to be superregular if Γ is a REGULAR GRAPH and both Γ_x and Δ_x are superregular for all x.

The superregular graphs are precisely C_5, mK_n ($m, n \geq 1$), G_n ($n \geq 1$), and the complements of these graphs, where C_n is a CYCLIC GRAPH, K_n is a COMPLETE GRAPH and mK_n is m disjoint copies of K_n, and G_n is the Cartesian product of K_n with itself (the graph whose VERTEX set consists of n^2 VERTICES arranged in an $n \times n$ square with two VERTICES adjacent IFF they are in the same row or column).

see also COMPLETE GRAPH, CYCLIC GRAPH, REGULAR GRAPH

References

Vince, A. "The Superregular Graph." Problem 6617. *Amer. Math. Monthly* **103**, 600–603, 1996.

West, D. B. "The Superregular Graphs." *J. Graph Th.* **23**, 289–295, 1996.

Superscript

A quantity displayed above the normal line of text (and generally in a smaller point size), as the "i" in x^i, is called a superscript. Superscripts are commonly used to indicate raising to a POWER (x^3 means $x \cdot x \cdot x$ or x CUBED), multiple differentiation ($f^{(3)}(x)$ is an abbreviation for $f'''(x) = d^3 f/dx^3$), and a host of other operations and notations in mathematics.

see also SUBSCRIPT

Superset

A SET containing all elements of a smaller SET. If B is a SUBSET of A, then A is a superset of B, written $A \supseteq B$. If A is a PROPER SUPERSET of B, this is written $A \supset B$.

see also PROPER SUBSET, PROPER SUPERSET, SUBSET

Supplementary Angle

Two ANGLES α and $\pi - \alpha$ which together form a STRAIGHT ANGLE are said to be supplementary.

see also ANGLE, COMPLEMENTARY ANGLE, DIGON, STRAIGHT ANGLE

Support

The CLOSURE of the SET of arguments of a FUNCTION f for which f is not zero.

see also CLOSURE

Support Function

Let M be an oriented REGULAR SURFACE in $\mathbb{R}^3$ with normal $\mathbf{N}$. Then the support function of M is the function $h : M \to \mathbb{R}$ defined by

$$h(\mathbf{p}) = \mathbf{p} \cdot \mathbf{N}(\mathbf{p}).$$

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 293, 1993.

Supremum

The supremum of a set is the least upper bound of the set. It is denoted

$$\sup_S .$$

On the REAL LINE, the supremum of a set is the same as the supremum of its CLOSURE.

see also INFIMUM, SUPREMUM LIMIT

Supremum Limit

The limit supremum is used for sequences and nets (as opposed to sets) and is denoted

$$\limsup_S .$$

see also SUPREMUM

Surd

An archaic term for a SQUARE ROOT.

see also QUADRATIC SURD, SQUARE ROOT

Surface

The word "surface" is an important term in mathematics and is used in many ways. The most common and straightforward use of the word is to denote a 2-D SUBMANIFOLD of 3-D EUCLIDEAN SPACE. Surfaces can range from the very complicated (e.g., FRACTALS such as the MANDELBROT SET) to the very simple (such as the PLANE). More generally, the word "surface" can be used to denote an $(n-1)$-D SUBMANIFOLD of an n-D MANIFOLD, or in general, any co-dimension 1 subobject in an object (like a BANACH SPACE or an infinite-dimensional MANIFOLD).

Even simple surfaces can display surprisingly counterintuitive properties. For example, the SURFACE OF REVOLUTION of $y = 1/x$ around the x-AXIS for $x \geq 1$ (called GABRIEL'S HORN) has FINITE VOLUME but INFINITE SURFACE AREA.

see also ALGEBRAIC SURFACE, BARTH DECIC, BARTH SEXTIC, BERNSTEIN MINIMAL SURFACE THEOREM, BOHEMIAN DOME, BOY SURFACE, CATALAN'S SURFACE, CAYLEY'S RULED SURFACE, CHAIR, CLEBSCH DIAGONAL CUBIC, COMPACT SURFACE, CONE, CONICAL WEDGE, CONOCUNEUS OF WALLIS, CORK PLUG, CORKSCREW SURFACE, CORNUCOPIA, COSTA MINIMAL SURFACE, CROSS-CAP, CROSSED TROUGH, CUBIC SURFACE, CYCLIDE, CYLINDER, CYLINDROID, DARWIN-DE SITTER SPHEROID, DECIC SURFACE, DEL PEZZO SURFACE, DERVISH, DESMIC SURFACE, DEVELOPABLE SURFACE, DINI'S SURFACE, EIGHT SURFACE, ELLIPSOID, ELLIPTIC CONE, ELLIPTIC CYLINDER, ELLIPTIC HELICOID, ELLIPTIC HYPERBOLOID, ELLIPTIC PARABOLOID, ELLIPTIC TORUS, ENNEPER'S SURFACES, ENRIQUES SURFACES, ETRUSCAN VENUS SURFACE, FLAT SURFACE, FRESNEL'S ELASTICITY SURFACE, GABRIEL'S HORN, HANDKERCHIEF SURFACE, HELICOID, HENNEBERG'S MINIMAL SURFACE, HOFFMAN'S MINIMAL SURFACE, HORN CYCLIDE, HORN TORUS, HUNT'S SURFACE, HYPERBOLIC CYLINDER, HYPERBOLIC PARABOLOID, HYPERBOLOID, IDA SURFACE, IMMERSED MINIMAL SURFACE, KISS SURFACE, KLEIN BOTTLE, KUEN SURFACE, KUMMER SURFACE, LICHTENFELS SURFACE, MAEDER'S OWL MINIMAL SURFACE, MANIFOLD, MENN'S SURFACE, MINIMAL SURFACE, MITER SURFACE, MÖBIUS STRIP, MONGE'S FORM, MONKEY SADDLE, NONORIENTABLE SURFACE, NORDSTRAND'S WEIRD SURFACE, NURBS SURFACE, OBLATE SPHEROID, OCTIC SURFACE, ORIENTABLE SURFACE, PARABOLIC CYLINDER, PARABOLIC HORN CYCLIDE, PARABOLIC RING CYCLIDE, PARABOLIC SPINDLE CYCLIDE, PARABOLOID, PEANO SURFACE, PIRIFORM, PLANE, PLÜCKER'S CONOID, POLYHEDRON, PRISM, PRISMATOID, PROLATE SPHEROID, PSEUDOCROSSCAP, QUADRATIC SURFACE, QUARTIC SURFACE, QUINTIC SURFACE, REGULAR SURFACE, REMBS' SURFACES, RIEMANN SURFACE, RING CYCLIDE, RING TORUS, ROMAN SURFACE, RULED SURFACE, SCHERK'S MINIMAL SURFACES, SEIFERT SURFACE, SEXTIC SURFACE, SHOE SURFACE, SIEVERT'S SURFACE, SMOOTH SURFACE, SOLID, SPHERE, SPHEROID, SPINDLE CYCLIDE, SPINDLE TORUS, STEINBACH SCREW, STEINER SURFACE, SWALLOWTAIL CATASTROPHE, SYMMETROID, TANGLECUBE, TETRAHEDRAL SURFACE, TOGLIATTI SURFACE, TOOTH SURFACE, TRINOID, UNDULOID, VERONESE SURFACE, VERONESE VARIETY, WALLIS'S CONICAL EDGE, WAVE SURFACE, WEDGE, WHITNEY UMBRELLA

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Surface Area

Surface area is the AREA of a given surface. Roughly speaking, it is the "amount" of a surface, and has units of distance squares. It is commonly denoted S for a surface in 3-D, or A for a region of the plane (in which case it is simply called "the" AREA).

If the surface is PARAMETERIZED using u and v, then

$$S = \int_S |\mathbf{T}_u \times \mathbf{T}_v|\, du\, dv, \tag{1}$$

where $\mathbf{T}_u$ and $\hat{\mathbf{T}}_v$ are tangent vectors and $\mathbf{a} \times \mathbf{b}$ is the CROSS PRODUCT.

The surface area given by rotating the curve $y = f(x)$ from $x = a$ to $x = b$ about the x-axis is

$$S = \int_b^a 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx. \tag{2}$$

If $z = f(x, y)$ is defined over a region R, then

$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA, \tag{3}$$

where the integral is taken over the entire surface.

The following tables gives surface areas for some common SURFACES. In the first table, S denotes the lateral surface, and in the second, T denotes the total surface. In both tables, r denotes the RADIUS, h the height, p the base PERIMETER, and s the SLANT HEIGHT (Beyer 1987).

Surface	S
cone	$\pi r\sqrt{r^2 + h^2}$
conical frustum	$\pi(R_1 + R_2)\sqrt{(R_1 - R_2)^2 + h^2}$
cube	$6a^2$
cylinder	$2\pi rh$
lune	$2r^2\theta$
oblate spheroid	$2\pi a^2 + \frac{\pi b^2}{e}\ln\left(\frac{1+e}{1-e}\right)$
prolate spheroid	$2\pi b^2 + \frac{2\pi ab}{e}\sin^{-1} e$
pyramid	$\frac{1}{2}ps$
pyramidal frustum	$\frac{1}{2}ps$
sphere	$4\pi r^2$
torus	$4\pi^2 Rr$
zone	$2\pi rh$

Surface	T
cone	$\pi r(r + \sqrt{r^2 + h^2}\,)$
conical frustum	$\pi[{R_1}^2 + {R_2}^2 + (R_1 + R_2)\sqrt{(R_1 - R_2)^2 + h^2}\,]$
cylinder	$2\pi r(r + h)$

Even simple surfaces can display surprisingly counterintuitive properties. For instance, the surface of revolution of $y = 1/x$ around the x-AXIS for $x \geq 1$ is called GABRIEL'S HORN, and has FINITE VOLUME but INFINITE surface AREA.

see also AREA, SURFACE INTEGRAL, SURFACE OF REVOLUTION, VOLUME

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 127–132, 1987.

Surface Integral

For a SCALAR FUNCTION f over a surface parameterized by u and v, the surface integral is given by

$$\Phi = \int_S f\, da = \int_S f(u, v)\, |\mathbf{T}_u \times \mathbf{T}_v|\, du\, dv, \tag{1}$$

where $\mathbf{T}_u$ and $\hat{\mathbf{T}}_v$ are tangent vectors and $\mathbf{a} \times \mathbf{b}$ is the CROSS PRODUCT.

For a VECTOR FUNCTION over a surface, the surface integral is given by

$$\Phi = \int_S \mathbf{F} \cdot d\mathbf{a} = \int_S (\mathbf{F} \cdot \hat{\mathbf{n}})\, da \tag{2}$$

$$= \int_S f_x\, dy\, dz + f_y\, dz\, dx + f_z\, dx\, dy, \tag{3}$$

where $\mathbf{a} \cdot \mathbf{b}$ is a DOT PRODUCT and $\hat{\mathbf{n}}$ is a unit NORMAL VECTOR. If $z = f(x, y)$, then $d\mathbf{a}$ is given explicitly by

$$d\mathbf{a} = \pm\left(-\frac{\partial z}{\partial x}\hat{\mathbf{x}} - \frac{\partial z}{\partial y}\hat{\mathbf{y}} + \hat{\mathbf{z}}\right) dx\, dy. \tag{4}$$

If the surface is SURFACE PARAMETERIZED using u and v, then

$$\Phi = \int_S \mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv. \tag{5}$$

see also SURFACE PARAMETERIZATION

Surface Parameterization

A surface in 3-SPACE can be parameterized by two variables (or coordinates) u and v such that

$$x = x(u,v) \tag{1}$$
$$y = y(u,v) \tag{2}$$
$$z = z(u,v). \tag{3}$$

If a surface is parameterized as above, then the tangent VECTORS

$$\mathbf{T}_u = \frac{\partial x}{\partial u}\hat{\mathbf{x}} + \frac{\partial y}{\partial u}\hat{\mathbf{y}} + \frac{\partial z}{\partial u}\hat{\mathbf{z}} \tag{4}$$
$$\mathbf{T}_v = \frac{\partial x}{\partial v}\hat{\mathbf{x}} + \frac{\partial y}{\partial v}\hat{\mathbf{y}} + \frac{\partial z}{\partial v}\hat{\mathbf{z}} \tag{5}$$

are useful in computing the SURFACE AREA and SURFACE INTEGRAL.

see also SMOOTH SURFACE, SURFACE AREA, SURFACE INTEGRAL

Surface of Revolution

A surface of revolution is a SURFACE generated by rotating a 2-D CURVE about an axis. The resulting surface therefore always has azimuthal symmetry. Examples of surfaces of revolution include the APPLE, CONE (excluding the base), CONICAL FRUSTUM (excluding the ends), CYLINDER (excluding the ends), DARWIN-DE SITTER SPHEROID, GABRIEL'S HORN, HYPERBOLOID, LEMON, OBLATE SPHEROID, PARABOLOID, PROLATE SPHEROID, PSEUDOSPHERE, SPHERE, SPHEROID, and TORUS (and its generalization, the TOROID).

The standard parameterization of a surface of revolution is given by

$$x(u,v) = \phi(v)\cos u \tag{1}$$
$$y(u,v) = \phi(v)\sin u \tag{2}$$
$$z(u,v) = \psi(v). \tag{3}$$

For a curve so parameterized, the first FUNDAMENTAL FORM has

$$E = \psi^2 \tag{4}$$
$$F = 0 \tag{5}$$
$$G = \phi'^2 + \psi'^2. \tag{6}$$

Wherever ϕ and $\phi'^2 + \psi'^2$ are nonzero, then the surface is regular and the second FUNDAMENTAL FORM has

$$e = -\frac{|\phi|\psi'}{\sqrt{\phi'^2+\psi'^2}} \tag{7}$$
$$f = 0 \tag{8}$$
$$g = \frac{\operatorname{sgn}(\phi)(\phi''\psi' - \phi'\psi'')}{\sqrt{\phi'^2+\psi'^2}}. \tag{9}$$

Furthermore, the unit NORMAL VECTOR is

$$\hat{\mathbf{N}}(u,v) = \frac{\operatorname{sgn}(\phi)}{\sqrt{\phi'^2+\psi'^2}}\begin{bmatrix}\phi'\cos u\\ \psi'\sin u\\ \phi'\end{bmatrix}, \tag{10}$$

and the PRINCIPAL CURVATURES are

$$\kappa_1 = \frac{g}{G} = \frac{\operatorname{sgn}(\phi)(\phi''\psi' - \phi'\psi'')}{(\phi'^2+\psi'^2)^{3/2}} \tag{11}$$
$$\kappa_2 = \frac{e}{E} = -\frac{\psi'}{|\phi|\sqrt{\phi'^2+\psi'^2}}. \tag{12}$$

The GAUSSIAN and MEAN CURVATURES are

$$K = \frac{-\psi'^2\phi'' + \phi'\psi'\psi''}{\phi(\phi'^2+\psi'^2)^2} \tag{13}$$
$$H = \frac{\phi(\phi''\psi' - \phi'\psi'') - \psi'(\phi'^2+\psi'^2)}{2|\phi|(\phi'^2+\psi'^2)^{3/2}} \tag{14}$$

(Gray 1993).

PAPPUS'S CENTROID THEOREM gives the VOLUME of a solid of rotation as the cross-sectional AREA times the distance traveled by the centroid as it is rotated.

CALCULUS OF VARIATIONS can be used to find the curve from a point (x_1, y_1) to a point (x_2, y_2) which, when revolved around the x-AXIS, yields a surface of smallest SURFACE AREA A (i.e., the MINIMAL SURFACE). This is equivalent to finding the MINIMAL SURFACE passing through two circular wire frames. The AREA element is

$$dA = 2\pi y\,ds = 2\pi y\sqrt{1+y'^2}\,dx, \tag{15}$$

so the SURFACE AREA is

$$A = 2\pi\int y\sqrt{1+y'^2}\,dx, \tag{16}$$

and the quantity we are minimizing is

$$f = y\sqrt{1+y'^2}. \tag{17}$$

This equation has $f_x = 0$, so we can use the BELTRAMI IDENTITY

$$f - y_x\frac{\partial f}{\partial y_x} = a \tag{18}$$

to obtain

$$y\sqrt{1+y'^2} - y'\frac{yy'}{\sqrt{1+y'^2}} = a \tag{19}$$

$$y(1+y'^2) - yy'^2 = a\sqrt{1+y'^2} \tag{20}$$
$$y = a\sqrt{1+y'^2} \tag{21}$$
$$\frac{y}{\sqrt{1+y'^2}} = a \tag{22}$$

$$\frac{y^2}{a} - 1 = y'^2 \tag{23}$$

$$\frac{dx}{dy} = \frac{1}{y'} = \frac{a}{\sqrt{y^2 - a^2}} \tag{24}$$

$$x = a \int \frac{dy}{\sqrt{y^2 - a^2}} = a \cosh^{-1}\left(\frac{y}{a}\right) + b \tag{25}$$

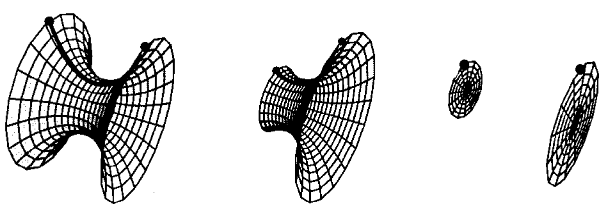
$$y = a \cosh\left(\frac{x - b}{a}\right), \tag{26}$$

which is called a CATENARY, and the surface generated by rotating it is called a CATENOID. The two constants a and b are determined from the two implicit equations

$$y_1 = a \cosh\left(\frac{x_1 - b}{a}\right) \tag{27}$$

$$y_2 = a \cosh\left(\frac{x_2 - b}{a}\right), \tag{28}$$

which cannot be solved analytically.



The general case is somewhat more complicated than this solution suggests. To see this, consider the MINIMAL SURFACE between two rings of equal RADIUS y_0. Without loss of generality, take the origin at the midpoint of the two rings. Then the two endpoints are located at $(-x_0, y_0)$ and (x_0, y_0), and

$$y_0 = a \cosh\left(\frac{-x_0 - b}{a}\right) = a \cosh\left(\frac{x_0 - b}{a}\right). \tag{29}$$

But $\cosh(-x) = \cosh(x)$, so

$$\cosh\left(\frac{-x_0 - b}{a}\right) = \cosh\left(\frac{-x_0 + b}{a}\right). \tag{30}$$

Inverting each side

$$-x_0 - b = -x_0 + b, \tag{31}$$

so $b = 0$ (as it must by symmetry, since we have chosen the origin between the two rings), and the equation of the MINIMAL SURFACE reduces to

$$y = a \cosh\left(\frac{x}{a}\right). \tag{32}$$

At the endpoints

$$y_0 = a \cosh\left(\frac{x_0}{a}\right), \tag{33}$$

but for certain values of x_0 and y_0, this equation has no solutions. The physical interpretation of this fact is that the surface breaks and forms circular disks in each ring to minimize AREA. CALCULUS OF VARIATIONS cannot be used to find such discontinuous solutions (known in this case as GOLDSCHMIDT SOLUTIONS). The minimal surfaces for several choices of endpoints are shown above. The first two cases are CATENOIDS, while the third case is a GOLDSCHMIDT SOLUTION.

To find the maximum value of x_0/y_0 at which CATENARY solutions can be obtained, let $p \equiv 1/a$. Then (31) gives

$$y_0 p = \cosh(p x_0). \tag{34}$$

Now, denote the maximum value of x_0 as x_0^*. Then it will be true that $dx_0/dp = 0$. Take d/dp of (34),

$$y_0 = \sinh(p x_0)\left(x_0 + p\frac{dx_0}{dp}\right). \tag{35}$$

Now set $dx_0/dp = 0$

$$y_0 = x_0 \sinh(p x_0^*). \tag{36}$$

From (34),

$$p y_0{}^* = \cosh(p x_0{}^*). \tag{37}$$

Take (37) ÷ (36),

$$p x_0^* = \coth(p x_0^*). \tag{38}$$

Defining $u \equiv p x_0{}^*$,

$$u = \coth u. \tag{39}$$

This has solution $u = 1.1996789403\ldots$. From (36), $y_0 p = \cosh u$. Divide this by (39) to obtain $y_0/x_0 = \sinh u$, so the maximum possible value of x_0/y_0 is

$$\frac{x_0}{y_0} = \operatorname{csch} u = 0.6627434193\ldots. \tag{40}$$

Therefore, only Goldschmidt ring solutions exist for $x_0/y_0 > 0.6627\ldots$.

The SURFACE AREA of the minimal CATENOID surface is given by

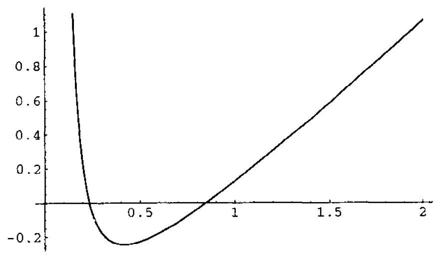
$$A = 2(2\pi) \int_0^{x_0} y\sqrt{1 + y'^2}\, dx, \tag{41}$$

but since

$$y = \sqrt{1 + y'^2}\, a \tag{42}$$

$$y = a \cosh\left(\frac{x}{a}\right), \tag{43}$$

$$\begin{aligned} A &= \frac{4\pi}{a}\int_0^{x_0} y^2\,dx = 4\pi a\int_0^{x_0}\cosh^2\left(\frac{x}{a}\right)dx \\ &= 4\pi a\int_0^{x_0}\tfrac{1}{2}\left[\cosh\left(\frac{2x}{a}\right)+1\right]dx \\ &= 2\pi a\left[\int_0^{x_0}\cosh\left(\frac{2x}{a}\right)dx+\int_0^{x_0}dx\right] \\ &= 2\pi a\left[\frac{a}{2}\sinh\left(\frac{2x}{a}\right)+x\right]_0^{x_0} \\ &= \pi a^2\left[\sinh\left(\frac{2x}{a}\right)+\frac{2x}{a}\right]_0^{x_0} \\ &= \pi a^2\left[\sinh\left(\frac{2x_0}{a}\right)+\frac{2x_0}{a}\right]. \end{aligned} \tag{44}$$



Some caution is needed in solving (33) for a. If we take $x_0 = 1/2$ and $y_0 = 1$ then (33) becomes

$$1 = a\cosh\left(\frac{1}{2a}\right), \tag{45}$$

which has *two* solutions: $a_1 = 0.2350\ldots$ ("deep"), and $a_2 = 0.8483\ldots$ ("flat"). However, upon plugging these into (44) with $x_0 = 1/2$, we find $A_1 = 6.8456\ldots$ and $A_2 = 5.9917\ldots$. So A_1 is *not*, in fact, a local minimum, and A_2 is the only *true* minimal solution.

The SURFACE AREA of the CATENOID solution equals that of the GOLDSCHMIDT SOLUTION when (44) equals the AREA of two disks,

$$\pi a^2\left[\sinh\left(\frac{2x_0}{a}\right)+\frac{2x_0}{a}\right] = 2\pi {y_0}^2 \tag{46}$$

$$a^2\left[2\sinh\left(\frac{x_0}{a}\right)\cosh\left(\frac{x_0}{a}\right)+\frac{2x_0}{a}\right]-2{y_0}^2 = 0 \tag{47}$$

$$a^2\left[\cosh\left(\frac{x_0}{a}\right)\sqrt{\cosh^2\left(\frac{x_0}{a}\right)-1}+\frac{x_0}{a}\right]-{y_0}^2 = 0. \tag{48}$$

Plugging in

$$\frac{y_0}{a} = \cosh\left(\frac{x_0}{a}\right), \tag{49}$$

$$\frac{y_0}{a}\sqrt{\left(\frac{y_0}{a}\right)^2-1}+\cosh^{-1}\left(\frac{y_0}{a}\right)-\left(\frac{y_0}{a}\right)^2 = 0. \tag{50}$$

Defining

$$u \equiv \frac{y_0}{a} \tag{51}$$

gives

$$u\sqrt{u^2-1}+\cosh^{-1}u-u^2 = 0. \tag{52}$$

This has a solution $u = 1.2113614259$. The value of x_0/y_0 for which

$$A_{\text{catenary}} = A_{2\ \text{disks}} \tag{53}$$

is therefore

$$\frac{x_0}{y_0} = \frac{\frac{x_0}{a}}{\frac{y_0}{a}} = \frac{\cosh^{-1}\left(\frac{y_0}{a}\right)}{\frac{y_0}{a}} = \frac{\cosh^{-1}u}{u} = 0.5276973967. \tag{54}$$

For $x_0/y_0 \in (0.52770, 0.6627)$, the CATENARY solution has larger AREA than the two disks, so it exists only as a RELATIVE MINIMUM.

There also exist solutions with a disk (of radius r) between the rings supported by two CATENOIDS of revolution. The AREA is larger than that for a simple CATENOID, but it is a RELATIVE MINIMUM. The equation of the POSITIVE half of this curve is

$$y = c_1\cosh\left(\frac{x}{c_1}+c_3\right). \tag{55}$$

At $(0, r)$,

$$r = c_1\cosh(c_3). \tag{56}$$

At (x_0, y_0),

$$y_0 = c_1\cosh\left(\frac{x_0}{c_1}+c_3\right). \tag{57}$$

The AREA of the two CATENOIDS is

$$\begin{aligned} A_{\text{catenoids}} &= 2(2\pi)\int_0^{x_0} y\sqrt{1+y'^2}\,dx = \frac{4\pi}{c_1}\int_0^{x_0} y^2\,dx \\ &= 4\pi c_1\int_0^{x_0}\cosh^2\left(\frac{x}{c_1}+c_3\right)dx. \end{aligned} \tag{58}$$

Now let $u \equiv x/c_1 + c_3$, so $du = dx/c_1$

$$\begin{aligned} A &= 4\pi {c_1}^2\int_{c_3}^{x_0/x_1+c_3}\cosh^2 u\,du \\ &= 4\pi {c_1}^2\tfrac{1}{2}\int_{c_3}^{x_0/x_1+c_3}[\cosh(2u)+1]\,du \\ &= 2\pi {c_1}^2\left[\tfrac{1}{2}\sinh(2u)+u\right]_{c_3}^{x_0/x_1+c_3} \\ &= 2\pi {c_1}^2\left\{\tfrac{1}{2}\sinh\left[2\left(\frac{x_0}{c_1}+c_3\right)\right]-\tfrac{1}{2}\sinh(2c_3)+\frac{x_0}{c_1}\right\} \\ &= \pi {c_1}^2\left\{\sinh\left[2\left(\frac{x_0}{c_1}+c_3\right)\right]-\sinh(2c_3)+\frac{2x_0}{c_1}\right\}. \end{aligned} \tag{59}$$

The AREA of the central DISK is

$$A_{\text{disk}} = \pi r^2 = \pi {c_1}^2\cosh^2 c_3, \tag{60}$$

so the total AREA is

$$A = \pi {c_1}^2 \left\{ \sinh\left[2\left(\frac{x_0}{c_1} + c_3\right)\right] + \left[\cosh^2 c_3 - \sinh(2c_3)\right] + \frac{2x_0}{c_1} \right\}. \quad (61)$$

By PLATEAU'S LAWS, the CATENOIDS meet at an ANGLE of $120°$, so

$$\tan 30° = \left[\frac{dy}{dx}\right]_{x=0} = \left[\sinh\left(\frac{x}{c_1} + c_3\right)\right]_{x=0} = \sinh c_3 = \frac{1}{\sqrt{3}} \quad (62)$$

and

$$c_3 = \sinh^{-1}\left(\frac{1}{\sqrt{3}}\right). \quad (63)$$

This means that

$$\begin{aligned}\cosh^2 c_3 - \sinh(2c_3) &= [1 + \sinh^2 c_3] - 2\sinh c_3 \sqrt{1 + \sinh^2 c_3} \\ &= (1 + \tfrac{1}{3}) - 2\left(\frac{1}{\sqrt{3}}\right)\sqrt{1 + \tfrac{1}{3}} \\ &= \frac{4}{3} - \frac{2}{\sqrt{3}}\frac{2}{\sqrt{3}} = 0, \end{aligned} \quad (64)$$

so

$$A = \pi {c_1}^2 \left\{ \sinh\left[2\left(\frac{x_0}{c_1} + c_3\right)\right] + \frac{2x_0}{c_1} \right\}. \quad (65)$$

Now examine x_0/y_0,

$$\frac{x_0}{y_0} = \frac{\frac{x_0}{c_1}}{\frac{y_0}{c_1}} = \frac{\frac{x_0}{c_1}}{\cosh\left(\frac{x_0}{c_1} + c_3\right)} = u \operatorname{sech}(u + c_3), \quad (66)$$

where $u \equiv x_0/c_1$. Finding the maximum ratio of x_0/y_0 gives

$$\frac{d}{du}\left(\frac{x_0}{y_0}\right) = \operatorname{sech}(u+c_3) - u\tanh(u+c_3)\operatorname{sech}(u+c_3) = 0 \quad (67)$$

$$u\tanh(u + c_3) = 1, \quad (68)$$

with $c_3 = \sinh^{-1}(1/\sqrt{3})$ as given above. The solution is $u = 1.0799632187$, so the maximum value of x_0/y_0 for two CATENOIDS with a central disk is $y_0 = 0.4078241702$.

If we are interested instead in finding the curve from a point (x_1, y_1) to a point (x_2, y_2) which, when revolved around the y-AXIS (as opposed to the x-AXIS), yields a surface of smallest SURFACE AREA A, we proceed as above. Note that the solution is physically equivalent to that for rotation about the x-AXIS, but takes on a different mathematical form. The AREA element is

$$dA = 2\pi x\, ds = 2\pi x\sqrt{1 + y'^2}\, dx \quad (69)$$

$$A = 2\pi \int x\sqrt{1 + y'^2}\, dx, \quad (70)$$

and the quantity we are minimizing is

$$f = x\sqrt{1 + y'^2}. \quad (71)$$

Taking the derivatives gives

$$\frac{\partial f}{\partial y} = 0 \quad (72)$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{d}{dx}\left(\frac{xy'}{\sqrt{1 + y'^2}}\right), \quad (73)$$

so the EULER-LAGRANGE DIFFERENTIAL EQUATION becomes

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{d}{dx}\left(\frac{xy'}{\sqrt{1 + y'^2}}\right) = 0. \quad (74)$$

$$\frac{xy'}{\sqrt{1 + y'^2}} = a \quad (75)$$

$$x^2 y'^2 = a^2(1 + y'^2) \quad (76)$$

$$y'^2(x^2 - a^2) = a^2 \quad (77)$$

$$\frac{dy}{dx} = \frac{a}{\sqrt{x^2 - a^2}} \quad (78)$$

$$y = a\int \frac{dx}{\sqrt{x^2 - a^2}} + b = a\cosh^{-1}\left(\frac{x}{a}\right) + b. \quad (79)$$

Solving for x then gives

$$x = a\cosh\left(\frac{y - b}{a}\right), \quad (80)$$

which is the equation for a CATENARY. The SURFACE AREA of the CATENOID product by rotation is

$$\begin{aligned} A &= 2\pi \int x\sqrt{1 + y'^2}\, dx = 2\pi \int x\sqrt{1 + \frac{a^2}{x^2 - a^2}}\, dx \\ &= 2\pi \int \frac{x}{\sqrt{x^2 - a^2}}\sqrt{(x^2 - a^2) + a^2}\, dx \\ &= 2\pi \int \frac{x^2\, dx}{\sqrt{x^2 - a^2}} \\ &= \left[\frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2}\ln\left(x + \sqrt{x^2 - a^2}\right)\right]_{x_1}^{x_2} \\ &= \frac{1}{2}\left[x_2\sqrt{{x_2}^2 - a^2} - x_1\sqrt{{x_1}^2 - a^2} + a^2 \ln\left(\frac{x_2 + \sqrt{{x_2}^2 - a^2}}{x_1 + \sqrt{{x_1}^2 - a^2}}\right)\right]. \end{aligned} \quad (81)$$

Isenberg (1992, p. 80) discusses finding the MINIMAL SURFACE passing through two rings with axes offset from each other.

see also APPLE, CATENOID, CONE CONICAL FRUSTUM, CYLINDER, DARWIN-DE SITTER SPHEROID, EIGHT SURFACE, GABRIEL'S HORN, HYPERBOLOID, LEMON, MERIDIAN, OBLATE SPHEROID, PAPPUS'S CENTROID THEOREM, PARABOLOID, PARALLEL (SURFACE OF REVOLUTION), PROLATE SPHEROID, PSEUDOSPHERE, SINCLAIR'S SOAP FILM PROBLEM, SOLID OF REVOLUTION, SPHERE, SPHEROID, TOROID, TORUS

References
Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 931–937, 1985.
Goldstein, H. *Classical Mechanics, 2nd ed.* Reading, MA: Addison-Wesley, p. 42, 1980.
Gray, A. "Surfaces of Revolution." Ch. 18 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 357–375, 1993.
Isenberg, C. *The Science of Soap Films and Soap Bubbles.* New York: Dover, pp. 79–80 and Appendix III, 1992.

Surface of Section

A surface (or "space") of section is a way of presenting a trajectory in n-D PHASE SPACE in an $(n-1)$-D SPACE. By picking one phase element constant and plotting the values of the other elements each time the selected element has the desired value, an intersection surface is obtained. If the equations of motion can be formulated as a MAP in which an explicit FORMULA gives the values of the other elements at successive passages through the selected element value, the time required to compute the surface of section is greatly reduced.

see also PHASE SPACE

Surgery

In the process of attaching a k-HANDLE to a MANIFOLD M, the BOUNDARY of M is modified by a process called $(k-1)$-surgery. Surgery consists of the removal of a TUBULAR NEIGHBORHOOD of a $(k-1)$-SPHERE $\mathbb{S}^{k-1}$ from the BOUNDARIES of M and the $\dim(M)-1$ standard SPHERE, and the gluing together of these two scarred-up objects along their common BOUNDARIES.

see also BOUNDARY, DEHN SURGERY, HANDLE, MANIFOLD, SPHERE, TUBULAR NEIGHBORHOOD

Surjection

An ONTO (SURJECTIVE) MAP.

see also BIJECTION, INJECTION, ONTO

Surjective

see ONTO

Surprise Examination Paradox

see UNEXPECTED HANGING PARADOX

Surreal Number

The most natural collection of numbers which includes both the REAL NUMBERS and the infinite ORDINAL NUMBERS of Georg Cantor. They were invented by John H. Conway in 1969. Every REAL NUMBER is surrounded by surreals, which are closer to it than any REAL NUMBER. Knuth (1974) describes the surreal numbers in a work of fiction.

The surreal numbers are written using the NOTATION $\{a|b\}$, where $\{|\}=0$, $\{0|\}=1$ is the simplest number greater than 0, $\{1|\}=2$ is the simplest number greater than 1, etc. Similarly, $\{|0\}=-1$ is the simplest number less than 1, etc. However, 2 can also be represented by $\{1|3\}$, $\{3/2|4\}$, $\{1|\omega\}$, etc.

see also OMNIFIC INTEGER, ORDINAL NUMBER, REAL NUMBER

References
Berlekamp, E. R.; Conway, J. H.; and Guy, R. K. *Winning Ways, For Your Mathematical Plays, Vol. 1: Games in General.* London: Academic Press, 1982.
Conway, J. H. *On Numbers and Games.* New York: Academic Press, 1976.
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 283–284, 1996.
Conway, J. H. and Jackson, A. "Budding Mathematician Wins Westinghouse Competition." *Not. Amer. Math. Soc.* **43**, 776–779, 1996.
Gonshor, H. *An Introduction to Surreal Numbers.* Cambridge: Cambridge University Press, 1986.
Knuth, D. *Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness.* Reading, MA: Addison-Wesley, 1974. http://www-cs-faculty.stanford.edu/~knuth/sn.html.

Surrogate

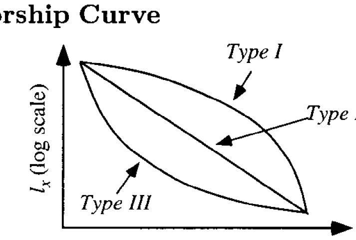
Surrogate data are artificially generated data which mimic statistical properties of real data. Isospectral surrogates have identical POWER SPECTRA as real data but with randomized phases. Scrambled surrogates have the same probability distribution as real data, but with white noise POWER SPECTRA.

see also POWER SPECTRUM

Surveying Problems

see HANSEN'S PROBLEM, SNELLIUS-POTHENOT PROBLEM

Survivorship Curve



Plotting l_x from a LIFE EXPECTANCY table on a logarithmic scale versus x gives a curve known as a survivorship curve. There are three general classes of survivorship curves, illustrated above.

1. Type I curves are typical of populations in which most mortality occurs among the elderly (e.g., humans in developed countries).
2. Type II curves occur when mortality is not dependent on age (e.g., many species of large birds and fish). For an infinite type II population, $e_0 = e_1 = \ldots$, but this cannot hold for a finite population.
3. Type III curves occur when juvenile mortality is extremely high (e.g., plant and animal species producing many offspring of which few survive). In type III populations, it is often true that $e_{i+1} > e_i$ for small i. In other words, life expectancy increases for individuals who survive their risky juvenile period.

see also LIFE EXPECTANCY

Suslin's Theorem

A SET in a POLISH SPACE is a BOREL SET IFF it is both ANALYTIC and COANALYTIC. For subsets of w, a set is δ_1^1 IFF it is "hyperarithmetic."

see also ANALYTIC SET, BOREL SET, COANALYTIC SET, POLISH SPACE

Suspended Knot

An ordinary KNOT in 3-D suspended in 4-D to create a knotted 2-sphere. Suspended knots are not smooth at the poles.

see also SPUN KNOT, TWIST-SPUN KNOT

Suspension

The JOIN of a TOPOLOGICAL SPACE X and a pair of points S^0, $\Sigma(X) = X * S^0$.

see also JOIN (SPACES), TOPOLOGICAL SPACE

References

Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, p. 6, 1976.

Suzanne Set

The nth Suzanne set S_n is defined as the set of COMPOSITE NUMBERS x for which $n|S(x)$ and $n|S_p(x)$, where

$$x = a_0 + a_1(10^1) + \ldots + a_d(10^d) = p_1 p_2 \cdots p_n,$$

and

$$S(x) = \sum_{j=0}^{d} a_j$$

$$S_p(x) = \sum_{i=1}^{m} S(p_i).$$

Every Suzanne set has an infinite number of elements. The Suzanne set S_n is a superset of the MONICA SET M_n.

see also MONICA SET

References

Smith, M. "Cousins of Smith Numbers: Monica and Suzanne Sets." *Fib. Quart.* **34**, 102–104, 1996.

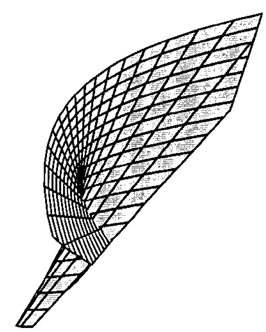
Suzuki Group

The SPORADIC GROUP *Suz.*

References

Wilson, R. A. "ATLAS of Finite Group Representation." `http://for.mat.bham.ac.uk/atlas/Suz.html`.

Swallowtail Catastrophe



A CATASTROPHE which can occur for three control factors and one behavior axis. The equations

$$\begin{aligned} x &= uv^2 + 3v^4 \\ y &= -2uv - 4v^3 \\ z &= u \end{aligned}$$

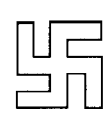
display such a catastrophe (von Seggern 1993, Nordstrand). The above surface uses $u \in [-2, 2]$ and $v \in [-0.8, 0.8]$.

References

Nordstrand, T. "Swallowtail." `http://www.uib.no/people/nfytn/stltxt.htm`.

von Seggern, D. *CRC Standard Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 94, 1993.

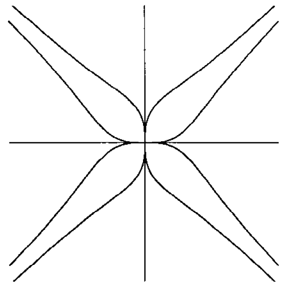
Swastika



An irregular ICOSAGON, also called the gammadion or fylfot, which symbolized good luck in ancient Arabic and Indian cultures. In more recent times, it was adopted as the symbol of the Nazi Party in Hitler's Germany and has thence come to symbolize anti-Semitism.

see also CROSS, DISSECTION

Swastika Curve



The plane curve with Cartesian equation

$$y^4 - x^4 = xy$$

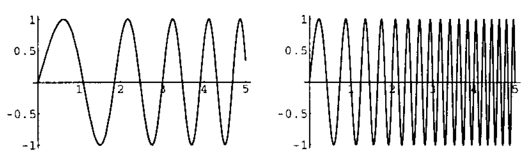
and polar equation

$$r^2 = \frac{\sin\theta\cos\theta}{\sin^4\theta - \cos^4\theta}.$$

References

Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., p. 71, 1989.

Sweep Signal



The general function

$$y(a,b,c,d) = c\sin\left\{\frac{\pi}{b-a}\left[\left((b-a)\frac{x}{d}+a\right)^2 - a^2\right]\right\}.$$

References

von Seggern, D. *CRC Standard Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 160, 1993.

Swinnerton-Dyer Conjecture

In the early 1960s, B. Birch and H. P. F. Swinnerton-Dyer conjectured that if a given ELLIPTIC CURVE has an infinite number of solutions, then the associated L-function has value 0 at a certain fixed point. In 1976, Coates and Wiles showed that elliptic curves with COMPLEX multiplication having an infinite number of solutions have L-functions which are zero at the relevant fixed point (COATES-WILES THEOREM), but they were unable to prove the converse. V. Kolyvagin extended this result to modular curves.

see also COATES-WILES THEOREM, ELLIPTIC CURVE

References

Cipra, B. "Fermat Prover Points to Next Challenges." *Science* **271**, 1668–1669, 1996.

Ireland, K. and Rosen, M. "New Results on the Birch–Swinnerton-Dyer Conjecture." §20.5 in *A Classical Introduction to Modern Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 353–357, 1990.

Mazur, B. and Stevens, G. (Eds.). *p-Adic Monodromy and the Birch and Swinnerton-Dyer Conjecture.* Providence, RI: Amer. Math. Soc., 1994.

Swinnerton-Dyer Polynomial

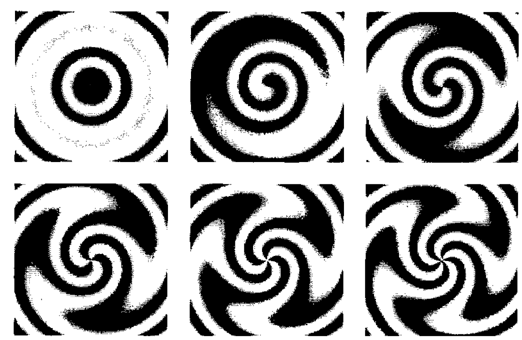
The minimal POLYNOMIAL $S_n(x)$ whose ROOTS are sums and differences of the SQUARE ROOTS of the first n PRIMES,

$$S_n(x) = \prod(x \pm \sqrt{2} \pm \sqrt{3} \pm \sqrt{5} \pm \ldots \pm \sqrt{p_n}).$$

References

Vardi, I. *Computational Recreations in Mathematica.* Redwood City, CA: Addison-Wesley, pp. 11 and 225–226, 1991.

Swirl



A swirl is a generic word to describe a function having arcs which double back swirl around each other. The plots above correspond to the function

$$f(r,\theta) = \sin(6\cos r - n\theta)$$

for $n = 0, 1, \ldots, 5$.

see also DAISY, WHIRL

Sylow p-Subgroup

If p^k is the highest POWER of a PRIME p dividing the ORDER of a finite GROUP G, then a SUBGROUP of G of ORDER p^k is called a Sylow p-subgroup of G.

see also ABHYANKAR'S CONJECTURE, SUBGROUP, SYLOW THEOREMS

Sylow Theorems

Let p be a PRIME NUMBER, G a GROUP, and $|G|$ the order of G.

1. If p divides $|G|$, then G has a SYLOW p-SUBGROUP.
2. In a FINITE GROUP, all the SYLOW p-SUBGROUPS are isomorphic for some fixed p.
3. The number of SYLOW p-SUBGROUPS for a fixed p is CONGRUENT to 1 (mod p).

Sylvester Cyclotomic Number

Given a LUCAS SEQUENCE with parameters P and Q, discriminant $D \neq 0$, and roots α and β, the Sylvester cyclotomic numbers are

$$Q_n = \prod_r (\alpha - \zeta^r \beta),$$

where

$$\zeta \equiv \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$$

is a PRIMITIVE ROOT OF UNITY and the product is over all exponents r RELATIVELY PRIME to n such that $r \in [1, n)$.

see also LUCAS SEQUENCE

References

Ribenboim, P. *The Book of Prime Number Records, 2nd ed.* New York: Springer-Verlag, p. 69, 1989.

Sylvester's Determinant Identity

$$|A|\,|A_{rs,pq}| = |A_{r,p}|\,|A_{s,q}| - |A_{r,q}|\,|A_{s,p}|,$$

where $A_{u,w}$ is the submatrix of A formed by the intersection of the subset w of columns and u of rows.

Sylvester's Four-Point Problem

Let $q(R)$ be the probability that four points chosen at random in a region R have a CONVEX HULL which is a QUADRILATERAL. For an open, convex subset of the PLANE of finite AREA,

$$0.667 \approx \tfrac{2}{3} \leq q(R) \leq 1 - \frac{35}{12\pi^2} \approx 0.704.$$

References

Schneinerman, E. and Wilf, H. S. "The Rectilinear Crossing Number of a Complete Graph and Sylvester's 'Four Point' Problem of Geometric Probability." *Amer. Math. Monthly* **101**, 939–943, 1994.

Sylvester Graph

The Sylvester graph of a configuration is the set of ORDINARY POINTS and ORDINARY LINES.

see also ORDINARY LINE, ORDINARY POINT

References

Guy, R. K. "Monthly Unsolved Problems, 1969–1987." *Amer. Math. Monthly* **94**, 961–970, 1987.

Guy, R. K. "Unsolved Problems Come of Age." *Amer. Math. Monthly* **96**, 903–909, 1989.

Sylvester's Inertia Law

The numbers of EIGENVALUES that are POSITIVE, NEGATIVE, or 0 do not change under a congruence transformation. Gradshteyn and Ryzhik (1979) state it as follows: when a QUADRATIC FORM Q in n variables is reduced by a nonsingular linear transformation to the form

$$Q = {y_1}^2 + {y_2}^2 + \ldots + {y_p}^2 - {p_{p+1}}^2 - {y_{p_2}}^2 - \ldots - {y_r}^2,$$

the number p of POSITIVE SQUARES appearing in the reduction is an invariant of the QUADRATIC FORM Q and does not depend on the method of reduction.

see also EIGENVALUE, QUADRATIC FORM

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1105, 1979.

Sylvester's Line Problem

It is not possible to arrange a finite number of points so that a LINE through every two of them passes through a third, unless they are all on a single LINE.

see also COLLINEAR, SYLVESTER'S FOUR-POINT PROBLEM

Sylvester Matrix

For POLYNOMIALS of degree m and n, the Sylvester matrix is an $(m+n) \times (m+n)$ matrix whose DETERMINANT is the RESULTANT of the two POLYNOMIALS.

see also RESULTANT

Sylvester's Sequence

The sequence defined by $e_0 = 2$ and the RECURRENCE RELATION

$$e_n = 1 + \prod_{i=0}^{n-1} e_i = {e_{n-1}}^2 - e_{n-1} + 1. \tag{1}$$

This sequence arises in Euclid's proof that there are an INFINITE number of PRIMES. The proof proceeds by constructing a sequence of PRIMES using the RECURRENCE RELATION

$$e_{n+1} = e_0 e_1 \cdots e_n + 1 \tag{2}$$

(Vardi 1991). Amazingly, there is a constant

$$E \approx 1.264084735306 \tag{3}$$

such that

$$e_n = \left\lfloor E^{2^{n+1}} + \tfrac{1}{2} \right\rfloor \tag{4}$$

(Vardi 1991, Graham *et al.* 1994). The first few numbers in Sylvester's sequence are 2, 3, 7, 43, 1807, 3263443, 10650056950807, ... (Sloane's A000058). The e_n satisfy

$$\sum_{n=0}^{\infty} \frac{1}{e_n} = 1. \quad (5)$$

In addition, if $0 < x < 1$ is an IRRATIONAL NUMBER, then the nth term of an infinite sum of unit fractions used to represent x as computed using the GREEDY ALGORITHM must be smaller than $1/e_n$.

The n of the first few PRIME e_n are 0, 1, 2, 3, 5, ... Vardi (1991) gives a lists of factors less than 5×10^7 of e_n for $n \leq 200$ and shows that e_n is COMPOSITE for $6 \leq n \leq 17$. Furthermore, all numbers less than 2.5×10^{15} in Sylvester's sequence are SQUAREFREE, and no SQUAREFUL numbers in this sequence are known (Vardi 1991).

see also EUCLID'S THEOREMS, GREEDY ALGORITHM, SQUAREFREE, SQUAREFUL

References

Graham, R. L.; Knuth, D. E.; and Patashnik, O. Research problem 4.65 in *Concrete Mathematics: A Foundation for Computer Science, 2nd ed.* Reading, MA: Addison-Wesley, 1994.

Sloane, N. J. A. Sequence A000058/M0865 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Vardi, I. "Are All Euclid Numbers Squarefree?" and "PowerMod to the Rescue." §5.1 and 5.2 in *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, pp. 82–89, 1991.

Sylvester's Signature

Diagonalize a form over the RATIONALS to

$$\mathrm{diag}[p^a \cdot A, p^b \cdot B, \ldots],$$

where all the entries are INTEGERS and A, B, ... are RELATIVELY PRIME to p. Then Sylvester's signature is the sum of the -1-parts of the entries.

see also p-SIGNATURE

Sylvester's Triangle Problem

The resultant of the vectors represented by the three RADII from the center of a TRIANGLE'S CIRCUMCIRCLE to its VERTICES is the segment extending from the CIRCUMCENTER to the ORTHOCENTER.

see also CIRCUMCENTER, CIRCUMCIRCLE, ORTHOCENTER, TRIANGLE

References

Dörrie, H. *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, p. 142, 1965.

Symbolic Logic

The study of the meaning and relationships of statements used to represent precise mathematical ideas. Symbolic logic is also called FORMAL LOGIC.

see also FORMAL LOGIC, LOGIC, METAMATHEMATICS

References

Carnap, R. *Introduction to Symbolic Logic and Its Applications.* New York: Dover, 1958.

Symmedian Line

The lines ISOGONAL to the MEDIANS of a TRIANGLE are called the triangle's symmedian lines. The symmedian lines are concurrent in a point called the LEMOINE POINT.

see also ISOGONAL CONJUGATE, LEMOINE POINT, MEDIAN (TRIANGLE)

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 213–218, 1929.

Symmedian Point

see LEMOINE POINT

Symmetric

A quantity which remains unchanged in SIGN when indices are reversed. For example, $A_{ij} \equiv a_i + a_j$ is symmetric since $A_{ij} = A_{ji}$.

see also ANTISYMMETRIC

Symmetric Block Design

A symmetric design is a BLOCK DESIGN (v, k, λ, r, b) with the same number of blocks as points, so $b = v$ (or, equivalently, $r = k$). An example of a symmetric block design is a PROJECTIVE PLANE.

see also BLOCK DESIGN, PROJECTIVE PLANE

References

Dinitz, J. H. and Stinson, D. R. "A Brief Introduction to Design Theory." Ch. 1 in *Contemporary Design Theory: A Collection of Surveys* (Ed. J. H. Dinitz and D. R. Stinson). New York: Wiley, pp. 1–12, 1992.

Symmetric Design

see SYMMETRIC BLOCK DESIGN

Symmetric Function

A symmetric function on n variables x_1, ..., x_n is a function that is unchanged by any PERMUTATION of its variables. In most contexts, the term "symmetric function" refers to a polynomial on n variables with this feature (more properly called a "symmetric polynomial"). Another type of symmetric functions is symmetric rational functions, which are the RATIONAL FUNCTIONS that are unchanged by PERMUTATION of variables.

The symmetric polynomials (respectively, symmetric rational functions) can be expressed as polynomials (respectively, rational functions) in the ELEMENTARY SYMMETRIC FUNCTIONS. This is called the FUNDAMENTAL THEOREM OF SYMMETRIC FUNCTIONS.

A function $f(x)$ is sometimes said to be symmetric about the y-AXIS if $f(-x) = f(x)$. Examples of such functions include $|x|$ (the ABSOLUTE VALUE) and x^2 (the PARABOLA).

see also ELEMENTARY SYMMETRIC FUNCTION, FUNDAMENTAL THEOREM OF SYMMETRIC FUNCTIONS, RATIONAL FUNCTION

References

Macdonald, I. G. *Symmetric Functions and Hall Polynomials, 2nd ed.* Oxford, England: Oxford University Press, 1995.

Macdonald, I. G. *Symmetric Funtions and Orthogonal Polynomials.* Providence, RI: Amer. Math. Soc., 1997.

Petkovšek, M.; Wilf, H. S.; and Zeilberger, D. "Symmetric Function Identities." §1.7 in *A=B.* Wellesley, MA: A. K. Peters, pp. 12–13, 1996.

Symmetric Group

The symmetric group S_n of DEGREE n is the GROUP of all PERMUTATIONS on n symbols. S_n is therefore of ORDER $n!$ and contains as SUBGROUPS every GROUP of ORDER n. The number of CONJUGACY CLASSES of S_n is given by the PARTITION FUNCTION P.

NETTO'S CONJECTURE states that the probability that two elements P_1 and P_2 of a symmetric group generate the entire group tends to 3/4 as $n \to \infty$. This was proven by Dixon in 1967.

see also ALTERNATING GROUP, CONJUGACY CLASS, FINITE GROUP, NETTO'S CONJECTURE, PARTITION FUNCTION P, SIMPLE GROUP

References

Lomont, J. S. "Symmetric Groups." Ch. 7 in *Applications of Finite Groups.* New York: Dover, pp. 258–273, 1987.

Wilson, R. A. "ATLAS of Finite Group Representation." `http://for.mat.bham.ac.uk/atlas#alt`.

Symmetric Matrix

A symmetric matrix is a SQUARE MATRIX which satisfies $\mathsf{A}^{\mathrm{T}} = \mathsf{A}$ where A^{T} denotes the TRANSPOSE, so $a_{ij} = a_{ji}$. This also implies

$$\mathsf{A}^{-1}\mathsf{A}^{\mathrm{T}} = \mathsf{I}, \tag{1}$$

where I is the IDENTITY MATRIX. Written explicitly,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}. \tag{2}$$

The symmetric part of any MATRIX may be obtained from

$$\mathsf{A}_s = \tfrac{1}{2}(\mathsf{A} + \mathsf{A}^{\mathrm{T}}). \tag{3}$$

A MATRIX A is symmetric if it can be expressed in the form

$$\mathsf{A} = \mathsf{QDQ}^{\mathrm{T}}, \tag{4}$$

where Q is an ORTHOGONAL MATRIX and D is a DIAGONAL MATRIX. This is equivalent to the MATRIX equation

$$\mathsf{AQ} = \mathsf{QD}, \tag{5}$$

which is equivalent to

$$\mathsf{A}\mathbf{Q}_n = \lambda_n \mathbf{Q}_n \tag{6}$$

for all n, where $\lambda_n = D_{nn}$. Therefore, the diagonal elements of D are the EIGENVALUES of A, and the columns of Q are the corresponding EIGENVECTORS.

see also ANTISYMMETRIC MATRIX, SKEW SYMMETRIC MATRIX

References

Nash, J. C. "Real Symmetric Matrices." Ch. 10 in *Compact Numerical Methods for Computers: Linear Algebra and Function Minimisation, 2nd ed.* Bristol, England: Adam Hilger, pp. 119–134, 1990.

Symmetric Points

Two points z and $z^S \in \mathbb{C}^*$ are symmetric with respect to a CIRCLE or straight LINE L if all CIRCLES and straight LINES passing through z and z^S are orthogonal to L. MÖBIUS TRANSFORMATIONS preserve symmetry. Let a straight line be given by a point z_0 and a unit VECTOR $e^{i\theta}$, then

$$z^S = e^{2i\theta}(z - z_0)^* + z_0.$$

Let a CIRCLE be given by center z_0 and RADIUS r, then

$$z^S = z_0 + \frac{r^2}{(z - z_0)^*}.$$

see also MÖBIUS TRANSFORMATION

Symmetric Relation

A RELATION R on a SET S is symmetric provided that for every x and y in S we have xRy IFF yRx.

see also RELATION

Symmetric Tensor

A second-RANK symmetric TENSOR is defined as a TENSOR A for which

$$A^{mn} = A^{nm}. \tag{1}$$

Any TENSOR can be written as a sum of symmetric and ANTISYMMETRIC parts

$$\begin{aligned} A^{mn} &= \tfrac{1}{2}(A^{mn} + A^{nm}) + \tfrac{1}{2}(A^{mn} - A^{nm}) \\ &= \tfrac{1}{2}({B_S}^{mn} + {B_A}^{mn}). \end{aligned} \tag{2}$$

The symmetric part of a TENSOR is denoted by parentheses as follows:

$$T_{(a,b)} \equiv \tfrac{1}{2}(T_{ab} + T_{ba}) \tag{3}$$

$$T_{(a_1,a_2,\ldots,a_n)} \equiv \frac{1}{n!} \sum_{\text{permutations}} T_{a_1a_2\cdots a_n}. \qquad (4)$$

The product of a symmetric and an ANTISYMMETRIC TENSOR is 0. This can be seen as follows. Let $a^{\alpha\beta}$ be ANTISYMMETRIC, so

$$a^{11} = a^{22} = 0 \qquad (5)$$

$$a^{21} = -a^{12}. \qquad (6)$$

Let $b_{\alpha\beta}$ be symmetric, so

$$b_{12} = b_{21}. \qquad (7)$$

Then

$$\begin{aligned} a^{\alpha\beta} b_{\alpha\beta} &= a^{11}b_{11} + a^{12}b_{12} + a^{21}b_{21} + a^{22}b_{22} \\ &= 0 + a^{12}b_{12} - a^{12}b_{12} + 0 = 0. \end{aligned} \qquad (8)$$

A symmetric second-RANK TENSOR A_{mn} has SCALAR invariants

$$s_1 = A_{11} + A_{22} + A_{22} \qquad (9)$$

$$\begin{aligned} s_2 = A_{22}A_{33} + A_{33}A_{11} + A_{11}A_{22} - A_{23}{}^2 \\ - A_{31}{}^2 - A_{12}{}^2. \end{aligned} \qquad (10)$$

Symmetroid

A QUARTIC SURFACE which is the locus of zeros of the DETERMINANT of a SYMMETRIC 4×4 matrix of linear forms. A general symmetroid has 10 ORDINARY DOUBLE POINTS (Jessop 1916, Hunt 1996).

References

Hunt, B. "Algebraic Surfaces." http://www.mathematik.uni-kl.de/~wwwagag/Galerie.html.

Hunt, B. "Symmetroids and Weddle Surfaces." §B.5.3 in *The Geometry of Some Special Arithmetic Quotients.* New York: Springer-Verlag, pp. 315–319, 1996.

Jessop, C. *Quartic Surfaces with Singular Points.* Cambridge, England: Cambridge University Press, p. 166, 1916.

Symmetry

An intrinsic property of a mathematical object which causes it to remain invariant under certain classes of transformations (such as ROTATION, REFLECTION, INVERSION, or more abstract operations). The mathematical study of symmetry is systematized and formalized in the extremely powerful and beautiful AREA of mathematics called GROUP THEORY.

Symmetry can be present in the form of coefficients of equations as well as in the physical arrangement of objects. By classifying the symmetry of polynomial equations using the machinery of GROUP THEORY, for example, it is possible to prove the unsolvability of the general QUINTIC EQUATION.

In physics, an extremely powerful theorem of Noether states that each symmetry of a system leads to a physically conserved quantity. Symmetry under TRANSLATION corresponds to momentum conservation, symmetry under ROTATION to angular momentum conservation, symmetry in time to energy conservation, etc.

see also GROUP THEORY

References

Eppstein, D. "Symmetry and Group Theory." http://www.ics.uci.edu/~eppstein/junkyard/sym.html.

Farmer, D. *Groups and Symmetry.* Providence, RI: Amer. Math. Soc., 1995.

Pappas, T. "Art & Dynamic Symmetry." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, pp. 154–155, 1989.

Rosen, J. *Symmetry in Science: An Introduction to the General Theory.* New York: Springer-Verlag, 1995.

Schattschneider, D. *Visions of Symmetry: Notebooks, Periodic Drawings, and Related Work of M. C. Escher.* New York: W. H. Freeman, 1990.

Stewart, I. and Golubitsky, M. *Fearful Symmetry.* New York: Viking Penguin, 1993.

Symmetry Group

see GROUP

Symmetry Operation

Symmetry operations include the IMPROPER ROTATION, INVERSION OPERATION, MIRROR PLANE, and ROTATION. Together, these operations create 32 crystal classes corresponding to the 32 POINT GROUPS.

The INVERSION OPERATION takes

$$(x, y, z) \to (-x, -y, -z)$$

and is denoted i. When used in conjunction with a ROTATION, it becomes an IMPROPER ROTATION. An IMPROPER ROTATION by $360°/n$ is denoted $\bar{n}$ (or S_n). For periodic crystals, the CRYSTALLOGRAPHY RESTRICTION allows only the IMPROPER ROTATIONS $\bar{1}$, $\bar{2}$, $\bar{3}$, $\bar{4}$, and $\bar{6}$.

The MIRROR PLANE symmetry operation takes

$$(x, y, z) \to (x, y, -z), (x, -y, z) \to (x, -y, z),$$

etc., which is equivalent to $\bar{2}$. Invariance under reflection can be denoted $n\sigma_v$ or $n\sigma_h$. The ROTATION symmetry operation for $360°/n$ is denoted n (or C_n). For periodic crystals, CRYSTALLOGRAPHY RESTRICTION allows only 1, 2, 3, 4, and 6.

Symmetry operations can be indicated with symbols such as C_n, S_n, E, i, $n\sigma_v$, and $n\sigma_h$.

1. C_n indicates ROTATION about an n-fold symmetry axis.
2. S_n indicates IMPROPER ROTATION about an n-fold symmetry axis.
3. E (or I) indicates invariance under TRANSLATION.
4. i indicates a center of symmetry under INVERSION.

5. $n\sigma_v$ indicates invariance under n vertical REFLECTIONS.
6. $n\sigma_h$ indicates invariance under n horizontal REFLECTIONS.

see also CRYSTALLOGRAPHY RESTRICTION, IMPROPER ROTATION, INVERSION OPERATION, MIRROR PLANE, POINT GROUPS, ROTATION, SYMMETRY

Symmetry Principle

SYMMETRIC POINTS are preserved under a MÖBIUS TRANSFORMATION.

see also MÖBIUS TRANSFORMATION, SYMMETRIC POINTS

Symplectic Diffeomorphism

A MAP $T : (M_1, \omega_1) \to (M_2, \omega_2)$ between the SYMPLECTIC MANIFOLDS (M_1, ω_1) and (M_2, ω_2) which is a DIFFEOMORPHISM and $T^*(\omega_2) = \omega_1$ (where T^* is the PULLBACK MAP induced by T, i.e., the derivative of the DIFFEOMORPHISM T acting on tangent vectors). A symplectic diffeomorphism is also known as a SYMPLECTOMORPHISM or CANONICAL TRANSFORMATION.

see also DIFFEOMORPHISM, PULLBACK MAP, SYMPLECTIC MANIFOLD

References

Guillemin, V. and Sternberg, S. *Symplectic Techniques in Physics.* New York: Cambridge University Press, p. 34, 1984.

Symplectic Form

A symplectic form on a SMOOTH MANIFOLD M is a smooth closed 2-FORM ω on M which is nondegenerate such that at every point m, the alternating bilinear form ω_m on the TANGENT SPACE $T_m M$ is nondegenerate.

A symplectic form on a VECTOR SPACE V over F_q is a function $f(x,y)$ (defined for all $x, y \in V$ and taking values in F_q) which satisfies

$$f(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 f(x_1, y) + \lambda_2 f(x_2, y),$$

$$f(y,x) = -f(x,y),$$

and

$$f(x,x) = 0.$$

Symplectic forms can exist on M (or V) only if M (or V) is EVEN-dimensional.

Symplectic Group

The symplectic group $Sp_n(q)$ for n EVEN is the GROUP of elements of the GENERAL LINEAR GROUP GL_n that preserve a given nonsingular SYMPLECTIC FORM. Any such MATRIX has DETERMINANT 1.

see also GENERAL LINEAR GROUP, LIE-TYPE GROUP, PROJECTIVE SYMPLECTIC GROUP, SYMPLECTIC FORM

References

Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; and Wilson, R. A. "The Groups $Sp_n(q)$ and $PSp_n(q) = S_n(q)$." §2.3 in *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford, England: Clarendon Press, pp. x–xi, 1985.

Wilson, R. A. "ATLAS of Finite Group Representation." `http://for.mat.bham.ac.uk/atlas#symp`.

Symplectic Manifold

A pair (M, ω), where M is a MANIFOLD and ω is a SYMPLECTIC FORM on M. The PHASE SPACE $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ is a symplectic manifold. Near every point on a symplectic manifold, it is possible to find a set of local "Darboux coordinates" in which the SYMPLECTIC FORM has the simple form

$$\omega = \sum_k dq_k \wedge dp_k$$

(Sjamaar 1996), where $dq_k \wedge dp_k$ is a WEDGE PRODUCT.

see also MANIFOLD, SYMPLECTIC DIFFEOMORPHISM, SYMPLECTIC FORM

References

Sjamaar, R. "Symplectic Reduction and Riemann-Roch Formulas for Multiplicities." *Bull. Amer. Math. Soc.* **33**, 327–338, 1996.

Symplectic Map

A MAP which preserves the sum of AREAS projected onto the set of (p_i, q_i) planes. It is the generalization of an AREA-PRESERVING MAP.

see also AREA-PRESERVING MAP, LIOUVILLE'S PHASE SPACE THEOREM

Symplectomorphism

see SYMPLECTIC DIFFEOMORPHISM

Synclastic

A surface on which the GAUSSIAN CURVATURE K is everywhere POSITIVE. When K is everywhere NEGATIVE, a surface is called ANTICLASTIC. A point at which the GAUSSIAN CURVATURE is POSITIVE is called an ELLIPTIC POINT.

see also ANTICLASTIC, ELLIPTIC POINT, GAUSSIAN QUADRATURE, HYPERBOLIC POINT, PARABOLIC POINT, PLANAR POINT

Synergetics

Synergetics deals with systems composed of many subsystems which may each be of a very different nature. In particular, synergetics treats systems in which cooperation among subsystems creates organized structure on macroscopic scales (Haken 1993). Examples of problems treated by synergetics include BIFURCATIONS, phase transitions in physics, convective instabilities, coherent oscillations in lasers, nonlinear oscillations in electrical circuits, population dynamics, etc.

see also BIFURCATION, CHAOS, DYNAMICAL SYSTEM

References

Haken, H. *Synergetics, an Introduction: Nonequilibrium Phase Transitions and Self-Organization in Physics, Chemistry, and Biology, 3rd rev. enl. ed.* New York: Springer-Verlag, 1983.

Haken, H. *Advanced Synergetics: Instability Hierarchies of Self-Organizing Systems and Devices.* New York: Springer-Verlag, 1993.

Mikhailov, A. S. *Foundations of Synergetics: Distributed Active Systems, 2nd ed.* New York: Springer-Verlag, 1994.

Mikhailov, A. S. and Loskutov, A. Y. *Foundations of Synergetics II: Complex Patterns, 2nd ed., enl. rev.* New York: Springer-Verlag, 1996.

Synthesized Beam

see DIRTY BEAM

Syntonic Comma

see COMMA OF DIDYMUS

Syracuse Algorithm

see COLLATZ PROBLEM

Syracuse Problem

see COLLATZ PROBLEM

System of Differential Equations

see ORDINARY DIFFERENTIAL EQUATION

System of Equations

Let a linear system of equations be denoted

$$\mathsf{A}\mathbf{X} = \mathbf{Y}, \tag{1}$$

where A is a MATRIX and $\mathbf{X}$ and $\mathbf{Y}$ are VECTORS. As shown by CRAMER'S RULE, there is a unique solution if A has a MATRIX INVERSE A^{-1}. In this case,

$$\mathbf{X} = \mathsf{A}^{-1}\mathbf{Y}. \tag{2}$$

If $\mathbf{Y} = \mathbf{0}$, then the solution is $\mathbf{X} = \mathbf{0}$. If A has no MATRIX INVERSE, then the solution SUBSPACE is either a LINE or the EMPTY SET. If two equations are multiples of each other, solutions are of the form

$$\mathbf{X} = \mathbf{A} + t\mathbf{B}, \tag{3}$$

for t a REAL NUMBER.

see also CRAMER'S RULE, MATRIX INVERSE

Syzygies Problem

The problem of finding all independent irreducible algebraic relations among any finite set of QUANTICS.

see also QUANTIC

Syzygy

A technical mathematical object defined in terms of a POLYNOMIAL RING of n variables over a FIELD k.

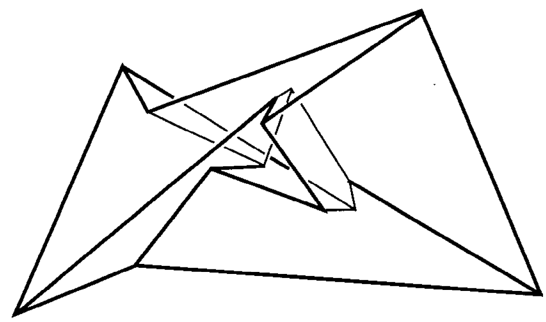
see also FUNDAMENTAL SYSTEM, HILBERT BASIS THEOREM, SYZYGIES PROBLEM

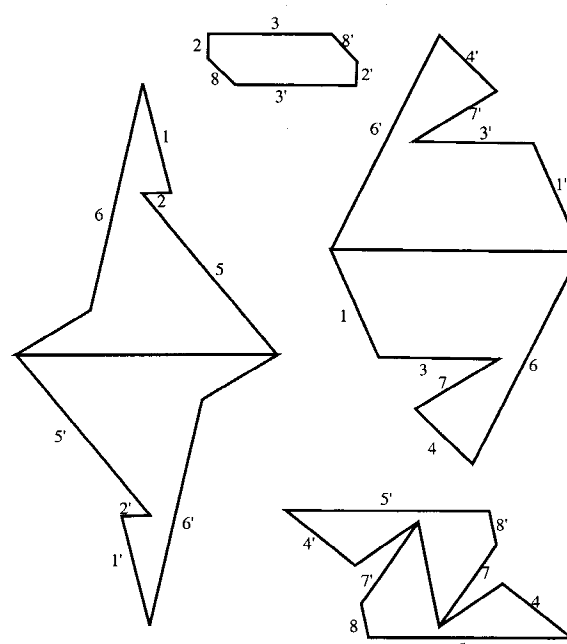
References

Hilbert, D. "Über die Theorie der algebraischen Formen." *Math. Ann.* **36**, 473–534, 1890.

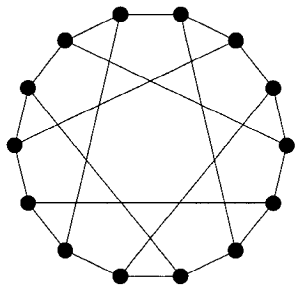
Iyanaga, S. and Kawada, Y. (Eds.). "Syzygy Theory." §364F in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1140, 1980.

Szilassi Polyhedron





A POLYHEDRON which is topologically equivalent to a TORUS and for which every pair of faces has an EDGE in common. This polyhedron was discovered by L. Szilassi in 1977. Its SKELETON is equivalent to the seven-color torus map illustrated below.



The Szilassi polyhedron has 14 VERTICES, seven faces, and 21 EDGES, and is the DUAL POLYHEDRON of the CSÁSZÁR POLYHEDRON.

see also CSÁSZÁR POLYHEDRON, TOROIDAL POLYHEDRON

References

Eppstein, D. "Polyhedra and Polytopes." http://www.ics.uci.edu/~eppstein/junkyard/polytope.html.

Gardner, M. *Fractal Music, Hypercards, and More Mathematical Recreations from Scientific American Magazine.* New York: W. H. Freeman, pp. 118–120, 1992.

Hart, G. "Toroidal Polyhedra." http://www.li.net/~george/virtual-polyhedra/toroidal.html.

Szpiro's Conjecture

A conjecture which relates the minimal DISCRIMINANT of an ELLIPTIC CURVE to the CONDUCTOR. If true, it would imply FERMAT'S LAST THEOREM for sufficiently large exponents.

see also CONDUCTOR, DISCRIMINANT (ELLIPTIC CURVE), ELLIPTIC CURVE

References

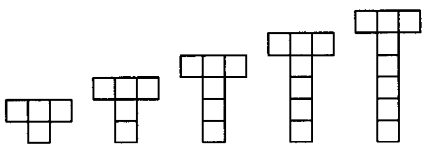
Cox, D. A. "Introduction to Fermat's Last Theorem." *Amer. Math. Monthly* **101**, 3–14, 1994.

T

t-Distribution

see STUDENT'S t-DISTRIBUTION

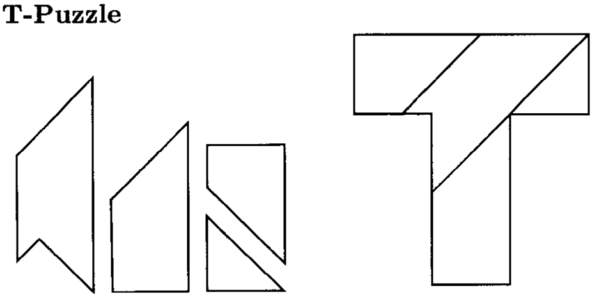
T-Polyomino



The order n T-polyomino consists of a vertical line of $n-3$ squares capped by a horizontal line of three squares centered on the line.

see also L-POLYOMINO, SKEW POLYOMINO, SQUARE POLYOMINO, STRAIGHT POLYOMINO

T-Puzzle



The DISSECTION of the four pieces shown at left into the capital letter "T" shown at right.

see also DISSECTION

References

Pappas, T. "The T Problem." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, pp. 35 and 230, 1989.

T2-Separation Axiom

Finite SUBSETS are CLOSED.

see also CLOSURE

Tableau

see YOUNG TABLEAU

Tabu Search

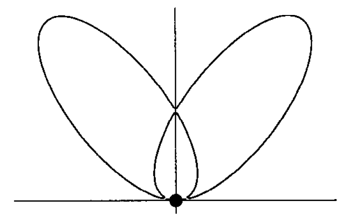
A heuristic procedure which has proven efficient at solving COMBINATORIAL optimization problems.

References

Glover, F.; Taillard, E.; and De Werra, D. "A User's Guide to Tabu Search." *Ann. Oper. Res.* **41**, 3–28, 1993.

Piwakowski, K. "Applying Tabu Search to Determine New Ramsey Numbers." *Electronic J. Combinatorics* **3**, R6, 1–4, 1996. `http://www.combinatorics.org/Volume_3/volume3.html#R6`.

Tacnode



A DOUBLE POINT at which two OSCULATING CURVES are tangent. The above plot shows the tacnode of the curve $2x^4 - 3x^2y + y^2 - 2y^3 + y^4 = 0$. The LINKS CURVE also has a tacnode at the origin.

see also ACNODE, CRUNODE, DOUBLE POINT SPINODE

References

Walker, R. J. *Algebraic Curves.* New York: Springer-Verlag, pp. 57–58, 1978.

Tacpoint

A tangent point of two similar curves.

Tactix

see NIM

Tail Probability

Define T as the set of all points t with probabilities $P(x)$ such that $a > t \Rightarrow P(a \leq x \leq a + da) < P_0$ or $a < t \Rightarrow P(a \leq x \leq a + da < P_0$, where P_0 is a POINT PROBABILITY (often, the likelihood of an observed event). Then the associated tail probability is given by $\int_T P(x)\,dx$.

see also P-VALUE, POINT PROBABILITY

Tait Coloring

A 3-coloring of GRAPH EDGES so that no two EDGES of the same color meet at a VERTEX (Ball and Coxeter 1987, pp. 265–266).

see also EDGE (GRAPH), TAIT CYCLE, VERTEX (GRAPH)

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, 1987.

Tait Cycle

A set of circuits going along the EDGES of a GRAPH, each with an EVEN number of EDGES, such that just one of the circuits passes through each VERTEX (Ball and Coxeter 1987, pp. 265–266).

see also EDGE (GRAPH), EULERIAN CYCLE, HAMILTONIAN CYCLE, TAIT COLORING, VERTEX (GRAPH)

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, 1987.

Tait Flyping Conjecture

see FLYPING CONJECTURE

Tait's Hamiltonian Graph Conjecture

Every 3-connected cubic GRAPH (each VERTEX has VALENCY 3) has a HAMILTONIAN CIRCUIT. Proposed by Tait in 1880 and refuted by W. T. Tutte in 1946 with a counterexample, TUTTE'S GRAPH. If it had been true, it would have implied the FOUR-COLOR THEOREM. A simpler counterexample was later given by Kozyrev and Grinberg.

see also HAMILTONIAN CIRCUIT, TUTTE'S GRAPH, VERTEX (GRAPH)

References
Honsberger, R. *Mathematical Gems I.* Washington, DC: Math. Assoc. Amer., pp. 82–89, 1973.

Tait's Knot Conjectures

P. G. Tait undertook a study of KNOTS in response to Kelvin's conjecture that the atoms were composed of knotted vortex tubes of ether (Thomson 1869). He categorized KNOTS in terms of the number of crossings in a plane projection. He also made some conjectures which remained unproven until the discovery of JONES POLYNOMIALS.

Tait's FLYPING CONJECTURE states that the number of crossings is the same for any diagram of an ALTERNATING KNOT. This was proved true in 1986.

see also ALTERNATING KNOT, FLYPING CONJECTURE, JONES POLYNOMIAL, KNOT

References
Tait, P. G. "On Knots I, II, III." *Scientific Papers, Vol. 1.* London: Cambridge University Press, pp. 273–347, 1900.
Thomson, W. H. "On Vortex Motion." *Trans. Roy. Soc. Edinburgh* **25**, 217–260, 1869.

TAK Function

A RECURSIVE FUNCTION devised by I. Takeuchi. For INTEGERS x, y, and z, and a function h, it is

$$\begin{aligned}&\mathrm{TAK}_h(x,y,z)\\ &\quad=\begin{cases} h(x,y,z) & \text{for } x \leq y\\ h(h(x-1,y,z),h(y-1,z,x), & \text{for } x > y.\\ \quad h(z-1,x,y)) & \end{cases}\end{aligned}$$

The number of function calls $F_0(a,b)$ required to compute $\mathrm{TAK}_0(a,b,0)$ for $a > b > 0$ is

$$\begin{aligned}F_0(a,b) &= 4\sum_{k=0}^{b}\frac{a-b}{a+b-2k}\binom{a+b-2k}{b-k}-3\\ &= 1+4\sum_{k=0}^{b-1}\frac{a-b}{a+b-2k}\binom{a+b-2k}{b-k}\end{aligned}$$

(Vardi 1991).

The TAK function is also connected with the BALLOT PROBLEM (Vardi 1991).

see also ACKERMANN FUNCTION, BALLOT PROBLEM

References
Gabriel, R. P. *Performance and Implementation of Lisp Systems.* Cambridge, MA: MIT Press, 1985.
Knuth, D. E. *Textbook Examples of Recursion.* Preprint 1990.
Vardi, I. "The Running Time of TAK." Ch. 9 in *Computational Recreations in Mathematica.* Redwood City, CA: Addison-Wesley, pp. 179–199, 1991.

Takagi Fractal Curve

see BLANCMANGE FUNCTION

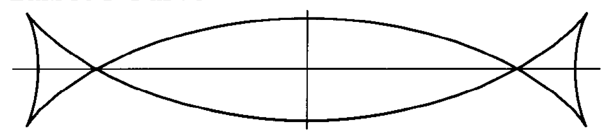
Take-Away Game

see NIM-HEAP

Takeuchi Function

see TAK FUNCTION

Talbot's Curve



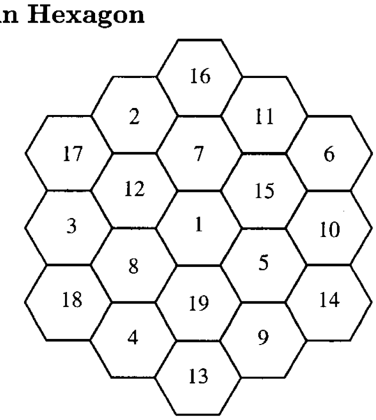
A curve investigated by Talbot which is the NEGATIVE PEDAL CURVE of an ELLIPSE with respect to its center. It has four CUSPS and two NODES, provided the ECCENTRICITY of the ELLIPSE is greater than $1/\sqrt{2}$. Its CARTESIAN EQUATION is

$$\begin{aligned}x &= \frac{(a^2+f^2\sin^2 t)\cos t}{a}\\ y &= \frac{(a^2-2f^2+f^2\sin^2 t)\sin t}{b},\end{aligned}$$

where f is a constant.

References
Lockwood, E. H. *A Book of Curves.* Cambridge, England: Cambridge University Press, p. 157, 1967.
MacTutor History of Mathematics Archive. "Talbot's Curve." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Talbots.html`.

Talisman Hexagon



An (n,k)-talisman hexagon is an arrangement of nested hexagons containing the integers 1, 2, ..., $H_n = 3n(n-$

1) + 1, where H_n is the nth HEX NUMBER, such that the difference between all adjacent hexagons is at least as large as k. The hexagon illustrated above is a (3, 5)-talisman hexagon.

see also HEX NUMBER, MAGIC SQUARE, TALISMAN SQUARE

References
Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, pp. 111–112, 1979.

Talisman Square

1	5	3	7
9	11	13	15
2	6	4	8
10	12	14	16

5	15	9	12
10	1	6	3
13	16	11	14
2	8	4	7

15	1	12	4	9
20	7	22	18	24
16	2	13	5	10
21	8	23	19	25
17	3	14	6	11

28	10	31	13	34	16
19	1	22	4	25	7
29	11	32	14	35	17
20	2	23	5	26	8
30	12	33	15	36	18
21	3	24	6	27	9

An $n \times n$ ARRAY of the integers from 1 to n^2 such that the difference between any one integer and its neighbor (horizontally, vertically, or diagonally, without wrapping around) is greater than or equal to some value k is called a (n, k)-talisman square. The above illustrations show (4, 2)-, (4, 3)-, (5, 4)-, and (6, 8)-talisman squares.

see also ANTIMAGIC SQUARE, HETEROSQUARE, MAGIC SQUARE, TALISMAN HEXAGON

References
Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, pp. 110–113, 1979.
Weisstein, E. W. "Magic Squares." http://www.astro.virginia.edu/~eww6n/math/notebooks/MagicSquares.m.

Tame Algebra

Let A denote an $\mathbb{R}$-algebra, so that A is a VECTOR SPACE over R and

$$A \times A \to A$$

$$(x, y) \mapsto x \cdot y,$$

where $x \cdot y$ is vector multiplication which is assumed to be BILINEAR. Now define

$$Z \equiv \{x \in a : x \cdot y = 0 \text{ for some nonzero } y \in A\},$$

where $0 \in Z$. A is said to be tame if Z is a finite union of SUBSPACES of A. A 2-D 0-ASSOCIATIVE algebra is tame, but a 4-D 4-ASSOCIATIVE algebra and a 3-D 1-ASSOCIATIVE algebra need not be tame. It is conjectured that a 3-D 2-ASSOCIATIVE algebra is tame, and proven that a 3-D 3-ASSOCIATIVE algebra is tame if it possesses a multiplicative IDENTITY ELEMENT.

References
Finch, S. "Zero Structures in Real Algebras." http://www.mathsoft.com/asolve/zerodiv/zerodiv.html.

Tame Knot

A KNOT equivalent to a POLYGONAL KNOT. Knots which are not tame are called WILD KNOTS.

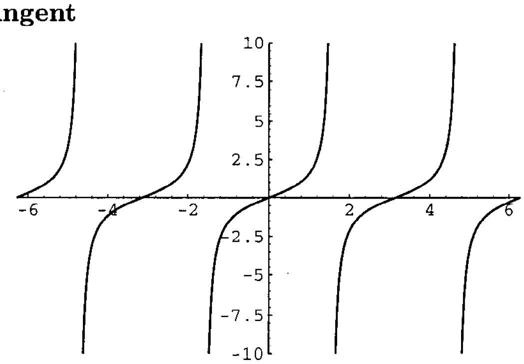
References
Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, p. 49, 1976.

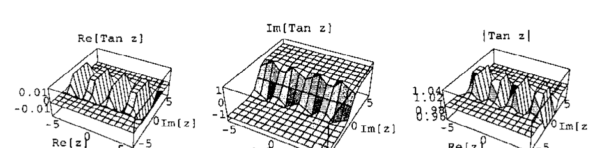
Tangency Theorem

The external (internal) SIMILARITY POINT of two fixed CIRCLES is the point at which all the CIRCLES homogeneously (nonhomogeneously) tangent to the fixed CIRCLES have the same POWER and at which all the tangency secants intersect.

References
Dörrie, H. *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, p. 157, 1965.

Tangent





The tangent function is defined by

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}, \tag{1}$$

where $\sin x$ is the SINE function and $\cos x$ is the COSINE function. The word "tangent," however, also has an important related meaning as a LINE or PLANE which touches a given curve or solid at a single point. These geometrical objects are then called a TANGENT LINE or TANGENT PLANE, respectively.

The MACLAURIN SERIES for the tangent function is

$$\begin{aligned}\tan x &= \sum \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1}+\ldots \\ &= x+\tfrac{1}{3}x^3+\tfrac{2}{15}x^5+\tfrac{17}{315}x^7+\tfrac{62}{2835}x^9+\ldots, \quad (2)\end{aligned}$$

where B_n is a BERNOULLI NUMBER.

$\tan x$ is IRRATIONAL for any RATIONAL $x \neq 0$, which can be proved by writing $\tan x$ as a CONTINUED FRACTION

$$\tan x = \cfrac{x}{1-\cfrac{x^2}{3-\cfrac{x^2}{5-\cfrac{x^2}{7-\ldots}}}}. \quad (3)$$

Lambert derived another CONTINUED FRACTION expression for the tangent,

$$\tan x = \cfrac{1}{\frac{1}{x}-\cfrac{1}{\frac{3}{x}-\cfrac{1}{\frac{5}{x}-\cfrac{1}{\frac{7}{x}-\ldots}}}}. \quad (4)$$

An interesting identity involving the PRODUCT of tangents is

$$\prod_{k=1}^{\lfloor (n-1)/2 \rfloor} \tan\left(\frac{k\pi}{n}\right) = \begin{cases} \sqrt{n} & \text{for } n \text{ odd} \\ 1 & \text{for } n \text{ even,} \end{cases} \quad (5)$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION. Another tangent identity is

$$\tan(n\tan^{-1}x) = \frac{1}{i}\frac{(1+ix)^n-(1-ix)^n}{(1+ix)^n+(1-ix)^m} \quad (6)$$

(Beeler *et al.* 1972, Item 16).

see also ALTERNATING PERMUTATION, COSINE, COTANGENT, INVERSE TANGENT, MORRIE'S LAW, SINE, TANGENT LINE, TANGENT PLANE

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Circular Functions." §4.3 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 71–79, 1972.

Beeler, M.; Gosper, R. W.; and Schroeppel, R. *HAKMEM.* Cambridge, MA: MIT Artificial Intelligence Laboratory, Memo AIM-239, Feb. 1972.

Spanier, J. and Oldham, K. B. "The Tangent $\tan(x)$ and Cotangent $\cot(x)$ Functions." Ch. 34 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 319–330, 1987.

Tangent Bifurcation

see FOLD BIFURCATION

Tangent Bundle

The tangent bundle TM of a SMOOTH MANIFOLD M is the SPACE of TANGENT VECTORS to points in the manifold, i.e., it is the set (x, v) where $x \in M$ and v is tangent to $x \in M$. For example, the tangent bundle to the CIRCLE is the CYLINDER.

see also COTANGENT BUNDLE, TANGENT VECTOR

Tangent Developable

A RULED SURFACE M is a tangent developable of a curve $\mathbf{y}$ if M can be parameterized by $\mathbf{x}(u,v) = \mathbf{y}(u) + v\mathbf{y}'(u)$. A tangent developable is a FLAT SURFACE.

see also BINORMAL DEVELOPABLE, NORMAL DEVELOPABLE

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 341–343, 1993.

Tangent Hyperbolas Method

see HALLEY'S METHOD

Tangent Indicatrix

Let the SPEED $\boldsymbol{\sigma}$ of a closed curve on the unit sphere S^2 never vanish. Then the tangent indicatrix

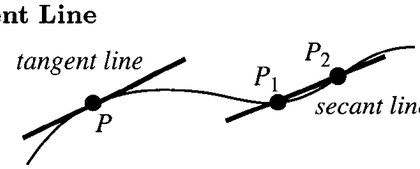
$$\boldsymbol{\tau} \equiv \frac{\dot{\boldsymbol{\sigma}}}{|\dot{\boldsymbol{\sigma}}|}$$

is another closed curve on S^2. It is sometimes called the TANTRIX. If $\boldsymbol{\sigma}$ IMMERSES in S^2, then so will $\boldsymbol{\tau}$.

References

Solomon, B. "Tantrices of Spherical Curves." *Amer. Math. Monthly* **103**, 30–39, 1996.

Tangent Line



A tangent line is a LINE which meets a given curve at a single POINT.

see also CIRCLE TANGENTS, SECANT LINE, TANGENT, TANGENT PLANE, TANGENT SPACE, TANGENT VECTOR

References

Yates, R. C. "Instantaneous Center of Rotation and the Construction of Some Tangents." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 119–122, 1952.

Tangent Map

If $f : M \to N$, then the tangent map Tf associated to f is a VECTOR BUNDLE HOMEOMORPHISM $Tf : TM \to TN$ (i.e., a MAP between the TANGENT BUNDLES of M and N respectively). The tangent map corresponds to DIFFERENTIATION by the formula

$$Tf(v) = (f \circ \phi)'(0), \tag{1}$$

where $\phi'(0) = v$ (i.e., ϕ is a curve passing through the base point to v in TM at time 0 with velocity v). In this case, if $f : M \to N$ and $g : N \to O$, then the CHAIN RULE is expressed as

$$T(f \circ g) = Tf \circ Tg. \tag{2}$$

In other words, with this way of formalizing differentiation, the CHAIN RULE can be remembered by saying that "the process of taking the tangent map of a map is functorial." To a topologist, the form

$$(f \circ g)'(a) = f'(g(a)) \circ g'(a), \tag{3}$$

for all a, is more intuitive than the usual form of the CHAIN RULE.

see also DIFFEOMORPHISM

References

Gray, A. "Tangent Maps." §9.3 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 168–171, 1993.

Tangent Number

A number also called a ZAG NUMBER giving the number of EVEN ALTERNATING PERMUTATIONS. The first few are 1, 2, 16, 272, 7936, ... (Sloane's A000182).

see also ALTERNATING PERMUTATION, EULER ZIGZAG NUMBER, SECANT NUMBER

References

Knuth, D. E. and Buckholtz, T. J. "Computation of Tangent, Euler, and Bernoulli Numbers." *Math. Comput.* **21**, 663–688, 1967.

Sloane, N. J. A. Sequence A000182/M2096 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Tangent Plane

A tangent plane is a PLANE which meets a given SURFACE at a single POINT. Let (x_0, y_0) be any point of a surface function $z = f(x, y)$. The surface has a nonvertical tangent plane at (x_0, y_0) with equation

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

see also NORMAL VECTOR, TANGENT, TANGENT LINE, TANGENT SPACE, TANGENT VECTOR

Tangent Space

Let x be a point in an n-dimensional COMPACT MANIFOLD M, and attach at x a copy of $\mathbb{R}^n$ tangential to M. The resulting structure is called the TANGENT SPACE of M at x and is denoted $T_x M$. If γ is a smooth curve passing through x, then the derivative of γ at x is a VECTOR in $T_x M$.

see also TANGENT, TANGENT BUNDLE, TANGENT PLANE, TANGENT VECTOR

Tangent Vector

For a curve with POSITION VECTOR $\mathbf{r}(t)$, the unit tangent vector $\hat{\mathbf{T}}(t)$ is defined by

$$\hat{\mathbf{T}}(t) \equiv \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\frac{d\mathbf{r}}{dt}}{\left|\frac{d\mathbf{r}}{dt}\right|} \tag{1}$$

$$= \frac{\frac{d\mathbf{r}}{dt}}{\frac{ds}{dt}} \tag{2}$$

$$= \frac{d\mathbf{r}}{ds}, \tag{3}$$

where t is a parameterization variable and s is the ARC LENGTH. For a function given parametrically by $(f(t), g(t))$, the tangent vector relative to the point $(f(t), g(t))$ is therefore given by

$$x(t) = \frac{f'}{\sqrt{f'^2 + g'^2}} \tag{4}$$

$$y(t) = \frac{g'}{\sqrt{f'^2 + g'^2}}. \tag{5}$$

To actually place the vector tangent to the curve, it must be displaced by $(f(t), g(t))$. It is also true that

$$\frac{d\hat{\mathbf{T}}}{ds} = \kappa \hat{\mathbf{N}} \tag{6}$$

$$\frac{d\hat{\mathbf{T}}}{dt} = \kappa \frac{ds}{dt} \hat{\mathbf{N}} \tag{7}$$

$$[\dot{\mathbf{T}}, \ddot{\mathbf{T}}, \dddot{\mathbf{T}}] = \kappa^5 \frac{d}{ds}\left(\frac{\tau}{\kappa}\right), \tag{8}$$

where $\mathbf{N}$ is the NORMAL VECTOR, κ is the CURVATURE, and τ is the TORSION.

see also CURVATURE, NORMAL VECTOR, TANGENT, TANGENT BUNDLE, TANGENT PLANE, TANGENT SPACE, TORSION (DIFFERENTIAL GEOMETRY)

References

Gray, A. "Tangent and Normal Lines to Plane Curves." §5.5 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 85–90, 1993.

Tangential Angle

For a PLANE CURVE, the tangential angle ϕ is defined by

$$\rho\, d\phi = ds, \qquad (1)$$

where s is the ARC LENGTH and ρ is the RADIUS OF CURVATURE. The tangential angle is therefore given by

$$\phi = \int_0^t s'(t)\kappa(t)\,dt, \qquad (2)$$

where $\kappa(t)$ is the CURVATURE. For a plane curve $\mathbf{r}(t)$, the tangential angle $\phi(t)$ can also be defined by

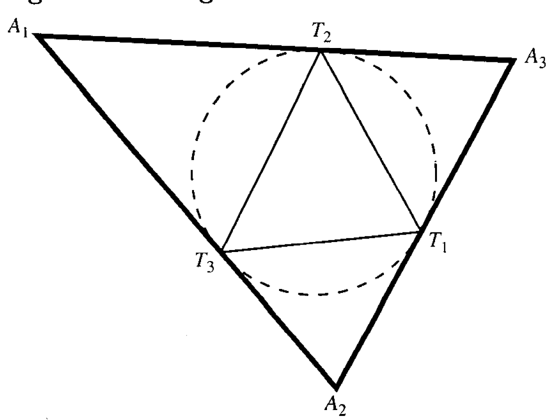
$$\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \begin{bmatrix} \cos[\phi(t)] \\ \sin[\phi(t)] \end{bmatrix}. \qquad (3)$$

Gray (1993) calls ϕ the TURNING ANGLE instead of the tangential angle.

see also ARC LENGTH, CURVATURE, PLANE CURVE, RADIUS OF CURVATURE, TORSION (DIFFERENTIAL GEOMETRY)

References
Gray, A. "The Turning Angle." §1.6 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 13–14, 1993.

Tangential Triangle



The TRIANGLE $\Delta T_1T_2T_3$ formed by the lines tangent to the CIRCUMCIRCLE of a given TRIANGLE $\Delta A_1A_2A_3$ at its VERTICES. It is the PEDAL TRIANGLE of $\Delta A_1A_2A_3$ with the CIRCUMCENTER as the PEDAL POINT. The TRILINEAR COORDINATES of the VERTICES of the tangential triangle are

$$\begin{aligned} A' &= -a : b : c \\ B' &= a : -b : c \\ C' &= a : b : -c. \end{aligned}$$

The CONTACT TRIANGLE and tangential triangle are perspective from the GERGONNE POINT.

see also CIRCUMCIRCLE, CONTACT TRIANGLE, GERGONNE POINT, PEDAL TRIANGLE, PERSPECTIVE

Tangential Triangle Circumcenter

A POINT with TRIANGLE CENTER FUNCTION

$$\alpha = a[b^2\cos(2B) + c^2\cos(2C) - a^2\cos(2A)].$$

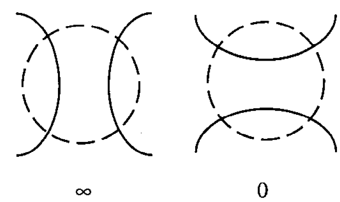
It lies on the EULER LINE.

References
Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.

Tangents Law

see LAW OF TANGENTS

Tangle



A region in a KNOT or LINK projection plane surrounded by a CIRCLE such that the KNOT or LINK crosses the circle exactly four times. Two tangles are equivalent if a sequence of REIDEMEISTER MOVES can be used to transform one into the other while keeping the four string endpoints fixed and not allowing strings to pass outside the CIRCLE.

The simplest tangles are the ∞-tangle and 0-tangle, shown above. A tangle with n left-handed twists is called an n-tangle, and one with n right-handed twists is called a $-n$-tangle. By placing tangles side by side, more complicated tangles can be built up such as $(-2, 3, 2)$, etc. The link created by connecting the ends of the tangles is now described by the sequence of tangle symbols, known as CONWAY'S KNOT NOTATION. If tangles are multiplied by 0 and then added, the resulting tangle symbols are separated by commas. Additional symbols which are used are the period, colon, and asterisk.

Amazingly enough, two tangles described in this NOTATION are equivalent IFF the CONTINUED FRACTIONS of the form

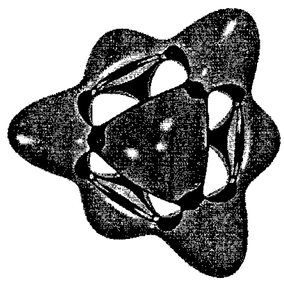
$$2 + \cfrac{1}{3 + \cfrac{1}{-2}}$$

are equal (Burde and Zieschang 1985)! An ALGEBRAIC TANGLE is any tangle obtained by ADDITIONS and MULTIPLICATIONS of rational tangles (Adams 1994). Not all tangles are ALGEBRAIC.

see also ALGEBRAIC LINK, FLYPE, PRETZEL KNOT

References
Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman pp. 41–51, 1994.
Burde, G. and Zieschang, H. *Knots.* Berlin: de Gruyter, 1985.

Tanglecube



A QUARTIC SURFACE given by the implicit equation

$$x^4 - 5x^2 + y^4 - 5y^2 + z^4 - 5z^2 + 11.8 = 0.$$

References
Banchoff, T. "The Best Homework Ever?" http:// www . brown . edu / Administration / Brown _ Alumni _ Monthly/ 12-96/features/homework.html.
Nordstrand, T. "Tangle." http://www.uib.no/people/ nfytn/tangltxt.htm.

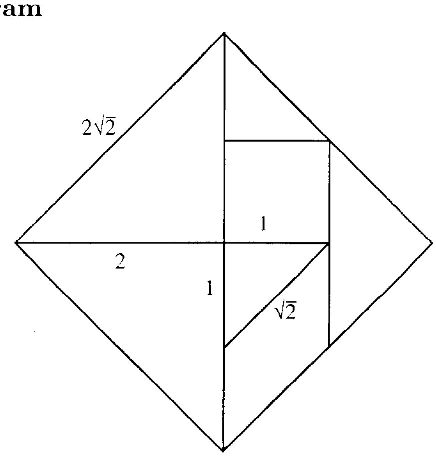
Tangled Hierarchy

A system in which a STRANGE LOOP appears.

see also STRANGE LOOP

References
Hofstadter, D. R. *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Vintage Books, p. 10, 1989.

Tangram



A combination of the above plane polygonal pieces such that the EDGES are coincident. There are 13 convex tangrams (where a "convex tangram" is a set of tangram pieces arranged into a CONVEX POLYGON).

see also ORIGAMI, STOMACHION

References
Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., pp. 19–20, 1989.
Gardner, M. "Tangrams, Parts 1 and 2." Ch. 3-4 in *Time Travel and Other Mathematical Bewilderments.* New York: W. H. Freeman, 1988.
Johnston, S. *Fun with Tangrams Kit: 120 Puzzles with Two Complete Sets of Tangram Pieces.* New York: Dover, 1977.
Pappas, T. "Tangram Puzzle." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 212, 1989.

Tanh

see HYPERBOLIC TANGENT

Taniyama Conjecture

see TANIYAMA-SHIMURA CONJECTURE

Taniyama-Shimura Conjecture

A conjecture which arose from several problems proposed by Taniyama in an international mathematics symposium in 1955. Let E be an ELLIPTIC CURVE whose equation has INTEGER COEFFICIENTS, let N be the CONDUCTOR of E and, for each n, let a_n be the number appearing in the L-function of E. Then there exists a MODULAR FORM of weight two and level N which is an eigenform under the HECKE OPERATORS and has a FOURIER SERIES $\sum a_n q^n$.

The conjecture says, in effect, that every rational ELLIPTIC CURVE is a MODULAR FORM in disguise. Stated formally, the conjecture suggests that, for every ELLIPTIC CURVE $y^2 = Ax^3 + Bx^2 + Cx + D$ over the RATIONALS, there exist nonconstant MODULAR FUNCTIONS $f(z)$ and $g(z)$ of the same level N such that

$$[f(z)]^2 = A[g(z)]^2 + Cg(z) + D.$$

Equivalently, for every ELLIPTIC CURVE, there is a MODULAR FORM with the same DIRICHLET L-SERIES.

In 1985, starting with a fictitious solution to FERMAT'S LAST THEOREM, G. Frey showed that he could create an unusual ELLIPTIC CURVE which appeared not to be modular. If the curve were not modular, then this would show that if FERMAT'S LAST THEOREM were false, then the Taniyama-Shimura conjecture would also be false. Furthermore, if the Taniyama-Shimura conjecture were true, then so would be FERMAT'S LAST THEOREM!

However, Frey did not actually prove whether his curve was modular. The conjecture that Frey's curve *was* modular came to be called the "epsilon conjecture," and was quickly proved by Ribet (RIBET'S THEOREM) in 1986, establishing a very close link between two mathematical structures (the Taniyama-Shimura conjecture and FERMAT'S LAST THEOREM) which appeared previously to be completely unrelated.

As of the early 1990s, most mathematicians believed that the Taniyama-Shimura conjecture was not accessible to proof. However, A. Wiles was not one of these. He attempted to establish the correspondence between the set of ELLIPTIC CURVES and the set of modular elliptic curves by showing that the number of each was the same. Wiles accomplished this by "counting" Galois representations and comparing them with the number of modular forms. In 1993, after a monumental seven-year effort, Wiles (almost) proved the Taniyama-Shimura conjecture for special classes of curves called SEMISTABLE ELLIPTIC CURVES.

Wiles had tried to use horizontal Iwasawa theory to create a so-called CLASS NUMBER formula, but was initially unsuccessful and therefore used instead an extension of a result of Flach based on ideas from Kolyvagin. However, there was a problem with this extension which was discovered during review of Wiles' manuscript in September 1993. Former student Richard Taylor came to Princeton in early 1994 to help Wiles patch up this error. After additional effort, Wiles discovered the reason that the Flach/Kolyvagin approach was failing, and also discovered that it was precisely what had prevented Iwasawa theory from working.

With this additional insight, he was able to successfully complete the erroneous portion of the proof using Iwasawa theory, proving the SEMISTABLE case of the Taniyama-Shimura conjecture (Taylor and Wiles 1995, Wiles 1995) and, at the same time, establishing FERMAT'S LAST THEOREM as a true theorem.

see also ELLIPTIC CURVE, FERMAT'S LAST THEOREM, MODULAR FORM, MODULAR FUNCTION, RIBET'S THEOREM

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Tank

see CYLINDRICAL SEGMENT

Tantrix

see TANGENT INDICATRIX

Tapering Function

see APODIZATION FUNCTION

Tarry-Escott Problem

For each POSITIVE INTEGER l, there exists a POSITIVE INTEGER n and a PARTITION of $\{1, \ldots, n\}$ as a disjoint union of two sets A and B, such that for $1 \leq i \leq l$,

$$\sum_{a\in A} a^i = \sum_{b\in B} b^i.$$

The results extended to three or more sets of INTEGERS are called PROUHET'S PROBLEM.

see also PROUHET'S PROBLEM

References

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Hahn, L. "The Tarry-Escott Problem." Problem 10284. *Amer. Math. Monthly* **102**, 843–844, 1995.

Tarry Point

The point at which the lines through the VERTICES of a TRIANGLE PERPENDICULAR to the corresponding sides of the first BROCARD TRIANGLE, are CONCURRENT. The Tarry point lies on the CIRCUMCIRCLE opposite the STEINER POINT. It has TRIANGLE CENTER FUNCTION

$$\alpha = \frac{bc}{b^4 + c^4 - a^2b^2 - a^2c^2} = \sec(A + \omega),$$

where ω is the BROCARD ANGLE. The SIMSON LINE of the Tarry point is PERPENDICULAR to the line OK, when O is the CIRCUMCENTER and K is the LEMOINE POINT.

see also BROCARD ANGLE, BROCARD TRIANGLES, CIRCUMCIRCLE, LEMOINE POINT, SIMSON LINE, STEINER POINTS

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Tarski's Theorem

Tarski's theorem states that the first-order theory of the FIELD of REAL NUMBERS is DECIDABLE. However, the best-known ALGORITHM for eliminating QUANTIFIERS is doubly exponential in the number of QUANTIFIER blocks (Heintz *et al.* 1989).

References

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Tau Conjecture

Also known as RAMANUJAN'S HYPOTHESIS. Ramanujan proposed that

$$\tau(n) \sim \mathcal{O}(n^{11/2+\epsilon}),$$

where $\tau(n)$ is the TAU FUNCTION, defined by

$$\sum_{n=1}^{\infty} \tau(n)x^n = x(1 - 3x + 5x^3 - 7x^6 + \ldots)^8.$$

This was proven by Deligne (1974), who was subsequently awarded the FIELDS MEDAL for his proof.

see also TAU FUNCTION

References

Deligne, P. "La conjecture de Weil. I." *Inst. Hautes Études Sci. Publ. Math.* **43**, 273–307, 1974.

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Hardy, G. H. *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work, 3rd ed.* New York: Chelsea, p. 169, 1959.

Tau-Dirichlet Series

$$\tau_{DS}(s) \equiv \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s},$$

where $\tau(n)$ is the TAU FUNCTION. Ramanujan conjectured that all nontrivial zeros of $f(z)$ lie on the line $\Re[s] = 6$, where

$$f(s) \equiv \sum_{n=1}^{\infty} \tau(n) n^{-z}$$

and $\tau(n)$ is the TAU FUNCTION.

see also TAU FUNCTION

References
Spira, R. "Calculation of the Ramanujan Tau-Dirichlet Series." *Math. Comput.* **27**, 379–385, 1973.
Yoshida, H. "On Calculations of Zeros of L-Functions Related with Ramanujan's Discriminant Function on the Critical Line." *J. Ramanujan Math. Soc.* **3**, 87–95, 1988.

Tau Function

A function $\tau(n)$ related to the DIVISOR FUNCTION $\sigma_k(n)$, also sometimes called RAMANUJAN'S TAU FUNCTION. It is given by the GENERATING FUNCTION

$$\sum_{n=1}^{\infty} \tau(n) x^n = \prod_{n=1}^{\infty} (1 - x^n)^{24}, \quad (1)$$

and the first few values are 1, -24, 252, -1472, 4380, ... (Sloane's A000594). $\tau(n)$ is also given by

$$g(-x) = \sum_{n=1}^{\infty} (-1)^n \tau(n) x^n \quad (2)$$

$$g(x^2) = \sum_{n=1}^{\infty} \tau(\tfrac{1}{2}n) x^n \quad (3)$$

$$\sum_{n=1}^{\infty} \tau(n) x^n = x(1 - 3x + 5x^3 - 7x^6 + \ldots)^8. \quad (4)$$

In ORE'S CONJECTURE, the tau function appears as the number of DIVISORS of n. Ramanujan conjectured and Mordell proved that if (n, n'), then

$$\tau(nn') = \tau(n)\tau(n'). \quad (5)$$

Ramanujan conjectured and Watson proved that $\tau(n)$ is divisible by 691 for almost all n. If

$$\tau(p) \equiv 0 \pmod{p}, \quad (6)$$

then

$$\tau(pn) \equiv 0 \pmod{p}. \quad (7)$$

Values of p for which the first equation holds are $p = 2$, 3, 5, 7, 23.

Ramanujan also studied

$$f(x) \equiv \sum_{n=1}^{\infty} \tau(n) n^{-s}, \quad (8)$$

which has properties analogous to the RIEMANN ZETA FUNCTION. It satisfies

$$\frac{f(s)\Gamma(s)}{(2\pi)^s} = \frac{f(12-s)}{(2\pi)^{12-s}}, \quad (9)$$

and Ramanujan's TAU-DIRICHLET SERIES conjecture alleges that all nontrivial zeros of $f(s)$ lie on the line $\Re[s] = 6$. f can be split up into

$$f(6 + it) = z(t) e^{-i\theta(t)}, \quad (10)$$

where

$$z(t) = \Gamma(6 + it) f(6 + it)(2\pi)^{-it} \times \sqrt{\frac{\sinh(\pi t)}{\pi t(1+t^2)(4+t^2)(9+t^2)(16+t^2)(25+t^2)}} \quad (11)$$

$$\theta(t) = -\tfrac{1}{2} i \ln\left[\frac{\Gamma(6+it)}{\Gamma(6-it)}\right] - t\ln(2\pi). \quad (12)$$

The SUMMATORY tau function is given by

$$T(n) = \sum_{n \leq x}{}' \tau(n). \quad (13)$$

Here, the prime indicates that when x is an INTEGER, the last term $\tau(x)$ should be replaced by $\frac{1}{2}\tau(x)$.

Ramanujan's tau theta function $Z(t)$ is a REAL function for REAL t and is analogous to the RIEMANN-SIEGEL FUNCTION Z. The number of zeros in the critical strip from $t = 0$ to T is given by

$$N(t) = \frac{\Theta(T) + \Im\{\ln[\tau_{DS}(6 + iT)]\}}{\pi}, \quad (14)$$

where Θ is the RIEMANN THETA FUNCTION and τ_{DS} is the TAU-DIRICHLET SERIES, defined by

$$\tau_{DS}(s) \equiv \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}. \quad (15)$$

Ramanujan conjectured that the nontrivial zeros of the function are all real.

Ramanujan's τ_z function is defined by

$$\tau_z(t) = \frac{\Gamma(6+it)(2\pi)^{-it}}{\tau_{DS}(6+it)\sqrt{\frac{\sinh(\pi t)}{\pi t \prod_{k=1}^{5} k^2+t^2}}}, \quad (16)$$

where $\tau_{DS}(z)$ is the TAU-DIRICHLET SERIES.

see also ORE'S CONJECTURE, TAU CONJECTURE, TAU-DIRICHLET SERIES

References

Hardy, G. H. "Ramanujan's Function $\tau(n)$." Ch. 10 in *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work, 3rd ed.* New York: Chelsea, 1959.

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Tauberian Theorem

A Tauberian theorem is a theorem which deduces the convergence of an INFINITE SERIES on the basis of the properties of the function it defines and any kind of auxiliary HYPOTHESIS which prevents the general term of the series from converging to zero too slowly.

see also HARDY-LITTLEWOOD TAUBERIAN THEOREM

Tautochrone Problem

Find the curve down which a bead placed anywhere will fall to the bottom in the same amount of time. The solution is a CYCLOID, a fact first discovered and published by Huygens in *Horologium oscillatorium* (1673). Huygens also constructed the first pendulum clock with a device to ensure that the pendulum was isochronous by forcing the pendulum to swing in an arc of a CYCLOID.

The parametric equations of the CYCLOID are

$$x = a(\theta - \sin\theta) \quad (1)$$

$$y = a(1 - \cos\theta). \quad (2)$$

To see that the CYCLOID satisfies the tautochrone property, consider the derivatives

$$x' = a(1 - \cos\theta) \quad (3)$$

$$y' = a\sin\theta, \quad (4)$$

and

$$\begin{aligned} x'^2 + y'^2 &= a^2[(1 - 2\cos\theta + \cos^2\theta) + \sin^2\theta] \\ &= 2a^2(1 - \cos\theta). \end{aligned} \quad (5)$$

Now

$$\tfrac{1}{2}mv^2 = mgy \quad (6)$$

$$v = \frac{ds}{dt} = \sqrt{2gy} \quad (7)$$

$$\begin{aligned} dt &= \frac{ds}{\sqrt{2gy}} = \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gy}} \\ &= \frac{a\sqrt{2(1-\cos\theta)}\,d\theta}{\sqrt{2ga(1-\cos\theta)}} = \sqrt{\frac{a}{g}}\,d\theta, \end{aligned} \quad (8)$$

so the time required to travel from the top of the CYCLOID to the bottom is

$$T = \int_0^\pi dt = \sqrt{\frac{a}{g}}\,\pi. \quad (9)$$

However, from an intermediate point θ_0,

$$v = \frac{ds}{dt} = \sqrt{2g(y - y_0)}, \quad (10)$$

so

$$\begin{aligned} T &= \int_{\theta_0}^{\pi} \frac{\sqrt{2a^2(1-\cos\theta)}}{2ag(\cos\theta_0 - \cos\theta)}\,d\theta \\ &= \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \sqrt{\frac{1-\cos\theta}{\cos\theta_0 - \cos\theta}}\,d\theta \\ &= \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \frac{\sin(\frac{1}{2}\theta)\,d\theta}{\sqrt{\cos^2(\frac{1}{2}\theta_0) - \cos^2(\frac{1}{2}\theta)}}. \end{aligned} \quad (11)$$

Now let

$$u = \frac{\cos(\frac{1}{2}\theta)}{\cos(\frac{1}{2}\theta_0)} \quad (12)$$

$$du = -\frac{\sin(\frac{1}{2}\theta)d\theta}{2\cos(\theta_0)}, \quad (13)$$

so

$$T = -2\sqrt{\frac{a}{g}} \int_1^0 \frac{du}{\sqrt{1-u^2}} = 2\sqrt{\frac{a}{g}}\,[\sin^{-1} u]_0^1 = \pi\sqrt{\frac{a}{g}}, \quad (14)$$

and the amount of time is the same from any point!

see also BRACHISTOCHRONE PROBLEM, CYCLOID

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Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, pp. 54–60 and 384–385, 1991.

Tautology

A logical statement in which the conclusion is equivalent to the premise. If p is a tautology, it is written $\models p$.

Taxicab Number

The nth taxicab number $\mathrm{Ta}(n)$ is the smallest number representable in n ways as a sum of POSITIVE CUBES. The numbers derive their name from the HARDY-RAMANUJAN NUMBER

$$\begin{aligned}\mathrm{Ta}(2) &= 1729\\ &= 1^3 + 12^3\\ &= 9^3 + 10^3,\end{aligned} \tag{1}$$

which is associated with the following story told about Ramanujan by G. H. Hardy. "Once, in the taxi from London, Hardy noticed its number, 1729. He must have thought about it a little because he entered the room where Ramanujan lay in bed and, with scarcely a hello, blurted out his disappointment with it. It was, he declared, 'rather a dull number,' adding that he hoped that wasn't a bad omen. 'No, Hardy,' said Ramanujan, 'it is a very interesting number. It is the smallest number expressible as the sum of two [POSITIVE] cubes in two different ways'" (Hofstadter 1989, Kanigel 1991, Snow 1993).

However, this property was also known as early as 1657 by F. de Bessy (Berndt and Bhargava 1993, Guy 1994). Leech (1957) found

$$\begin{aligned}\mathrm{Ta}(3) &= 87539319\\ &= 167^3 + 436^3\\ &= 228^3 + 423^3\\ &= 255^3 + 414^3.\end{aligned} \tag{2}$$

Rosenstiel *et al.* (1991) recently found

$$\begin{aligned}\mathrm{Ta}(4) &= 6963472309248\\ &= 2421^3 + 19083^3\\ &= 5436^3 + 18948^3\\ &= 10200^3 + 18072^3\\ &= 13322^3 + 16630^3.\end{aligned} \tag{3}$$

D. Wilson found

$$\begin{aligned}\mathrm{Ta}(5) &= 48988659276962496\\ &= 38787^3 + 365757^3\\ &= 107839^3 + 362753^3\\ &= 205292^3 + 342952^3\\ &= 221424^3 + 336588^3\\ &= 231518^3 + 331954^3.\end{aligned} \tag{4}$$

The first few taxicab numbers are therefore 2, 1729, 87539319, 6963472309248, ... (Sloane's A011541).

Hardy and Wright (Theorem 412, 1979) show that the number of such sums can be made arbitrarily large but, updating Guy (1994) with Wilson's result, the least example is not known for six or more equal sums.

Sloane defines a slightly different type of taxicab numbers, namely numbers which are sums of two cubes in two or more ways, the first few of which are 1729, 4104, 13832, 20683, 32832, 39312, 40033, 46683, 64232, ... (Sloane's A001235).

see also DIOPHANTINE EQUATION—CUBIC, HARDY-RAMANUJAN NUMBER

References

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Taylor Center

The center of the TAYLOR CIRCLE, which is the SPIEKER CENTER of $\Delta H_1H_2H_3$, where H_i are the ALTITUDES.

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, p. 277, 1929.

Taylor Circle

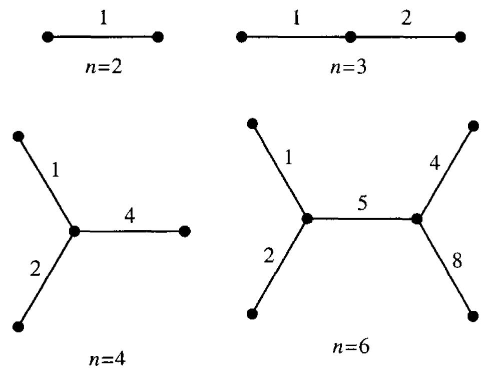
From the feet of each ALTITUDE of a TRIANGLE, draw lines PERPENDICULAR to the adjacent sides. Then the feet of these perpendiculars lie on a CIRCLE called the TAYLOR CIRCLE.

see also TUCKER CIRCLES

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, p. 277, 1929.

Taylor's Condition



For a given POSITIVE INTEGER n, does there exist a WEIGHTED TREE with n VERTICES whose paths have weights 1, 2, ..., $\binom{n}{2}$, where $\binom{n}{2}$ is a BINOMIAL COEFFICIENT? Taylor showed that no such TREE can exist unless it is a PERFECT SQUARE or a PERFECT SQUARE plus 2. No such TREES are known except $n = 2$, 3, 4, and 6.

see also GOLOMB RULER, PERFECT DIFFERENCE SET

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Taylor Expansion

see TAYLOR SERIES

Taylor-Greene-Chirikov Map

see STANDARD MAP

Taylor Polynomial

see TAYLOR SERIES

Taylor Series

A Taylor series is a series expansion of a FUNCTION about a point. A 1-D Taylor series is an expansion of a SCALAR FUNCTION $f(x)$ about a point $x = a$. If $a = 0$, the expansion is known as a MACLAURIN SERIES.

$$\int_a^x f^{(n)}(x)\,dx = [f^{(n-1)}(x)]_a^x = f^{(n-1)}(x) - f^{(n-1)}(a) \tag{1}$$

$$\int_a^x \left[\int_a^x f^{(n)}(x)\,dx\right] dx = \int_a^x [f^{(n-1)}(x) - f^{(n-1)}(a)]\,dx$$
$$= f^{(n-2)}(x) - f^{(n-2)}(a) - (x-a)f^{(n-1)}(a). \tag{2}$$

Continuing,

$$\iiint_a^x f^{(n)}(x)\,(dx)^3 = f^{(n-3)}(a) - (x-a)f^{(n-2)}(a)$$
$$-\tfrac{1}{2!}(x-a)^2 f^{(n-1)}(a) \tag{3}$$

$$\underbrace{\int\cdots\int_a^x}_{n} f^{(n)}(x)\,(dx)^n = f(x) - f(a) - (x-a)f'(a)$$
$$-\tfrac{1}{2!}(x-a)^2 f''(a) - \ldots - \tfrac{1}{(n-1)!}(x-a)^{n-1} f^{(n-1)}(a). \tag{4}$$

Therefore, we obtain the 1-D Taylor series

$$f(x) = f(a) + (x-a)f'(a) + \tfrac{1}{2!}(x-a)^2 f''(a) + \ldots$$
$$+ \tfrac{1}{(n-1)!}(x-a)^{n-1} f^{(n-1)}(a) + R_n, \tag{5}$$

where R_n is a remainder term defined by

$$R_n = \underbrace{\int\cdots\int_a^x}_{n} f^{(n)}(x)\,(dx)^n. \tag{6}$$

Using the MEAN-VALUE THEOREM for a function g, it must be true that

$$\int_a^x g(x)\,dx = (x-a)g(x^*) \tag{7}$$

for some $x^* \in [a, x]$. Therefore, integrating n times gives the result

$$R_n = \frac{(x-a)^n}{n!} f^{(n)}(x). \tag{8}$$

The maximum error is then the maximum value of (8) for all possible $x^* \in [a, x]$.

An alternative form of the 1-D Taylor series may be obtained by letting

$$x - a \equiv \Delta x \tag{9}$$

so that

$$x = a + \Delta x \equiv x_0 + \Delta x. \tag{10}$$

Substitute this result into (5) to give

$$f(x_0 + \Delta x) = f(x_0) + \Delta x f'(x_0) + \tfrac{1}{2!}(\Delta x)^2 f''(x_0) + \ldots. \tag{11}$$

A Taylor series of a FUNCTION in two variables $f(x, y)$ is given by

$$f(x+\Delta x, y+\Delta y) = f(x,y) + [f_x(x,y)\Delta x + f_y(x,y)\Delta y]$$
$$+ \tfrac{1}{2!}[(\Delta x)^2 f_{xx}(x,y) + 2\Delta x\Delta y f_{xy}(x,y) + (\Delta y)^2 f_{yy}(x,y)]$$
$$+ \tfrac{1}{3!}[(\Delta x)^3 f_{xxx}(x,y) + 3(\Delta x)^2 \Delta y f_{xxy}(x,y)$$
$$+ 3\Delta x(\Delta y)^2 f_{xyy}(x,y) + (\Delta y)^3 f_{yyy}(x,y)] + \ldots. \tag{12}$$

This can be further generalized for a FUNCTION in n variables,

$$f(x_1, \ldots, x_n)$$
$$= \sum_{j=0}^{\infty} \left\{ \frac{1}{j!} \left[\sum_{k=1}^{n} (x'_k - a_k) \frac{\partial}{\partial x'_k} \right]^j f(x'_1, \ldots, x'_n) \right\}_{x'_1 = a_1, \ldots, x'_n = a_n}. \tag{13}$$

Rewriting,

$$f(x_1+a_1,\ldots,x_n+a_n) = \sum_{j=0}^{\infty}\left\{\frac{1}{j!}\left(\sum_{k=1}^{n} a_k \frac{\partial}{\partial x'_k}\right)^j f(x'_1,\ldots,x'_n)\right\}_{x'_1=x_1,\ldots,x'_n=x_n}. \tag{14}$$

Taking $n=2$ in (13) gives

$$\begin{aligned}f(x_1,x_2) &= \sum_{j=0}^{\infty}\left\{\frac{1}{j!}\left[(x'_1-a_1)\frac{\partial}{\partial x'_1} + (x'_2-a_2)\frac{\partial}{\partial x'_2}\right]^j f(x'_1,x'_2)\right\}_{x'_1=x_1,x'_2=x_2}\\ &= f(a_1,a_2) + \left[(x_1-a_1)\frac{\partial f}{\partial x_1} + (x_2-a_2)\frac{\partial f}{\partial x_2}\right]\\ &\quad + \frac{1}{2!}\left[(x_1-a_1)^2\frac{\partial^2 f}{\partial x_1{}^2} + 2(x_1-a_1)(x_2-a_2)\frac{\partial^2 f}{\partial x_1 \partial x_2} + (x_2-a_2)^2\frac{\partial^2 f}{\partial x_2{}^2}\right] + \ldots.\end{aligned} \tag{15}$$

Taking $n=3$ in (14) gives

$$f(x_1+a_1,x_2+a_2,x_3+a_3) = \sum_{j=0}^{\infty}\left\{\frac{1}{j!}\left(a_1\frac{\partial}{\partial x'_1} + a_2\frac{\partial}{\partial x'_2} + a_3\frac{\partial}{\partial x'_3}\right)^j \times f(x'_1,x'_2,x'_3)\right\}_{x'_1=x_1,x'_2=x_2,x'_3=x_3}, \tag{16}$$

or, in VECTOR form

$$f(\mathbf{r}+\mathbf{a}) = \sum_{j=0}^{\infty}\left[\frac{1}{j!}(\mathbf{a}\cdot\nabla_{\mathbf{r}'})^j f(\mathbf{r}')\right]_{\mathbf{r}'=\mathbf{r}}. \tag{17}$$

The zeroth- and first-order terms are

$$f(\mathbf{r}) \tag{18}$$

and

$$(\mathbf{a}\cdot\nabla_{\mathbf{r}'})f(\mathbf{r}')|_{\mathbf{r}'=\mathbf{r}}, \tag{19}$$

respectively. The second-order term is

$$\begin{aligned}\tfrac{1}{2}(\mathbf{a}\cdot\nabla_{\mathbf{r}'})(\mathbf{a}\cdot\nabla_{\mathbf{r}'})f(\mathbf{r}')|_{\mathbf{r}'=\mathbf{r}} &= \tfrac{1}{2}\mathbf{a}\cdot\nabla_{\mathbf{r}'}[\mathbf{a}\cdot(\nabla f(\mathbf{r}'))]_{\mathbf{r}'=\mathbf{r}}\\ &= \tfrac{1}{2}\mathbf{a}\cdot[\mathbf{a}\cdot\nabla_{\mathbf{r}'}(\nabla_{\mathbf{r}'}f(\mathbf{r}'))]|_{\mathbf{r}'=\mathbf{r}},\end{aligned} \tag{20}$$

so the first few terms of the expansion are

$$f(\mathbf{r}+\mathbf{a}) = f(\mathbf{r}) + (\mathbf{a}\cdot\nabla_{\mathbf{r}'})f(\mathbf{r}')|_{\mathbf{r}'=\mathbf{r}} + \tfrac{1}{2}\mathbf{a}\cdot[\mathbf{a}\cdot\nabla_{\mathbf{r}'}(\nabla_{\mathbf{r}'}f(\mathbf{r}'))]|_{\mathbf{r}'=\mathbf{r}}. \tag{21}$$

Taylor series can also be defined for functions of a COMPLEX variable. By the CAUCHY INTEGRAL FORMULA,

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\int_C \frac{f(z')\,dz}{z'-z} = \frac{1}{2\pi i}\int_C \frac{f(z')\,dz'}{(z'-z_0)-(z-z_0)}\\ &= \frac{1}{2\pi i}\int_C \frac{f(z')\,dz'}{(z'-z_0)\left(1-\frac{z-z_0}{z'-z_0}\right)}.\end{aligned} \tag{22}$$

In the interior of C,

$$\frac{|z-z_0|}{|z'-z_0|} < 1 \tag{23}$$

so, using

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n, \tag{24}$$

it follows that

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\int_C \sum_{n=0}^{\infty}\frac{(z-z_0)^n f(z')\,dz'}{(z'-z_0)^{n+1}}\\ &= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \int_C \frac{f(z')\,dz}{(z'-z_0)^{n+1}}.\end{aligned} \tag{25}$$

Using the the CAUCHY INTEGRAL FORMULA for derivatives,

$$f(z) = \sum_{n=0}^{\infty}(z-z_0)^n \frac{f^{(n)}(z_0)}{n!}. \tag{26}$$

see also CAUCHY REMAINDER FORM, LAGRANGE EXPANSION, LAURENT SERIES, LEGENDRE SERIES, MACLAURIN SERIES, NEWTON'S FORWARD DIFFERENCE FORMULA

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Tchebycheff

see CHEBYSHEV APPROXIMATION FORMULA, CHEBYSHEV CONSTANTS, CHEBYSHEV DEVIATION, CHEBYSHEV DIFFERENTIAL EQUATION, CHEBYSHEV FUNCTION, CHEBYSHEV-GAUSS QUADRATURE, CHEBYSHEV INEQUALITY, CHEBYSHEV INEQUALITY, CHEBYSHEV INTEGRAL, CHEBYSHEV PHENOMENON, CHEBYSHEV POLYNOMIAL OF THE FIRST KIND, CHEBYSHEV POLYNOMIAL OF THE SECOND KIND, CHEBYSHEV QUADRATURE, CHEBYSHEV-RADAU QUADRATURE, CHEBYSHEV-SYLVESTER CONSTANT

Teardrop Curve

A plane curve given by the parametric equations

$$x = \cos t$$
$$y = \sin t \sin^m(\tfrac{1}{2}t).$$

see also PEAR-SHAPED CURVE

References

von Seggern, D. *CRC Standard Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 174, 1993.

Technique

A specific method of performing an operation. The terms ALGORITHM, METHOD, and PROCEDURE are also used interchangeably.

see also ALGORITHM, METHOD, PROCEDURE

Teichmüller Space

TEICHMÜLLER'S THEOREM asserts the EXISTENCE and UNIQUENESS of the extremal quasiconformal map between two compact RIEMANN SURFACES of the same GENUS modulo an EQUIVALENCE RELATION. The equivalence classes form the Teichmüller space T_p of compact RIEMANN SURFACES of GENUS p.

see also RIEMANN'S MODULI PROBLEM

Teichmüller's Theorem

Asserts the EXISTENCE and UNIQUENESS of the extremal quasiconformal map between two compact RIEMANN SURFACES of the same GENUS modulo an EQUIVALENCE RELATION.

see also TEICHMÜLLER SPACE

Telescoping Sum

A sum in which subsequent terms cancel each other, leaving only initial and final terms. For example,

$$\begin{aligned} S &= \sum_{i=1}^{n-1}\left(\frac{1}{a_i}-\frac{1}{a_{i+1}}\right) \\ &= \left(\frac{1}{a_1}-\frac{1}{a_2}\right)+\left(\frac{1}{a_2}-\frac{1}{a_3}\right)+\ldots \\ &\quad +\left(\frac{1}{a_{n-2}}-\frac{1}{a_{n-1}}\right)+\left(\frac{1}{a_{n-1}}-\frac{1}{a_n}\right) \\ &= \frac{1}{a_1}-\frac{1}{a_n} \end{aligned}$$

is a telescoping sum.

see also ZEILBERGER'S ALGORITHM

Temperature

The "temperature" of a curve Γ is defined as

$$T \equiv \frac{1}{\ln\left(\frac{2l}{2l-h}\right)},$$

where l is the length of Γ and h is the length of the PERIMETER of the CONVEX HULL. The temperature of a curve is 0 only if the curve is a straight line, and increases as the curve becomes more "wiggly."

see also CURLICUE FRACTAL

References

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Templar Magic Square

S	A	T	O	R
A	R	E	P	O
T	E	N	E	T
O	P	E	R	A
R	O	T	A	S

A MAGIC SQUARE-type arrangement of the words in the Latin sentence "Sator Arepo tenet opera rotas" ("the farmer Arepo keeps the world rolling"). This square has been found in excavations of ancient Pompeii.

see also MAGIC SQUARE

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Hocke, G. R. *Manierismus in der Literatur: Sprach-Alchimie und esoterische Kombinationskunst.* Hamburg, Germany: Rowohlt, p. 24, 1967.

Tennis Ball Theorem

A closed simple smooth spherical curve dividing the SPHERE into two parts of equal areas has at least four inflection points.

see also BALL, BASEBALL COVER

References

Arnold, V. I. *Topological Invariants of Plane Curves and Caustics.* Providence, RI: Amer. Math. Soc., 1994.

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Tensor

An nth-RANK tensor of order m is a mathematical object in m-dimensional space which has n indices and m^n components and obeys certain transformation rules. Each index of a tensor ranges over the number of dimensions of SPACE. If the components of any tensor of any RANK vanish in one particular coordinate system, they vanish in all coordinate systems.

Zeroth-RANK tensors are called SCALARS, and first-RANK tensors are called VECTORS. In tensor notation, a vector $\mathbf{v}$ would be written v_i, where $i = 1, \ldots, m$. Tensor notation can provide a very concise way of writing vector and more general identities. For example, in tensor notation, the DOT PRODUCT $\mathbf{u} \cdot \mathbf{v}$ is simply written

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i, \tag{1}$$

where repeated indices are summed over (EINSTEIN SUMMATION) so that $u_i v_i$ stands for $u_1 v_1 + \ldots + u_m v_m$. Similarly, the CROSS PRODUCT can be concisely written as

$$\mathbf{u} \times \mathbf{v} = \epsilon_{ijk} u^j v^k, \tag{2}$$

where ϵ_{ijk} is the LEVI-CIVITA TENSOR.

Second-RANK tensors resemble square MATRICES. CONTRAVARIANT second-RANK tensors are objects which transform as

$$A'^{ij} = \frac{\partial x'_i}{\partial x_k} \frac{\partial x'_j}{\partial x_l} A^{kl}. \tag{3}$$

COVARIANT second-RANK tensors are objects which transform as

$$C'_{ij} = \frac{\partial x_k}{\partial x'_i} \frac{\partial x_l}{\partial x'_i} C_{kl}. \tag{4}$$

MIXED second-RANK tensors are objects which transform as

$$B'^j_i = \frac{\partial x'_i}{\partial x_k} \frac{\partial x_l}{\partial x'_j} B^k_l. \tag{5}$$

If two tensors A and B have the same RANK and the same COVARIANT and CONTRAVARIANT indices, then

$$A^{ij} + B^{ij} = C^{ij} \tag{6}$$

$$A_{ij} + B_{ij} = C_{ij} \tag{7}$$

$$A^i_j + B^i_j = C^i_j. \tag{8}$$

A transformation of the variables of a tensor changes the tensor into another whose components are linear HOMOGENEOUS FUNCTIONS of the components of the original tensor.

see also ANTISYMMETRIC TENSOR, CURL, DIVERGENCE, GRADIENT, IRREDUCIBLE TENSOR, ISOTROPIC TENSOR, JACOBI TENSOR, RICCI TENSOR, RIEMANN TENSOR, SCALAR, SYMMETRIC TENSOR, TORSION TENSOR, VECTOR, WEYL TENSOR

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Morse, P. M. and Feshbach, H. "Vector and Tensor Formalism." §1.5 in *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 44–54, 1953.

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Sokolnikoff, I. S. *Tensor Analysis—Theory and Applications, 2nd ed.* New York: Wiley, 1964.

Synge, J. L. and Schild, A. *Tensor Calculus.* New York: Dover, 1978.

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Tensor Calculus

The set of rules for manipulating and calculating with TENSORS.

Tensor Density

A quantity which transforms like a TENSOR except for a scalar factor of a JACOBIAN.

Tensor Dual

see DUAL TENSOR

Tensor Product

see DIRECT PRODUCT (TENSOR)

Tensor Space

Let E be a linear space over a FIELD K. Then the DIRECT PRODUCT $\bigotimes_{\lambda=1}^k E$ is called a tensor space of degree k.

References

Yokonuma, T. *Tensor Spaces and Exterior Algebra.* Providence, RI: Amer. Math. Soc., 1992.

Tensor Spherical Harmonic

see DOUBLE CONTRACTION RELATION

Tensor Transpose

see TRANSPOSE

Tent Map

A piecewise linear, 1-D MAP on the interval $[0,1]$ exhibiting CHAOTIC dynamics and given by

$$x_{n+1} = \mu(1 - 2|x_n - \tfrac{1}{2}|).$$

The case $\mu = 1$ is equivalent to the LOGISTIC EQUATION WITH $r = 4$, so the NATURAL INVARIANT in this case is

$$\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}.$$

see also $2x$ MOD 1 MAP, LOGISTIC EQUATION, LOGISTIC EQUATION WITH $r = 4$

Terminal

see SINK (DIRECTED GRAPH)

Ternary

The BASE 3 method of counting in which only the digits 0, 1, and 2 are used. These digits have the following multiplication table.

$\times$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	11

Erdős and Graham (1980) conjectured that no POWER of 2, 2^n, is a SUM of distinct powers of 3 for $n > 8$. This is equivalent to the requirement that the ternary expansion of 2^n always contains a 2. This has been verified by Vardi (1991) up to $n = 2 \cdot 3^{20}$. N. J. A. Sloane has conjectured that any POWER of 2 has a 0 in its ternary expansion (Vardi 1991, p. 28).

see also BASE (NUMBER), BINARY, DECIMAL, HEXADECIMAL, OCTAL, QUATERNARY

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Erdős, P. and Graham, R. L. *Old and New Problems and Results in Combinatorial Number Theory.* Geneva, Switzerland: L'Enseignement Mathématique Université de Genève, Vol. 28, 1980.

Lauwerier, H. *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, pp. 10–11, 1991.

Vardi, I. "The Digits of 2^n in Base Three." *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, pp. 20–25, 1991.

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Tessellation

A regular TILING of POLYGONS (in 2-D), POLYHEDRA (3-D), or POLYTOPES (n-D) is called a tessellation. Tessellations can be specified using a SCHLÄFLI SYMBOL.

Consider a 2-D tessellation with q regular p-gons at each VERTEX. In the PLANE,

$$\left(1 - \frac{2}{p}\right)\pi = \frac{2\pi}{q} \tag{1}$$

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \tag{2}$$

so

$$(p-2)(q-2) = 4 \tag{3}$$

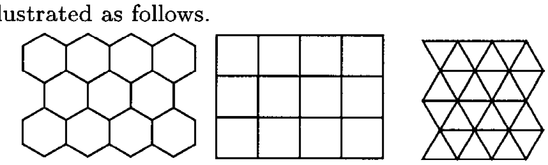
(Ball and Coxeter 1987), and the only factorizations are

$$4 = 4 \cdot 1 = (6-2)(3-2) \Rightarrow \{6,3\} \tag{4}$$

$$= 2 \cdot 2 = (4-2)(4-2) \Rightarrow \{4,4\} \tag{5}$$

$$= 1 \cdot 4 = (3-2)(6-2) \Rightarrow \{3,6\}. \tag{6}$$

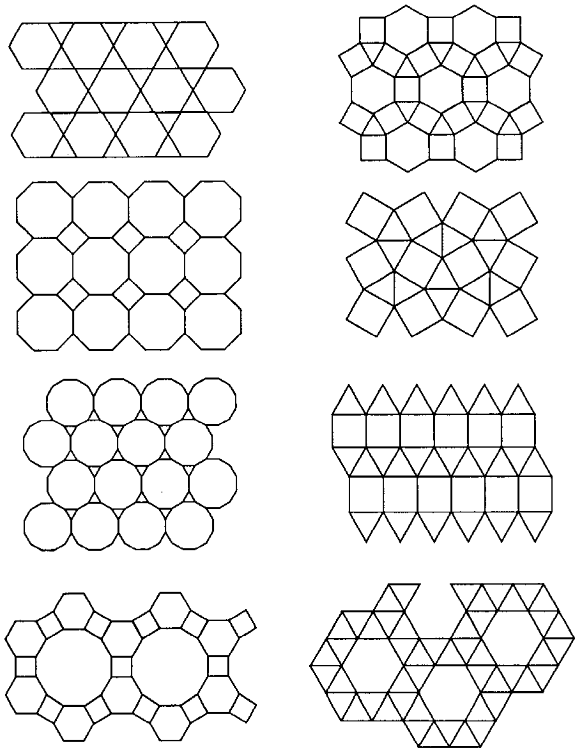
Therefore, there are only three regular tessellations (composed of the HEXAGON, SQUARE, and TRIANGLE), illustrated as follows.



$\{6,3\}$ $\{4,4\}$ $\{3,6\}$

There do not exist any regular STAR POLYGON tessellations in the PLANE. Regular tessellations of the SPHERE by SPHERICAL TRIANGLES are called TRIANGULAR SYMMETRY GROUPS.

Regular tilings of the plane by *two or more* convex regular POLYGONS such that the same POLYGONS in the same order surround each VERTEX are called semiregular tilings. In the plane, there are eight such tessellations, illustrated below.



In 3-D, a POLYHEDRON which is capable of tessellating space is called a SPACE-FILLING POLYHEDRON. Examples include the CUBE, RHOMBIC DODECAHEDRON, and TRUNCATED OCTAHEDRON. There is also a 16-sided space-filler and a convex POLYHEDRON known as the SCHMITT-CONWAY BIPRISM which fills space only aperiodically.

A tessellation of n-D polytopes is called a HONEYCOMB.

see also ARCHIMEDEAN SOLID, CELL, HONEYCOMB, SCHLÄFLI SYMBOL, SEMIREGULAR POLYHEDRON, SPACE-FILLING POLYHEDRON, TILING, TRIANGULAR SYMMETRY GROUP

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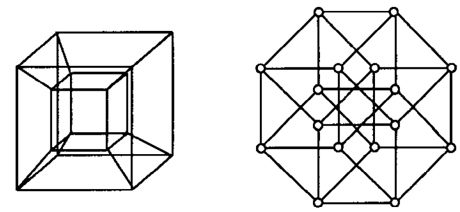
Tesseract



The HYPERCUBE in $\mathbb{R}^4$ is called a tesseract. It has the SCHLÄFLI SYMBOL $\{4,3,3\}$, and VERTICES $(\pm1,\pm1,\pm1,\pm1)$. The above figures show two visualizations of the TESSERACT. The figure on the left is a projection of the TESSERACT in 3-space (Gardner 1977), and the figure on the right is the GRAPH of the TESSERACT symmetrically projected into the PLANE (Coxeter 1973). A TESSERACT has 16 VERTICES, 32 EDGES, 4 SQUARES, and 8 CUBES.

see also HYPERCUBE, POLYTOPE

References

Coxeter, H. S. M. *Regular Polytopes, 3rd ed.* New York: Dover, p. 123, 1973.

Gardner, M. "Hypercubes." Ch. 4 in *Mathematical Carnival: A New Round-Up of Tantalizers and Puzzles from Scientific American.* New York: Vintage Books, 1977.

Geometry Center. "The Tesseract (or Hypercube)." `http://www.geom.umn.edu/docs/outreach/4-cube/`.

Tesseral Harmonic

A SPHERICAL HARMONIC which is expressible as products of factors linear in x^2, y^2, and z^2 multiplied by one of 1, x, y, z, yz, zx, xy, and xyz.

see also ZONAL HARMONIC

Tethered Bull Problem

Let a bull be tethered to a silo whose horizontal CROSS-SECTION is a CIRCLE of RADIUS R by a leash of length L. Then the AREA which the bull can graze if $L \leq R\pi$ is

$$A = \frac{\pi L^2}{2} + \frac{L^3}{3R}.$$

References

Hoffman, M. E. "The Bull and the Silo: An Application of Curvature." *Amer. Math. Monthly* **105**, 55–58, 1998.

Tetrabolo

A 4-POLYABOLO.

Tetrachoric Function

The function defined by

$$T_n \equiv \frac{(-1)^{n-1}}{\sqrt{n!}} Z^{(n-1)}(x),$$

where

$$Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

see also NORMAL DISTRIBUTION

References

Kenney, J. F. and Keeping, E. S. "Tetrachoric Correlation." §8.5 in *Mathematics of Statistics, Pt. 2, 2nd ed.* Princeton, NJ: Van Nostrand, pp. 205–207, 1951.

Tetracontagon

A 40-sided POLYGON.

Tetracuspid

see HYPOCYCLOID—4-CUSPED

Tetrad

A SET of four, also called a QUARTET.

see also HEXAD, MONAD, PAIR, QUARTET, QUINTET, TRIAD, TRIPLE, TWINS

Tetradecagon

A 14-sided POLYGON, sometimes called a TETRAKAIDECAGON.

Tetradecahedron

A 14-sided POLYHEDRON, sometimes called a TETRAKAIDECAHEDRON.

see also CUBOCTAHEDRON, TRUNCATED OCTAHEDRON

References

Ghyka, M. *The Geometry of Art and Life.* New York: Dover, p. 54, 1977.

Tetradic

Tetradics transform DYADICS in much the same way that DYADICS transform VECTORS. They are represented using Hebrew characters and have 81 components (Morse and Feshbach 1953, pp. 72–73). The use of tetradics is archaic, since TENSORS perform the same function but are notationally simpler.

References

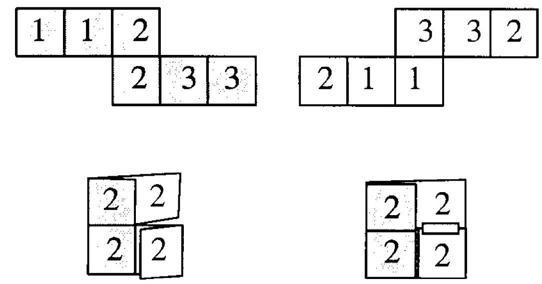
Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Vol. 1.* New York: McGraw-Hill, 1953.

Tetradyakis Hexahedron

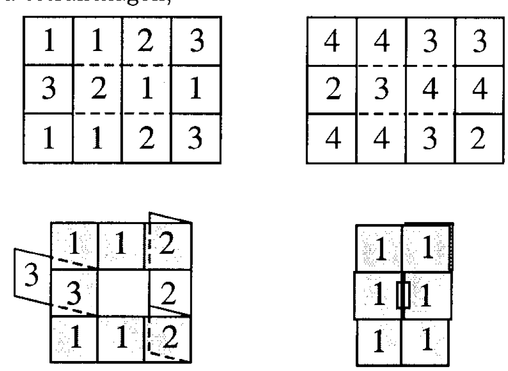
The DUAL POLYHEDRON of the CUBITRUNCATED CUBOCTAHEDRON.

Tetraflexagon

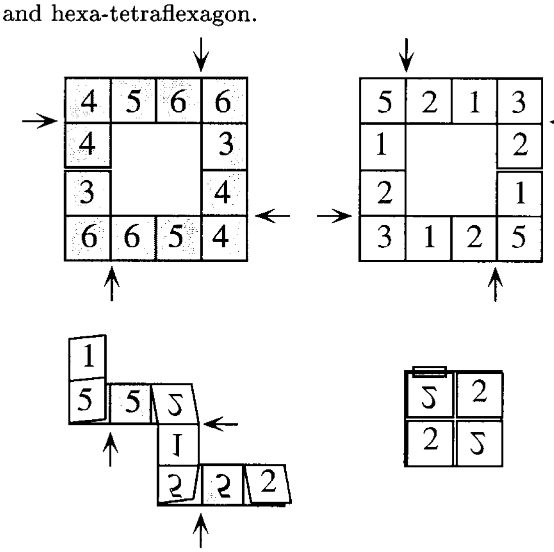
A FLEXAGON made with SQUARE faces. Gardner (1961) shows how to construct a tri-tetraflexagon,



tetra-tetraflexagon,



and hexa-tetraflexagon.



see also FLEXAGON, FLEXATUBE, HEXAFLEXAGON

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Gardner, M. Ch. 1 in *The Scientific American Book of Mathematical Puzzles & Diversions.* New York: Simon and Schuster, 1959.

Gardner, M. Ch. 2 in *The Second Scientific American Book of Mathematical Puzzles & Diversions: A New Selection.* New York: Simon and Schuster, 1961.

Pappas, T. "Making a Tri-Tetra Flexagon." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 107, 1989.

Tetragon

see QUADRILATERAL

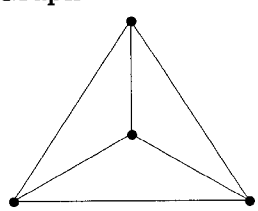
Tetrahedral Coordinates

Coordinates useful for plotting projective 3-D curves of the form $f(x_0, x_1, x_2, x_3) = 0$ which are defined by

$$\begin{aligned} x_0 &= 1 - z - \sqrt{2}\,x \\ x_1 &= 1 - z + \sqrt{2}\,x \\ x_2 &= 1 + z + \sqrt{2}\,y \\ x_3 &= 1 + z - \sqrt{2}\,y. \end{aligned}$$

see also CAYLEY CUBIC, KUMMER SURFACE

Tetrahedral Graph



A POLYHEDRAL GRAPH which is also the COMPLETE GRAPH K_4.

see also CUBICAL GRAPH, DODECAHEDRAL GRAPH, ICOSAHEDRAL GRAPH, OCTAHEDRAL GRAPH, TETRAHEDRON

Tetrahedral Group

The POINT GROUP of symmetries of the TETRAHEDRON, denoted T_d. The tetrahedral group has symmetry operations E, $8C_3$, $3C_2$, $6S_4$, and $6\sigma_d$ (Cotton 1990).

see also ICOSAHEDRAL GROUP, OCTAHEDRAL GROUP, POINT GROUPS, TETRAHEDRON

References

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Lomont, J. S. "Icosahedral Group." §3.10.C in *Applications of Finite Groups.* New York: Dover, p. 81, 1987.

Tetrahedral Number

A FIGURATE NUMBER Te_n of the form

$$Te_n = \sum_{i=1}^{n} T_n = \tfrac{1}{6}n(n+1)(n+2) = \binom{n+2}{3}, \qquad (1)$$

where T_n is the nth TRIANGULAR NUMBER and $\binom{n}{m}$ is a BINOMIAL COEFFICIENT. These numbers correspond to placing discrete points in the configuration of a TETRAHEDRON (triangular base pyramid). Tetrahedral numbers are PYRAMIDAL NUMBERS with $r = 3$, and are the sum of consecutive TRIANGULAR NUMBERS. The first few are 1, 4, 10, 20, 35, 56, 84, 120, ... (Sloane's A000292). The GENERATING FUNCTION of the tetrahedral numbers is

$$\frac{x}{(x-1)^4} = x + 4x^2 + 10x^3 + 20x^4 + \ldots. \qquad (2)$$

Tetrahedral numbers are EVEN, except for every fourth tetrahedral number, which is ODD (Conway and Guy 1996).

The only numbers which are simultaneously SQUARE and TETRAHEDRAL are $Te_1 = 1$, $Te_2 = 4$, and $Te_{48} = 19600$ (giving $S_1 = 1$, $S_2 = 4$, and $S_{140} = 19600$), as proved by Meyl (1878; cited in Dickson 1952, p. 25). Numbers which are simultaneously TRIANGULAR and tetrahedral satisfy the BINOMIAL COEFFICIENT equation

$$\binom{n}{2} = \binom{m}{3}, \qquad (3)$$

the only solutions of which are $(m, n) = (10, 16)$, (22, 56), and (36, 120) (Guy 1994, p. 147). Beukers (1988) has studied the problem of finding numbers which are simultaneously tetrahedral and PYRAMIDAL via INTEGER points on an ELLIPTIC CURVE, and finds that the only solution is the trivial $Te_1 = P_1 = 1$.

see also PYRAMIDAL NUMBER, TRUNCATED TETRAHEDRAL NUMBER

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Meyl, A.-J.-J. "Solution de Question 1194." *Nouv. Ann. Math.* **17**, 464–467, 1878.

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Tetrahedral Surface

A SURFACE given by the parametric equations

$$\begin{aligned} x &= A(u-a)^m(v-a)^n \\ y &= B(u-b)^m(v-b)^n \\ z &= C(u-c)^m(v-c)^n. \end{aligned}$$

References

Eisenhart, L. P. *A Treatise on the Differential Geometry of Curves and Surfaces.* New York: Dover, p. 267, 1960.

Tetrahedroid

A special case of a quartic KUMMER SURFACE.

References

Fischer, G. (Ed.). *Mathematical Models from the Collections of Universities and Museums.* Braunschweig, Germany: Vieweg, pp. 17–19, 1986.

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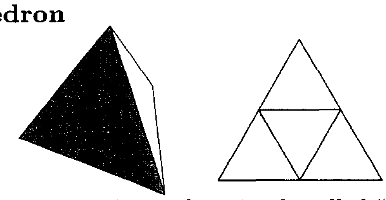
Tetrahedron



The regular tetrahedron, often simply called "the" tetrahedron, is the PLATONIC SOLID P_1 with four VERTICES, six EDGES, and four equivalent EQUILATERAL TRIANGULAR faces ($4\{3\}$). It is also UNIFORM POLYHEDRON U_1. It is described by the SCHLÄFLI SYMBOL $\{3,3\}$ and the WYTHOFF SYMBOL is $3\,|\,2\,3$. It is the prototype of the TETRAHEDRAL GROUP T_d,

The tetrahedron is its own DUAL POLYHEDRON. It is the only simple POLYHEDRON with no DIAGONALS, and cannot be STELLATED. The VERTICES of a tetrahedron are given by $(0,0,\sqrt{3})$, $(0,\frac{2}{3}\sqrt{6},-\frac{1}{3}\sqrt{3})$, $(-\sqrt{2},-\frac{1}{3}\sqrt{6},-\frac{1}{3}\sqrt{3})$, and $(\sqrt{2},-\frac{1}{3}\sqrt{6},-\frac{1}{3}\sqrt{3})$, or by (0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0). In the latter case, the face planes are

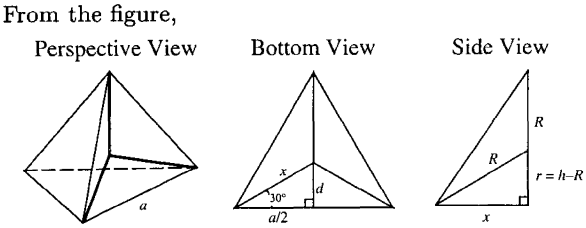
$$x+y+z=2 \tag{1}$$

$$x-y-z=0 \tag{2}$$

$$-x+y-z=0 \tag{3}$$

$$x+y-z=0. \tag{4}$$

Let a tetrahedron be length a on a side. The VERTICES are located at $(x, 0, 0)$, $(-d, \pm a/2, 0)$, and $(0, 0, h)$. From the figure,



$$x=\frac{\frac{a}{2}}{\cos(\frac{\pi}{6})}=\frac{1}{2}\frac{2}{\sqrt{3}}a=\tfrac{1}{3}\sqrt{3}\,a. \tag{5}$$

d is then

$$d=\sqrt{x^2-(\tfrac{1}{2}a)^2}=a\sqrt{\tfrac{1}{3}-\tfrac{1}{4}}=a\sqrt{\frac{4-3}{12}}=\frac{a}{\sqrt{12}}$$
$$=\tfrac{1}{6}\sqrt{3}\,a. \tag{6}$$

This gives the AREA of the base as

$$\begin{aligned}A&=\tfrac{1}{2}a(R+x)=\tfrac{1}{2}a\left(\frac{\sqrt{3}}{6}a+\frac{1}{\sqrt{3}}a\right)\\&=\tfrac{1}{2}a^2\left(\frac{\sqrt{3}}{6}+\frac{2\sqrt{3}}{6}\right)\\&=\tfrac{1}{2}a^2\frac{3\sqrt{3}}{6}=\tfrac{1}{4}\sqrt{3}\,a^2.\end{aligned} \tag{7}$$

The height is

$$h=\sqrt{a^2-x^2}=a\sqrt{1-\tfrac{1}{3}}=\tfrac{1}{3}\sqrt{6}\,a. \tag{8}$$

The CIRCUMRADIUS R is found from

$$x^2+(h-R)^2=R^2 \tag{9}$$

$$x^2+h^2-2hR+R^2=R^2. \tag{10}$$

Solving gives

$$R=\frac{x^2+h^2}{2h}=\frac{\frac{1}{3}+\frac{2}{3}}{2\sqrt{\frac{2}{3}}}=\frac{1}{2}\sqrt{\frac{3}{2}}=\tfrac{1}{4}\sqrt{6}\,a\approx 0.61237a. \tag{11}$$

The INRADIUS r is

$$r\equiv h-R=\sqrt{\frac{2}{3}}\,a-\frac{\sqrt{3}}{6}\,a=\tfrac{1}{12}\sqrt{6}\,a\approx 0.20412a, \tag{12}$$

which is also

$$r=\tfrac{1}{4}h=\tfrac{1}{3}R. \tag{13}$$

The MIDRADIUS is

$$\rho=\sqrt{r^2+d^2}=a\sqrt{\tfrac{6}{144}+\tfrac{3}{36}}=\sqrt{\tfrac{1}{8}}\,a=\tfrac{1}{4}\sqrt{2}\,a$$
$$\approx 0.35355a. \tag{14}$$

Plugging in for the VERTICES gives

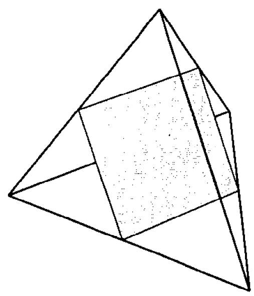
$$(a\sqrt{3},0,0),(-\tfrac{1}{6}\sqrt{3}\,a,\pm\tfrac{1}{2}a,0),\text{ and }(0,0,\tfrac{1}{3}\sqrt{6}\,a). \tag{15}$$

Since a tetrahedron is a PYRAMID with a triangular base, $V=\frac{1}{3}A_bh$, and

$$V=\tfrac{1}{3}\left(\tfrac{1}{4}\sqrt{3}\,a^2\right)\left(\sqrt{\frac{2}{3}}\,a\right)=\tfrac{1}{12}\sqrt{2}\,a^3. \tag{16}$$

The DIHEDRAL ANGLE is

$$\theta=\tan^{-1}(2\sqrt{2})=2\sin^{-1}(\tfrac{1}{3}\sqrt{6})=\cos^{-1}(\tfrac{1}{3}). \tag{17}$$



By slicing a tetrahedron as shown above, a SQUARE can be obtained. This cut divides the tetrahedron into two congruent solids rotated by 90°.

Now consider a general (not necessarily regular) tetrahedron, defined as a convex POLYHEDRON consisting of four (not necessarily identical) TRIANGULAR faces. Let the tetrahedron be specified by its VERTICES at (x_i, y_i) where $i = 1, \ldots, 4$. Then the VOLUME is given by

$$V = \frac{1}{3!}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}. \quad (18)$$

Specifying the tetrahedron by the three EDGE vectors **a**, **b**, and **c** from a given VERTEX, the VOLUME is

$$V = \tfrac{1}{3!}|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|. \quad (19)$$

If the faces are congruent and the sides have lengths a, b, and c, then

$$V = \sqrt{\frac{(a^2+b^2-c^2)(a^2+c^2-b^2)(b^2+c^2-a^2)}{72}} \quad (20)$$

(Klee and Wagon 1991, p. 205). Let a, b, c, and d be the areas of the four faces, and define

$$B \equiv \angle cd \quad (21)$$

$$C \equiv \angle bd \quad (22)$$

$$D \equiv \angle bc, \quad (23)$$

where $\angle jk$ means here the ANGLE between the PLANES formed by the FACES j and k, with VERTEX along their intersecting EDGE. Then

$$a^2 = b^2+c^2+d^2-2cd\cos B-2bd\cos C-2bc\cos D. \quad (24)$$

The analog of GAUSS'S CIRCLE PROBLEM can be asked for tetrahedra: how many LATTICE POINTS lie within a tetrahedron centered at the ORIGIN with a given INRADIUS (Lehmer 1940, Granville 1991, Xu and Yau 1992, Guy 1994).

see also AUGMENTED TRUNCATED TETRAHEDRON, BANG'S THEOREM, EHRHART POLYNOMIAL, HERONIAN TETRAHEDRON, HILBERT'S 3RD PROBLEM, ISOSCELES TETRAHEDRON, SIERPIŃSKI TETRAHEDRON, STELLA OCTANGULA, TETRAHEDRON 5-COMPOUND, TETRAHEDRON 10-COMPOUND, TRUNCATED TETRAHEDRON

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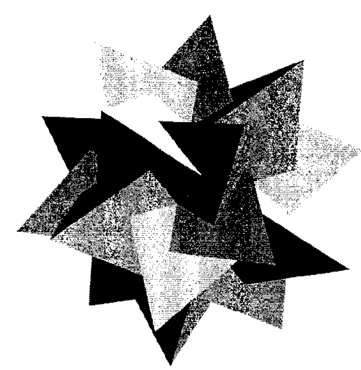
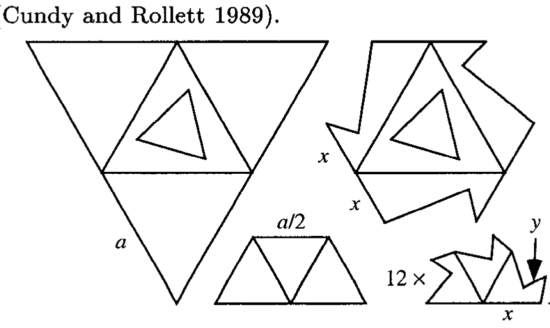
Tetrahedron 5-Compound



A POLYHEDRON COMPOUND composed of 5 TETRAHEDRA. Two tetrahedron 5-compounds of opposite CHIRALITY combine to make a TETRAHEDRON 10-COMPOUND. The following diagram shows pieces which can be assembled to form a tetrahedron 5-compound (Cundy and Rollett 1989).



see also POLYHEDRON COMPOUND, TETRAHEDRON 10-COMPOUND

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 135, 1987.

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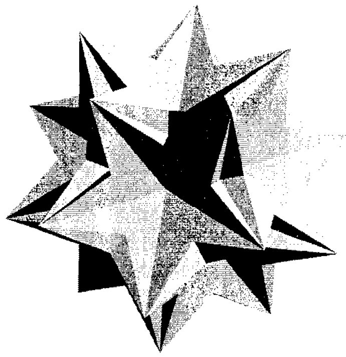
Tetrahedron 10-Compound



Two TETRAHEDRON 5-COMPOUNDS of opposite CHIRALITY combined.

see also POLYHEDRON COMPOUND, TETRAHEDRON 5-COMPOUND

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 135, 1987.

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Wenninger, M. J. *Polyhedron Models.* New York: Cambridge University Press, p. 45, 1989.

Tetrahedron Inscribing

Pick four points at random on the surface of a unit SPHERE. Find the distribution of possible volumes of (nonregular) TETRAHEDRA. Without loss of generality, the first point can be chosen as (1, 0, 0). Designate the other points **a**, **b**, and **c**. Then the distances from the first VERTEX are

$$\mathbf{a} = \begin{bmatrix} \cos\theta_1 - 1 \\ \sin\theta_1 \\ 0 \end{bmatrix} \tag{1}$$

$$\mathbf{b} = \begin{bmatrix} \cos\theta_2 \sin\phi_2 - 1 \\ \sin\theta_2 \sin\phi_2 \\ \cos\phi_2 \end{bmatrix} \tag{2}$$

$$\mathbf{c} = \begin{bmatrix} \cos\theta_3 \sin\phi_3 - 1 \\ \sin\theta_3 \sin\phi_3 \\ \cos\phi_3 \end{bmatrix}. \tag{3}$$

The average volume is then

$$\bar{V} = \frac{1}{C}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} \tfrac{1}{3!}|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| \, d\phi_3\, d\phi_2\, d\theta_3\, d\theta_2\, d\theta_1, \tag{4}$$

where

$$C = \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} d\phi_3\, d\phi_2\, d\theta_3\, d\theta_2\, d\theta_1 = 8\pi^5 \tag{5}$$

and

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = &-\cos\phi_2\sin\theta_1 + \cos\phi_3\sin\theta_1 \\ &- \cos\phi_3\cos\theta_2\sin\phi_2\sin\theta_1 + \cos\phi_2\cos\theta_3\sin\phi_3\sin\theta_1 \\ &- \cos\phi_3\sin\phi_2\sin\theta_2 + \cos\phi_3\cos\theta_1\sin\phi_2\sin\theta_2 \\ &+ \cos\phi_2\sin\phi_3\sin\theta_3 - \cos\phi_2\cos\theta_1\sin\phi_3\sin\theta_3. \end{aligned} \tag{6}$$

The integrals are difficult to compute analytically, but 10^7 computer TRIALS give

$$\langle V\rangle \approx 0.1080 \tag{7}$$

$$\langle V^2\rangle \approx 0.02128 \tag{8}$$

$$\sigma_V{}^2 = \langle V^2\rangle - \langle V\rangle^2 \approx 0.009937. \tag{9}$$

see also POINT-POINT DISTANCE—1-D, TRIANGLE INSCRIBING IN A CIRCLE, TRIANGLE INSCRIBING IN AN ELLIPSE

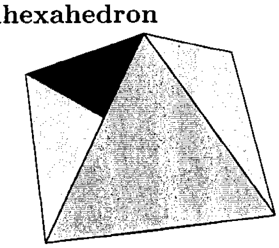
References

Buchta, C. "A Note on the Volume of a Random Polytope in a Tetrahedron." *Ill. J. Math.* **30**, 653–659, 1986.

Tetrahemihexacron

The DUAL POLYHEDRON of the TETRAHEMIHEXAHEDRON.

Tetrahemihexahedron



The UNIFORM POLYHEDRON U_4 whose DUAL POLYHEDRON is the TETRAHEMIHEXACRON. It has SCHLÄFLI SYMBOL $r'\{\frac{3}{3}\}$ and WYTHOFF SYMBOL $\frac{3}{2}\,3\,|\,2$. Its faces are $4\{3\} + 3\{4\}$. It is a faceted form of the OCTAHEDRON. Its CIRCUMRADIUS is

$$R = \tfrac{1}{2}\sqrt{2}.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 101–102, 1971.

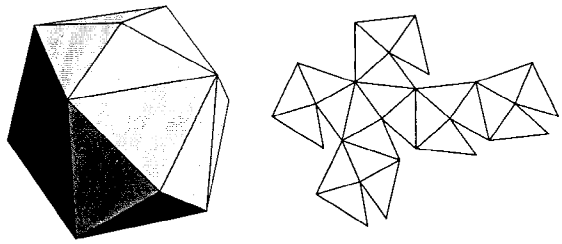
Tetrakaidecagon

see TETRADECAGON

Tetrakaidecahedron

see TETRADECAHEDRON

Tetrakis Hexahedron



The DUAL POLYHEDRON of the TRUNCATED OCTAHEDRON.

Tetranacci Number

The tetranacci numbers are a generalization of the FIBONACCI NUMBERS defined by $T_0 = 0$, $T_1 = 1$, $T_2 = 1$, $T_3 = 2$, and the RECURRENCE RELATION

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4}$$

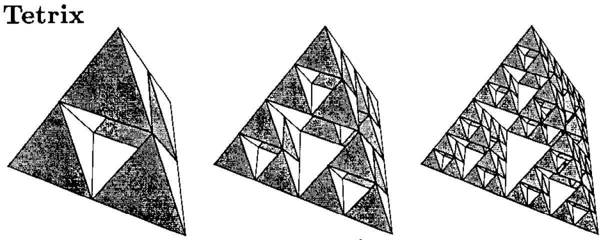
for $n \geq 4$. They represent the $n = 4$ case of the FIBONACCI n-STEP NUMBERS. The first few terms are 1, 1, 2, 4, 8, 15, 29, 56, 108, 208, ... (Sloane's A000078). The ratio of adjacent terms tends to 1.92756, which is the REAL ROOT of $x^5 - 2x^4 + 1 = 0$.

see also FIBONACCI n-STEP NUMBER, FIBONACCI NUMBER, TRIBONACCI NUMBER

References

Sloane, N. J. A. Sequence A000078/M1108 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Tetrix



The 3-D analog of the SIERPIŃSKI SIEVE illustrated above, also called the SIERPIŃSKI SPONGE or SIERPIŃSKI TETRAHEDRON. Let N_n be the number of tetrahedra, L_n the length of a side, and A_n the fractional VOLUME of tetrahedra after the nth iteration. Then

$$N_n = 4^n \quad (1)$$

$$L_n = (\tfrac{1}{2})^n = 2^{-n} \quad (2)$$

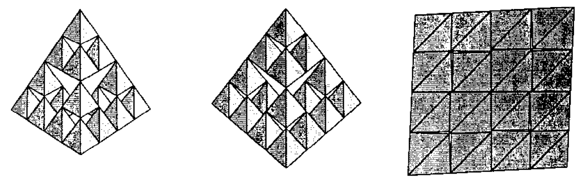
$$A_n = {L_n}^3 N_n = (\tfrac{1}{2})^n. \quad (3)$$

The CAPACITY DIMENSION is therefore

$$d_{\text{cap}} = -\lim_{n\to\infty} \frac{\ln N_n}{\ln L_n} = -\lim_{n\to\infty} \frac{\ln(4^n)}{\ln(2^{-n})} = \frac{\ln 4}{\ln 2} = \frac{2\ln 2}{\ln 2} = 2, \quad (4)$$

so the tetrix has an INTEGRAL CAPACITY DIMENSION (albeit one less than the DIMENSION of the 3-D TETRAHEDRA from which it is built), despite the fact that it is a FRACTAL.

The following illustration demonstrates how this counterintuitive fact can be true by showing three stages of the rotation of a tetrix, viewed along one of its edges. In the last frame, the tetrix "looks" like the 2-D PLANE.



see also MENGER SPONGE, SIERPIŃSKI SIEVE

References

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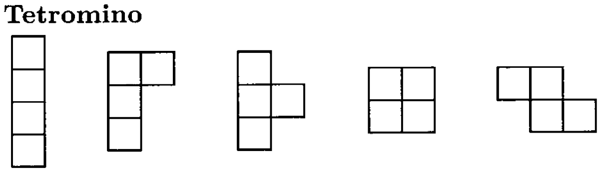
Tetromino



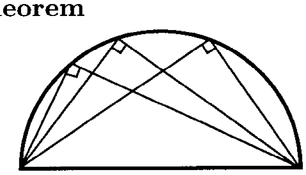
The five 4-POLYOMINOES, known as STRAIGHT, L-, T-, SQUARE, and SKEW.

References

Gardner, M. "Polyominoes." Ch. 13 in *The Scientific American Book of Mathematical Puzzles & Diversions.* New York: Simon and Schuster, pp. 124–140, 1959.

Hunter, J. A. H. and Madachy, J. S. *Mathematical Diversions.* New York: Dover, pp. 80–81, 1975.

Thales' Theorem



An ANGLE inscribed in a SEMICIRCLE is a RIGHT ANGLE.

see also RIGHT ANGLE, SEMICIRCLE

Theorem

A statement which can be demonstrated to be true by accepted mathematical operations and arguments. In general, a theorem is an embodiment of some general principle that makes it part of a larger theory.

According to the Nobel Prize-winning physicist Richard Feynman (1985), any theorem, no matter how difficult to prove in the first place, is viewed as "TRIVIAL" by mathematicians once it has been proven. Therefore, there are exactly two types of mathematical objects: TRIVIAL ones, and those which have not yet been proven.

see also AXIOM, AXIOMATIC SYSTEM, COROLLARY, DEEP THEOREM, PORISM, LEMMA, POSTULATE, PRINCIPLE, PROPOSITION

References

Feynman, R. P. and Leighton, R. *Surely You're Joking, Mr. Feynman!* New York: Bantam Books, 1985.

Theorema Egregium

see GAUSS'S THEOREMA EGREGIUM

Theta Function

The theta functions are the elliptic analogs of the EXPONENTIAL FUNCTION, and may be used to express the JACOBI ELLIPTIC FUNCTIONS. Let t be a constant COMPLEX NUMBER with $\Im[t] > 0$. Define the NOME

$$q \equiv e^{i\pi t} = e^{\pi K'(k)/K(k)}, \qquad (1)$$

where

$$t \equiv -i\frac{K'(k)}{K(k)}, \qquad (2)$$

and $K(k)$ is a complete ELLIPTIC INTEGRAL OF THE FIRST KIND, k is the MODULUS, and k' is the complementary MODULUS. Then the theta functions are, in the NOTATION of Whittaker and Watson,

$$\begin{aligned}\vartheta_1(z,q) &\equiv 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin[(2n+1)z]\\ &= zq^{1/4}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\sin[(2n+1)z] \qquad (3)\end{aligned}$$

$$\begin{aligned}\vartheta_2(z,q) &\equiv 2\sum_{n=0}^{\infty} q^{(n+1/2)^2}\cos[(2n+1)z]\\ &= 2q^{1/4}\sum_{n=0}^{\infty} q^{n(n+1)}\cos[(2n+1)z] \qquad (4)\end{aligned}$$

$$\vartheta_3(z,q) \equiv 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz) \qquad (5)$$

$$\begin{aligned}\vartheta_4(z,q) &\equiv \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{2niz}\\ &= 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2nz). \qquad (6)\end{aligned}$$

Written in terms of t,

$$\vartheta_2(t,q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} e^{\Im[t]} \qquad (7)$$

$$\vartheta_3(t,q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{\Im[t]}. \qquad (8)$$

These functions are sometimes denoted Θ_i or θ_i, and a number of indexing conventions have been used. For a summary of these notations, see Whittaker and Watson (1990). The theta functions are quasidoubly periodic, as illustrated in the following table.

ϑ_i	$\vartheta_i(z+\pi)/\vartheta_i(z)$	$\vartheta_i(z+t\pi)/\vartheta_i(z)$
ϑ_1	-1	$-N$
ϑ_2	-1	N
ϑ_3	1	N
ϑ_4	1	$-N$

Here,

$$N \equiv q^{-1}e^{-2iz}. \qquad (9)$$

The quasiperiodicity can be established as follows for the specific case of ϑ_4,

$$\begin{aligned}\vartheta_4(z+\pi,q) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{2niz}e^{2ni\pi}\\ &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{2niz} = \vartheta_4(z,q) \qquad (10)\end{aligned}$$

$$\begin{aligned}\vartheta_4(z+\pi t,q) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{2ni\pi t}e^{2niz}\\ &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} q^{2n} e^{2niz}\\ &= -q^{-1}e^{-2iz}\sum_{n=-\infty}^{\infty}(-1)^{n+1} q^{(n+1)^2} q^{2(n+1)iz}\\ &= -q^{-1}e^{-2iz}\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} q^{2niz}\\ &= -q^{-1}e^{-2iz}\vartheta_4(z,q). \qquad (11)\end{aligned}$$

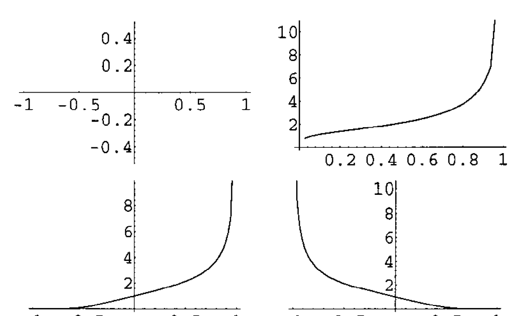
The theta functions can be written in terms of each other:

$$\vartheta_1(z,q) = -ie^{iz+\pi it/4}\vartheta_4(z+\tfrac{1}{4}\pi t,q) \qquad (12)$$

$$\vartheta_2(z,q) = \vartheta_1(z+\tfrac{1}{2}\pi,q) \qquad (13)$$

$$\vartheta_3(z,q) = \vartheta_4(z+\tfrac{1}{2}\pi,q). \qquad (14)$$

Any theta function of given arguments can be expressed in terms of any other two theta functions with the same arguments.



Define

$$\vartheta_i \equiv \vartheta_i(z=0), \tag{15}$$

which are plotted above. Then we have the identities

$$\vartheta_1{}^2(z)\vartheta_4{}^2 = \vartheta_3{}^2(z)\vartheta_2{}^2 - \vartheta_2{}^2(z)\vartheta_3{}^2 \tag{16}$$

$$\vartheta_2{}^2(z)\vartheta_4{}^2 = \vartheta_4{}^2(z)\vartheta_2{}^2 - \vartheta_1{}^2(z)\vartheta_3{}^2 \tag{17}$$

$$\vartheta_3{}^2(z)\vartheta_4{}^2 = \vartheta_4{}^2(z)\vartheta_3{}^2 - \vartheta_1{}^2(z)\vartheta_2{}^2 \tag{18}$$

$$\vartheta_4{}^2(z)\vartheta_4{}^2 = \vartheta_3{}^2(z)\vartheta_3{}^2 - \vartheta_2{}^2(z)\vartheta_2{}^2. \tag{19}$$

Taking $z=0$ in the last gives the special case

$$\vartheta_4{}^4 = \vartheta_3{}^4 - \vartheta_2{}^4. \tag{20}$$

In addition,

$$\vartheta_3(x) = \sum_{n=-\infty}^{\infty} x^{n^2} = 1 + 2x + 2x^4 + 2x^9 + \ldots \tag{21}$$

$$\vartheta_3{}^2(x) = 1 + 4\left(\frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \frac{x^7}{1-x^7} + \ldots\right) \tag{22}$$

$$\vartheta_3{}^4(x) = 1 + 8\left(\frac{x}{1-x} + \frac{2x^2}{1+x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1+x^4} + \ldots\right). \tag{23}$$

The theta functions obey addition rules such as

$$\vartheta_3(z+y)\vartheta_3(z-y)\vartheta_3{}^2 = \vartheta_3{}^2(y)\vartheta_3{}^2(z) + \vartheta_1{}^2(y)\vartheta_1{}^2(z). \tag{24}$$

Letting $y=z$ gives a duplication FORMULA

$$\vartheta_3(2z)\vartheta_3{}^3 = \vartheta_3{}^4(z) + \vartheta_1{}^4(z). \tag{25}$$

For more addition FORMULAS, see Whittaker and Watson (1990, pp. 487–488). Ratios of theta function derivatives to the functions themselves have the simple forms

$$\frac{\vartheta_1'(z)}{\vartheta_1(z)} = \cot z + 4\sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \sin(2nz) \tag{26}$$

$$\frac{\vartheta_2'(z)}{\vartheta_2(z)} = -\tan z + 4\sum_{n=1}^{\infty} (-1)^n \frac{q^{2n}}{1-q^{2n}} \sin(2nz) \tag{27}$$

$$\frac{\vartheta_3'(z)}{\vartheta_3(z)} = 4\sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1-q^{2n}} \sin(2nz) \tag{28}$$

$$\frac{\vartheta_4'(z)}{\vartheta_4(z)} = \sum_{n=1}^{\infty} \frac{q^{2n-1}\sin(2z)}{1-2q^{2n-1}\cos(2z)+q^{4n-2}} = \sum_{n=1}^{\infty} \frac{4q^n \sin(2nz)}{1-q^{2n}}. \tag{29}$$

The theta functions can be expressed as products instead of sums by

$$\vartheta_1(z) = 2Gq^{1/4}\sin z \prod_{n=1}^{\infty}[1 - 2q^{2n}\cos(2z) + q^{4n}] \tag{30}$$

$$\vartheta_2(z) = 2Gq^{1/4}\cos z \prod_{n=1}^{\infty}[1 + 2q^{2n}\cos(2z) + q^{4n}] \tag{31}$$

$$\vartheta_3(z) = G\prod_{n=1}^{\infty}[1 + 2q^{2n-1}\cos(2z) + q^{4n-2}] \tag{32}$$

$$\vartheta_4(z) = G\prod_{n=1}^{\infty}[1 - 2q^{2n-1}\cos(2z) + q^{4n-2}], \tag{33}$$

where

$$G \equiv \prod_{n=1}^{\infty}(1-q^{2n}) \tag{34}$$

(Whittaker and Watson 1990, pp. 469–470).

The theta functions satisfy the PARTIAL DIFFERENTIAL EQUATION

$$\tfrac{1}{4}\pi i \frac{\partial^2 y}{\partial z^2} + \frac{\partial y}{\partial t} = 0, \tag{35}$$

where $y \equiv \vartheta_j(z|t)$. Ratios of the theta functions with ϑ_4 in the DENOMINATOR also satisfy differential equations

$$\frac{d}{dz}\left[\frac{\vartheta_1(z)}{\vartheta_4(z)}\right] = \vartheta_4{}^2 \frac{\vartheta_2(z)\vartheta_3(z)}{\vartheta_4{}^2(z)} \tag{36}$$

$$\frac{d}{dz}\left[\frac{\vartheta_2(z)}{\vartheta_4(z)}\right] = -\vartheta_3{}^2 \frac{\vartheta_1(z)\vartheta_3(z)}{\vartheta_4{}^2(z)} \tag{37}$$

$$\frac{d}{dz}\left[\frac{\vartheta_3(z)}{\vartheta_4(z)}\right] = -\vartheta_2{}^2 \frac{\vartheta_1(z)\vartheta_2(z)}{\vartheta_4{}^2(z)}. \tag{38}$$

Some additional remarkable identities are

$$\vartheta_1' = \vartheta_2\vartheta_3\vartheta_4 \tag{39}$$

$$\vartheta_3(z,t) = -(it)^{1/2}e^{z^2/\pi i t}\vartheta_3\left(\frac{2}{t}, -\frac{1}{t}\right), \tag{40}$$

which were discovered by Poisson in 1827 and are equivalent to

$$\sum_{n=-\infty}^{\infty} e^{-t(x+n)^2} = \sqrt{\frac{\pi}{t}} \sum_{k=-\infty}^{\infty} 2^{2\pi i k x - (\pi^2 k^2/t)}. \tag{41}$$

Another amazing identity is

$$\begin{aligned}
&2\vartheta_1[\tfrac{1}{2}(-b+c+d+e)]\vartheta_2[\tfrac{1}{2}(b-c+d+e)]\vartheta_3[\tfrac{1}{2}(b+c-d+e)]\\
&\quad\times\vartheta_4[\tfrac{1}{2}(b+c+d-e)] = \vartheta_3(b)\vartheta_4(c)\vartheta_1(d)\vartheta_2(e)\\
&\quad+\vartheta_2(b)\vartheta_1(c)\vartheta_4(d)\vartheta_3(e) - \vartheta_1(b)\vartheta_2(c)\vartheta_3(d)\vartheta_4(e)\\
&\quad+\vartheta_4(b)\vartheta_3(c)\vartheta_2(d)\vartheta_1(e)
\end{aligned} \tag{42}$$

(Whittaker and Watson 1990, p. 469).

The complete ELLIPTIC INTEGRALS OF THE FIRST and SECOND KINDS can be expressed using theta functions. Let

$$\xi \equiv \frac{\vartheta_1(z)}{\vartheta_4(z)}, \tag{43}$$

and plug into (36)

$$\left(\frac{d\xi}{dz}\right)^2 = ({\vartheta_2}^2 - \xi^2{\vartheta_3}^2)({\vartheta_3}^2 - \xi^2{\vartheta_2}^2). \tag{44}$$

Now write

$$\xi\frac{\vartheta_3}{\vartheta_2} \equiv y \tag{45}$$

and

$$z{\vartheta_3}^2 \equiv u. \tag{46}$$

Then

$$\left(\frac{dy}{du}\right)^2 = (1-y^2)(1-k^2y^2), \tag{47}$$

where the MODULUS is defined by

$$k = k(q) = \frac{{\vartheta_2}^2(q)}{{\vartheta_3}^2(q)}. \tag{48}$$

Define also the complementary MODULUS

$$k' = k'(q) = \frac{{\vartheta_4}^2(-q)}{{\vartheta_3}^2(q)}. \tag{49}$$

Now, since

$${\vartheta_2}^4 + {\vartheta_4}^4 = {\vartheta_3}^4, \tag{50}$$

we have shown

$$k^2 + k'^2 = 1. \tag{51}$$

The solution to the equation is

$$y = \frac{\vartheta_3}{\vartheta_2}\frac{\vartheta_1(u{\vartheta_3}^{-2}|t)}{\vartheta_4(u{\vartheta_3}^{-2}|t)} \equiv \operatorname{sn}(u,k), \tag{52}$$

which is a JACOBI ELLIPTIC FUNCTION with periods

$$4K(k) = 2\pi{\vartheta_3}^2(q) \tag{53}$$

and

$$2iK'(k) = \pi t{\vartheta_3}^2(q). \tag{54}$$

Here, K is the complete ELLIPTIC INTEGRAL OF THE FIRST KIND,

$$K(k) = \tfrac{1}{2}\pi{\vartheta_3}^2(q). \tag{55}$$

see also BLECKSMITH-BRILLHART-GERST THEOREM, ELLIPTIC FUNCTION, ETA FUNCTION, EULER'S PENTAGONAL NUMBER THEOREM, JACOBI ELLIPTIC FUNCTIONS, JACOBI TRIPLE PRODUCT, LANDEN'S FORMULA, MOCK THETA FUNCTION, MODULAR EQUATION, MODULAR TRANSFORMATION, MORDELL INTEGRAL, NEVILLE THETA FUNCTION, NOME, POINCARÉ-FUCHS-KLEIN AUTOMORPHIC FUNCTION, PRIME THETA FUNCTION, QUINTUPLE PRODUCT IDENTITY, RAMANUJAN THETA FUNCTIONS, SCHRÖTER'S FORMULA, WEBER FUNCTIONS

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 577, 1972.

Bellman, R. E. *A Brief Introduction to Theta Functions.* New York: Holt, Rinehart and Winston, 1961.

Berndt, B. C. "Theta-Functions and Modular Equations." Ch. 25 in *Ramanujan's Notebooks, Part IV.* New York: Springer-Verlag, pp. 138–244, 1994.

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Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis, 4th ed.* Cambridge, England: Cambridge University Press, 1990.

Theta Operator

In the NOTATION of Watson (1966),

$$\vartheta \equiv z\frac{d}{dz}.$$

References

Watson, G. N. *A Treatise on the Theory of Bessel Functions, 2nd ed.* Cambridge, England: Cambridge University Press, 1966.

Theta Subgroup

see LAMBDA GROUP

Thiele's Interpolation Formula

Let ρ be a RECIPROCAL DIFFERENCE. Then Thiele's interpolation formula is the CONTINUED FRACTION

$$f(x) = f(x_1) + \cfrac{x-x_1}{\rho(x_1,x_2) + \cfrac{x-x_2}{\rho_2(x_1,x_2,x_3) - f(x_1) + \cfrac{x-x_3}{\rho_3(x_1,x_2,x_3,x_4) - \rho(x_1,x_2) + \ldots}}}.$$

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 881, 1972.

Milne-Thomson, L. M. *The Calculus of Finite Differences.* London: Macmillan, 1951.

Thiessen Polytope

see VORONOI POLYGON

Third Curvature

Also known as the TOTAL CURVATURE. The linear element of the INDICATRIX

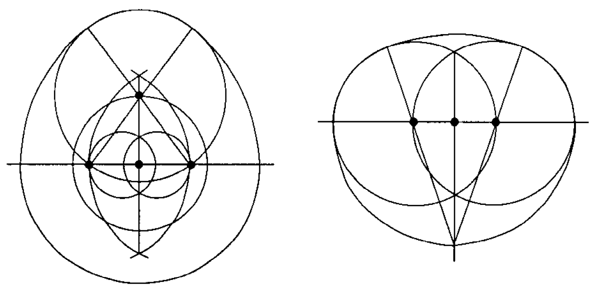
$$ds_P = \sqrt{{ds_T}^2 + {ds_B}^2}.$$

see also LANCRET EQUATION

Thirteenth

see FRIDAY THE THIRTEENTH

Thom's Eggs



EGG-shaped curves constructed using multiple CIRCLES which Thom (1967) used to model Megalithic stone rings in Britain.

see also EGG, OVAL

References

Dixon, R. *Mathographics.* New York: Dover, p. 6, 1991.

Thom, A. "Mathematical Background." Ch. 4 in *Megalithic Sites in Britain.* Oxford, England: Oxford University Press, pp. 27–33, 1967.

Thomae's Theorem

$$\frac{\Gamma(x+y+s+1)}{\Gamma(x+s+1)\Gamma(y+s+1)}\,{}_3F_2\left(\begin{matrix}-a,-b,x+y+s+1\\ x+s+1,y+s+1\end{matrix};1\right)$$
$$=\frac{\Gamma(a+b+s+1)}{\Gamma(a+s+1)\Gamma(b+s+1)}\,{}_3F_2\left(\begin{matrix}-x,-y,a+b+s+1\\ a+s+1,b+s+1\end{matrix};1\right),$$

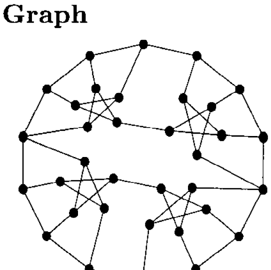
where $\Gamma(z)$ is the GAMMA FUNCTION and the function ${}_3F_2(a,b,c;d,e;z)$ is a GENERALIZED HYPERGEOMETRIC FUNCTION.

see also GENERALIZED HYPERGEOMETRIC FUNCTION

References

Hardy, G. H. *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work, 3rd ed.* New York: Chelsea, pp. 104–105, 1959.

Thomassen Graph



The GRAPH illustrated above.

see also THOMSEN GRAPH

Thompson's Functions

see BEI, BER, KELVIN FUNCTIONS

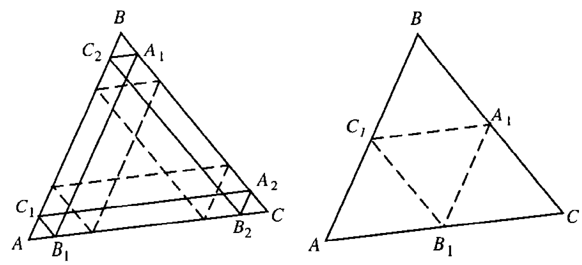
Thompson Group

The SPORADIC GROUP *Th.*

References

Wilson, R. A. "ATLAS of Finite Group Representation." `http://for.mat.bham.ac.uk/atlas/Th.html`.

Thomsen's Figure



Take any TRIANGLE with VERTICES A, B, and C. Pick a point A_1 on the side opposite A, and draw a line PARALLEL to AB. Upon reaching the side AC at B_1, draw the line PARALLEL to BC. Continue (left figure). Then $A_3 = A_1$ for any TRIANGLE. If A_1 is the MIDPOINT of BC, then $A_2 = A_1$ (right figure).

see also MIDPOINT, TRIANGLE

References

Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, pp. 234, 1979.

Thomsen Graph

The COMPLETE BIPARTITE GRAPH $K_{3,3}$, which is equivalent to the UTILITY GRAPH. It has a CROSSING NUMBER 1.

see also COMPLETE BIPARTITE GRAPH, CROSSING NUMBER (GRAPH), THOMASSEN GRAPH, UTILITY GRAPH

Thomson Lamp Paradox

A lamp is turned on for 1/2 minute, off for 1/4 minute, on for 1/8 minute, etc. At the end of one minute, the lamp switch will have been moved $\aleph_0$ times, where $\aleph_0$ is ALEPH-0. Will the lamp be on or off? This PARADOX is actually nonsensical, since it is equivalent to asking if the "last" INTEGER is EVEN or ODD.

References

Pickover, C. A. *Keys to Infinity.* New York: Wiley, pp. 19–23, 1995.

Thomson's Principle

see DIRICHLET'S PRINCIPLE

Thomson Problem

Determine the stable equilibrium positions of N classical electrons constrained to move on the surface of a SPHERE and repelling each other by an inverse square law. Exact solutions for $N = 2$ to 8 are known, but $N = 9$ and 11 are still unknown.

In reality, Earnshaw's theorem guarantees that no system of discrete electric charges can be held in stable equilibrium under the influence of their electrical interaction alone (Aspden 1987).

see also FEJES TÓTH'S PROBLEM

References

Altschuler, E. L.; Williams, T. J.; Ratner, E. R.; Dowla, F.; and Wooten, F. "Method of Constrained Global Optimization." *Phys. Rev. Let.* **72**, 2671–2674, 1994.

Altschuler, E. L.; Williams, T. J.; Ratner, E. R.; Dowla, F.; and Wooten, F. "Method of Constrained Global Optimization—Reply." *Phys. Rev. Let.* **74**, 1483, 1995.

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Aspden, H. "Earnshaw's Theorem." *Amer. J. Phys.* **55**, 199–200, 1987.

Berezin, A. A. "Spontaneous Symmetry Breaking in Classical Systems." *Amer. J. Phys.* **53**, 1037, 1985.

Calkin, M. G.; Kiang, D.; and Tindall, D. A. "Minimum Energy Configurations." *Nature* **319**, 454, 1986.

Erber, T. and Hockney, G. M. "Comment on 'Method of Constrained Global Optimization.'" *Phys. Rev. Let.* **74**, 1482–1483, 1995.

Marx, E. "Five Charges on a Sphere." *J. Franklin Inst.* **290**, 71–74, Jul. 1970.

Melnyk, T. W.; Knop, O.; and Smith, W. R. "Extremal Arrangements of Points and Unit Charges on a Sphere: Equilibrium Configurations Revisited." *Canad. J. Chem.* **55**, 1745–1761, 1977.

Whyte, L. L. "Unique Arrangement of Points on a Sphere." *Amer. Math. Monthly* **59**, 606–611, 1952.

Thousand

$1{,}000 = 10^3$. The word "thousand" appears in common expressions in a number of languages, for example, "a thousand pardons" in English and "tusen takk" ("a thousand thanks") in Norwegian.

see also HUNDRED, LARGE NUMBER, MILLION

Three

see 3

Three-Colorable

see COLORABLE

Three-In-A-Row

see TIC-TAC-TOE

Three Jug Problem

Given three jugs with x pints in the first, y in the second, and z in the third, obtain a desired amount in one of the vessels by completely filling up and/or emptying vessels into others. This problem can be solved with the aid of TRILINEAR COORDINATES.

References

Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 89–93, 1967.

Three-Valued Logic

A logical structure which does not assume the EXCLUDED MIDDLE LAW. Three possible truth values are possible: true, false, or undecided. There are 3072 such logics.

see also EXCLUDED MIDDLE LAW, FUZZY LOGIC, LOGIC

Threefoil Knot

see TREFOIL KNOT

Thue Constant

The base-2 TRANSCENDENTAL NUMBER

$$0.110110111110110111111\ldots_2,$$

where the nth bit is 1 if n is not divisible by 3 and is the complement of the $(n/3)$th bit if n is divisible by 3. It is also given by the SUBSTITUTION MAP

$$0 \to 111$$
$$1 \to 110.$$

In decimal, the Thue constant equals 0.8590997969....

see also RABBIT CONSTANT, THUE-MORSE CONSTANT

References

Thue-Morse Constant

The constant also called the PARITY CONSTANT and defined by

$$P \equiv \tfrac{1}{2}\sum_{n=0}^{\infty} P(n)2^{-n} = 0.4124540336401075977\ldots \qquad (1)$$

(Sloane's A014571), where $P(n)$ is the PARITY of n. Dekking (1977) proved that the Thue-Morse constant is TRANSCENDENTAL, and Allouche and Shallit give a complete proof correcting a minor error of Dekking.

The Thue-Morse constant can be written in base 2 by stages by taking the previous iteration a_n, taking the complement $\overline{a_n}$, and appending, producing

$$\begin{aligned} a_0 &= 0.0_2 \\ a_1 &= 0.01_2 \\ a_2 &= 0.0110_2 \\ a_3 &= 0.01101001_2 \\ a_4 &= 0.0110100110010110_2. \end{aligned} \qquad (2)$$

This can be written symbolically as

$$a_{n+1} = a_n + \overline{a_n}\cdot 2^{-2^n} \qquad (3)$$

with $a_0 = 0$. Here, the complement is the number $\overline{a_n}$ such that $a_n + \overline{a_n} = 0.11\ldots_2$, which can be found from

$$a_n + \overline{a_n} = \sum_{k=1}^{2^n} (\tfrac{1}{2})^k = \frac{1-(\frac{1}{2})^{2^n}}{1-\frac{1}{2}} - 1 = 1 - 2^{-2^n}. \qquad (4)$$

Therefore,

$$\overline{a_n} = 1 - a_n - 2^{-2^n}, \tag{5}$$

and

$$a_{n+1} = a_n + (1 - 2^{-2^n} - a_n)2^{-2^n}. \tag{6}$$

The regular CONTINUED FRACTION for the Thue-Morse constant is [0 2 2 2 1 4 3 5 2 1 4 2 1 5 44 1 4 1 2 4 1 1 1 5 14 1 50 15 5 1 1 1 4 2 1 4 1 43 1 4 1 2 1 3 16 1 2 1 2 1 50 1 2 424 1 2 5 2 1 1 1 5 5 2 22 5 1 1 1 1274 3 5 2 1 1 1 4 1 1 15 154 7 2 1 2 2 1 2 1 1 50 1 4 1 2 867374 1 1 1 5 5 1 1 6 1 2 7 2 1650 23 3 1 1 1 2 5 3 84 1 1 1 1284 ...] (Sloane's A014572), and seems to continue with sporadic large terms in suspicious-looking patterns. A nonregular CONTINUED FRACTION is

$$P = \cfrac{1}{3 - \cfrac{1}{2 - \cfrac{1}{4 - \cfrac{3}{16 - \cfrac{15}{256 - \cfrac{255}{65536 - \ldots}}}}}}. \tag{7}$$

A related infinite product is

$$4P = 2 - \frac{1 \cdot 3 \cdot 15 \cdot 255 \cdot 65535 \cdots}{2 \cdot 4 \cdot 16 \cdot 256 \cdot 65536 \cdots}. \tag{8}$$

The SEQUENCE $a_\infty = 0110100110010110100101100\ldots$ (Sloane's A010060) is known as the THUE-MORSE SEQUENCE.

see also RABBIT CONSTANT, THUE CONSTANT

References

Allouche, J. P.; Arnold, A.; Berstel, J.; Brlek, S.; Jockusch, W.; Plouffe, S.; and Sagan, B. "A Relative of the Thue-Morse Sequence." *Discr. Math.* **139**, 455–461, 1995.

Allouche, J. P. and Shallit, J. In preparation.

Beeler, M.; Gosper, R. W.; and Schroeppel, R. *HAKMEM.* Cambridge, MA: MIT Artificial Intelligence Laboratory, Memo AIM-239, Item 122, Feb. 1972.

Dekking, F. M. "Transcendence du nombre de Thue-Morse." *Comptes Rendus de l'Academie des Sciences de Paris* **285**, 157–160, 1977.

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/cntfrc/cntfrc.html`.

Sloane, N. J. A. Sequences A010060, A014571, and A014572 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Thue-Morse Sequence

The INTEGER SEQUENCE (also called the MORSE-THUE SEQUENCE)

$$0110100110010110100101100110100 1\ldots \tag{1}$$

(Sloane's A010060) which arises in the THUE-MORSE CONSTANT. It can be generated from the SUBSTITUTION MAP

$$0 \to 01 \tag{2}$$

$$1 \to 10 \tag{3}$$

starting with 0 as follows:

$$0 \to 01 \to 0110 \to 01101001 \to \ldots. \tag{4}$$

Writing the sequence as a POWER SERIES over the GALOIS FIELD GF(2),

$$F(x) = 0 + 1x + 1x^2 + 0x^3 + 1x^4 + \ldots, \tag{5}$$

then F satisfies the quadratic equation

$$(1+x)F^2 + F = \frac{x}{1+x^2} \pmod 2. \tag{6}$$

This equation has two solutions, F and F', where F' is the complement of F, i.e.,

$$F + F' = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1+x}, \tag{7}$$

which is consistent with the formula for the sum of the roots of a quadratic. The equality (6) can be demonstrated as follows. Let $(abcdef\ldots)$ be a shorthand for the POWER series

$$a + bx + cx^2 + dx^3 + \ldots, \tag{8}$$

so $F(x)$ is $(0110100110010110\ldots)$. To get F^2, simply use the rule for squaring POWER SERIES over GF(2)

$$(A+B)^2 = A^2 + B^2 \pmod 2, \tag{9}$$

which extends to the simple rule for squaring a POWER SERIES

$$(a_0 + a_1x + a_2x^2 + \ldots)^2 = a_0 + a_1x^2 + a_2x^4 + \ldots \pmod 2, \tag{10}$$

i.e., space the series out by a factor of 2, (0 1 1 0 1 0 0 1 ...), and insert zeros in the ODD places to get

$$F^2 = (00101000100000 10\ldots). \tag{11}$$

Then multiply by x (which just adds a zero at the front) to get

$$xF^2 = (00010100010000010\ldots). \tag{12}$$

Adding to F^2 gives

$$(1+x)F^2 = (0011110011000011\ldots). \tag{13}$$

This is the first term of the quadratic equation, which is the Thue-Morse sequence with each term doubled up. The next term is F, so we have

$$(1+x)F^2 = (0011110011000011\ldots) \tag{14}$$

$$F = (0110100110010110\ldots). \tag{15}$$

The sum is the above two sequences XORed together (there are no CARRIES because we're working over GF(2)), giving

$$(1+x)F^2 + F = (0101010101010101\ldots). \quad (16)$$

We therefore have

$$\begin{aligned}(1+x)F^2 + F &= \frac{x}{1+x^2} \\ &= x + x^3 + x^5 + x^7 + x^9 + x^{11} + \ldots \pmod 2. \quad (17)\end{aligned}$$

The Thue-Morse sequence is an example of a cube-free sequence on two symbols (Morse and Hedlund 1944), i.e., it contains no substrings of the form WWW, where W is any WORD. For example, it does not contain the WORDS 000, 010101 or 010010010. In fact, the following stronger statement is true: the Thue-Morse sequence does not contain any substrings of the form WWa, where a is the first symbol of W. We can obtain a SQUAREFREE sequence on three symbols by doing the following: take the Thue-Morse sequence 0110100110010110... and look at the sequence of WORDS of length 2 that appear: 01 11 10 01 10 00 01 11 10 .... Replace 01 by 0, 10 by 1, 00 by 2 and 11 by 2 to get the following: 021012021.... Then this SEQUENCE is SQUAREFREE (Morse and Hedlund 1944).

The Thue-Morse sequence has important connections with the GRAY CODE. Kindermann generates fractal music using the SELF-SIMILARITY of the Thue-Morse sequence.

see also GRAY CODE, PARITY CONSTANT, RABBIT SEQUENCE, THUE SEQUENCE

References

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Thue Sequence

The SEQUENCE of BINARY DIGITS of the THUE CONSTANT, $0.110110111110110111110110110\ldots_2$ (Sloane's A014578).

see also RABBIT CONSTANT, THUE CONSTANT

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Thue-Siegel-Roth Theorem

If α is a TRANSCENDENTAL NUMBER, it can be approximated by infinitely many RATIONAL NUMBERS m/n to within n^{-r}, where r is any POSITIVE number.

see also LIOUVILLE'S RATIONAL APPROXIMATION THEOREM, LIOUVILLE-ROTH CONSTANT, ROTH'S THEOREM

Thue-Siegel-Schneider-Roth Theorem

see THUE-SIEGEL-ROTH THEOREM

Thue's Theorem

If $n > 1$, $(a, n) = 1$ (i.e., a and n are RELATIVELY PRIME), and m is the least integer $> \sqrt{n}$, then there exist an x and y such that

$$ay \equiv \pm x \pmod n$$

where $0 < x < m$ and $0 < y < m$.

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Thurston's Geometrization Conjecture

Thurston's conjecture has to do with geometric structures on 3-D MANIFOLDS. Before stating Thurston's conjecture, some background information is useful. 3-dimensional MANIFOLDS possess what is known as a standard 2-level DECOMPOSITION. First, there is the CONNECTED SUM DECOMPOSITION, which says that every COMPACT 3-MANIFOLD is the CONNECTED SUM of a unique collection of PRIME 3-MANIFOLDS.

The second DECOMPOSITION is the JACO-SHALEN-JOHANNSON TORUS DECOMPOSITION, which states that irreducible orientable COMPACT 3-MANIFOLDS have a canonical (up to ISOTOPY) minimal collection of disjointly EMBEDDED incompressible TORI such that each component of the 3-MANIFOLD removed by the TORI is either "atoroidal" or "Seifert-fibered."

Thurston's conjecture is that, after you split a 3-MANIFOLD into its CONNECTED SUM and then JACO-SHALEN-JOHANNSON TORUS DECOMPOSITION, the remaining components each admit exactly one of the following geometries:

1. EUCLIDEAN GEOMETRY,
2. HYPERBOLIC GEOMETRY,
3. SPHERICAL GEOMETRY,
4. the GEOMETRY of $\mathbb{S}^2 \times \mathbb{R}$,
5. the GEOMETRY of $\mathbb{H}^2 \times \mathbb{R}$,
6. the GEOMETRY of SL_2R,
7. NIL GEOMETRY, or
8. SOL GEOMETRY.

Here, $\mathbb{S}^2$ is the 2-SPHERE and $\mathbb{H}^2$ is the HYPERBOLIC PLANE. If Thurston's conjecture is true, the truth of the POINCARÉ CONJECTURE immediately follows.

see also CONNECTED SUM DECOMPOSITION, EUCLIDEAN GEOMETRY, HYPERBOLIC GEOMETRY, JACO-SHALEN-JOHANNSON TORUS DECOMPOSITION, NIL GEOMETRY, POINCARÉ CONJECTURE, SOL GEOMETRY, SPHERICAL GEOMETRY

Thwaites Conjecture

see COLLATZ PROBLEM

Tic-Tac-Toe

The usual game of tic-tac-toe (also called TICKTACKTOE) is 3-in-a-row on a 3×3 board. However, a generalized n-IN-A-ROW on an $n \times m$ board can also be considered. For $n = 1$ and 2 the first player can always win. If the board is at least 3×4, the first player can win for $n = 3$.

However, for TIC-TAC-TOE which uses a 3×3 board, a draw can always be obtained. If the board is at least 4×30, the first player can win for $n = 4$. For $n = 5$, a draw can always be obtained on a 5×5 board, but the first player can win if the board is at least 15×15. The cases $n = 6$ and 7 have not yet been fully analyzed for an $n \times n$ board, although draws can always be forced for $n = 8$ and 9. On an $\infty \times \infty$ board, the first player can win for $n = 1$, 2, 3, and 4, but a tie can always be forced for $n \geq 8$. For $3 \times 3 \times 3$ and $4 \times 4 \times 4$, the first player can always win (Gardner 1979).

see also PONG HAU K'I

References

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Stewart, I. "A Shepherd Takes A Sheep Shot." *Sci. Amer.* **269**, 154–156, 1993.

Ticktacktoe

see TIC-TAC-TOE

Tight Closure

The application of characteristic p methods in COMMUTATIVE ALGEBRA, which is a synthesis of some areas of COMMUTATIVE ALGEBRA and ALGEBRAIC GEOMETRY.

see also ALGEBRAIC GEOMETRY, COMMUTATIVE ALGEBRA

References

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Huneke, C. "An Algebraist Commuting in Berkeley." *Math. Intell.* **11**, 40–52, 1989.

Tightly Embedded

Q is said to be tightly embedded if $|Q \cap Q^g|$ is ODD for all $g \in G - N_G(Q)$, where $N_G(Q)$ is the NORMALIZER of Q in G.

Tiling

A plane-filling arrangement of plane figures or its generalization to higher dimensions. Formally, a tiling is a collection of disjoint open sets, the closures of which cover the plane. Given a single tile, the so-called first CORONA is the set of all tiles that have a common boundary point with the tile (including the original tile itself).

WANG'S CONJECTURE (1961) stated that if a set of tiles tiled the plane, then they could always be arranged to do so periodically. A periodic tiling of the PLANE by POLYGONS or SPACE by POLYHEDRA is called a TESSELLATION. The conjecture was refuted in 1966 when R. Berger showed that an aperiodic set of 20,426 tiles exists. By 1971, R. Robinson had reduced the number to six and, in 1974, R. Penrose discovered an aperiodic set (when color-matching rules are included) of two tiles: the so-called PENROSE TILES. (Penrose also sued the Kimberly Clark Corporation over their quilted toilet paper, which allegedly resembles a Penrose aperiodic tiling; Mirsky 1997.)

It is not known if there is a single aperiodic tile.

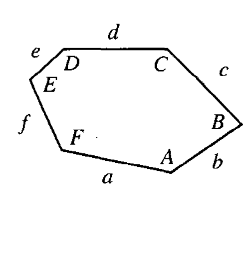
n-gon	tilings
3	any
4	any
5	14
6	3

The number of tilings possible for convex irregular POLYGONS are given in the above table. Any TRIANGLE or convex QUADRILATERAL tiles the plane. There are at least 14 classes of convex PENTAGONAL tilings. There are at least three aperiodic tilings of HEXAGONS, given by the following types:

$$\begin{array}{cc} A+B+C=360^\circ & a=d \\ A+B+D=360^\circ & a=d, c=e \\ A=C=E & a=b, c=d, e=f \end{array} \tag{1}$$

(Gardner 1988). Note that the periodic hexagonal TESSELLATION is a degenerate case of all three tilings with

$$A=B=C=D=E=F \qquad a=b=c=d=e=f. \tag{2}$$



There are no tilings for convex n-gons for $n \geq 7$.

see also ANISOHEDRAL TILING, CORONA (TILING), GOSPER ISLAND, HEESCH'S PROBLEM, ISOHEDRAL TILING, KOCH SNOWFLAKE, MONOHEDRAL TILING, PENROSE TILES, POLYOMINO TILING, SPACE-FILLING POLYHEDRON, TILING THEOREM, TRIANGLE TILING

References

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Tiling Theorem

Due to Lebesgue and Brouwer. If an n-D figure is covered in any way by sufficiently small subregions, then there will exist points which belong to at least $n+1$ of these subareas. Moreover, it is always possible to find a covering by arbitrarily small regions for which no point will belong to more than $n+1$ regions.

see also TESSELLATION, TILING

Times

The operation of MULTIPLICATION, i.e., a times b. Various notations are $a \times b$, $a \cdot b$, ab, and $(a)(b)$. The "multiplication sign" $\times$ is based on SAINT ANDREW'S CROSS (Bergamini 1969). Floating point MULTIPLICATION is sometimes denoted $\otimes$.

see also CROSS PRODUCT, DOT PRODUCT, MINUS, MULTIPLICATION, PLUS, PRODUCT

References

Bergamini, D. *Mathematics.* New York: Time-Life Books, p. 11, 1969.

Tit-for-Tat

A strategy for the iterated PRISONER'S DILEMMA in which a prisoner cooperates on the first move, and thereafter copies the previous move of the other prisoner. Any better strategy has more complicated rules.

see also PRISONER'S DILEMMA

References

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Titanic Prime

A PRIME with ≥ 1000 DIGITS. As of 1990, there were more than 1400 known (Ribenboim 1990). The table below gives the number of known titanic primes as a function of year end.

Year	Titanic Primes
1992	2254
1993	9166
1994	9779
1995	12391

References

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Morain, F. "Elliptic Curves, Primality Proving and Some Titanic Primes." *Astérique* **198–200**, 245–251, 1992.

Ribenboim, P. *The Little Book of Big Primes.* Berlin: Springer-Verlag, p. 97, 1990.

Yates, S. "Titanic Primes." *J. Recr. Math.* **16**, 250–262, 1983-84.

Yates, S. "Sinkers of the Titanics." *J. Recr. Math.* **17**, 268–274, 1984–85.

Titchmarsh Theorem

If $f(\omega)$ is SQUARE INTEGRABLE over the REAL ω axis, then any one of the following implies the other two:

1. The FOURIER TRANSFORM of $f(\omega)$ is 0 for $t < 0$.
2. Replacing ω by z, the function $f(z)$ is analytic in the COMPLEX PLANE z for $y > 0$ and approaches $f(x)$ almost everywhere as $y \to 0$. Furthermore, $\int_{-\infty}^{\infty} |f(x+iy)|^2\,dx < k$ for some number k and $y > 0$ (i.e., the integral is bounded).
3. The REAL and IMAGINARY PARTS of $f(z)$ are HILBERT TRANSFORMS of each other.

Tits Group

A finite SIMPLE GROUP which is a SUBGROUP of the TWISTED CHEVALLEY GROUP ${}^2F_4(2)$.

Toeplitz Matrix

Given $2N-1$ numbers r_k where $k = -N+1, \ldots, -1, 0, 1, \ldots, N-1$, a MATRIX of the form

$$\begin{bmatrix} r_0 & r_{-1} & r_{-2} & \cdots & r_{-n+1} \\ r_1 & r_0 & r_{-1} & \cdots & r_{-n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{n-1} & r_{n-2} & r_{n-3} & \cdots & r_0 \end{bmatrix}$$

is called a Toeplitz matrix. MATRIX equations of the form

$$\sum_{j=1}^{N} r_{i-j} x_j = y_i$$

can be solved with $\mathcal{O}(N^2)$ operations.

see also VANDERMONDE MATRIX

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Vandermonde Matrices and Toeplitz Matrices." §2.8 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 82–89, 1992.

Togliatti Surface

Togliatti (1940, 1949) showed that QUINTIC SURFACES having 31 ORDINARY DOUBLE POINTS exist, although he did not explicitly derive equations for such surfaces. Beauville (1978) subsequently proved that 31 double points are the maximum possible, and quintic surfaces having 31 ORDINARY DOUBLE POINTS are therefore sometimes called Togliatti surfaces. van Straten (1993) subsequently constructed a 3-D family of solutions and in 1994, Barth derived the example known as the DERVISH.

see also DERVISH, ORDINARY DOUBLE POINT, QUINTIC SURFACE

References

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Endraß, S. "Togliatti Surfaces." `http://www.mathematik.uni-mainz.de/Algebraische Geometrie/docs/Etogliatti.shtml`.

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Tomography

Tomography is the study of the reconstruction of 2- and 3-dimensional objects from 1-dimensional slices. The RADON TRANSFORM is an important tool in tomography.

Rather surprisingly, there exist certain sets of four directions in Euclidean n-space such that X-rays of a convex body in these directions distinguish it from all other convex bodies.

see also ALEKSANDROV'S UNIQUENESS THEOREM, BRUNN-MINKOWSKI INEQUALITY, BUSEMANN-PETTY PROBLEM, DVORETZKY'S THEOREM, RADON TRANSFORM, STEREOLOGY

References

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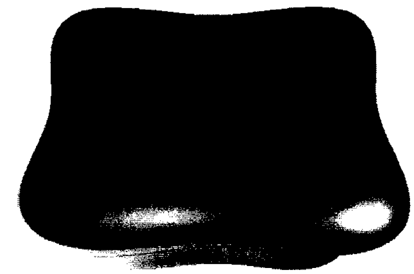
Tooth Surface



The QUARTIC SURFACE given by the equation

$$x^4 + y^4 + z^4 - (x^2 + y^2 + z^2) = 0.$$

References

Nordstrand, T. "Surfaces." `http://www.uib.no/people/nfytn/surfaces.htm`.

Topological Basis

A topological basis is a SUBSET B of a SET T in which all other OPEN SETS can be written as UNIONS or finite INTERSECTIONS of B. For the REAL NUMBERS, the SET of all OPEN INTERVALS is a basis.

Topological Completion

The topological completion C of a FIELD F with respect to the ABSOLUTE VALUE $|\cdot|$ is the smallest FIELD containing F for which all CAUCHY SEQUENCES or rationals converge.

References

Burger, E. B. and Struppeck, T. "Does $\sum_{n=0}^{\infty} \frac{1}{n!}$ Really Converge? Infinite Series and p-adic Analysis." *Amer. Math. Monthly* **103**, 565–577, 1996.

Topologically Conjugate

Two MAPS $\phi, \psi : M \to M$ are said to be topologically conjugate if there EXISTS a HOMEOMORPHISM $h : M \to M$ such that $\phi \circ h = h \circ \psi$, i.e., h maps ψ-orbits onto ϕ-orbits. Two maps which are topologically conjugate cannot be distinguished topologically.

see also ANOSOV DIFFEOMORPHISM, STRUCTURALLY STABLE

Topological Dimension

see LEBESGUE COVERING DIMENSION

Topological Entropy

The topological entropy of a MAP M is defined as

$$h_T(M) = \sup_{\{W_i\}} h(M, \{W_i\}),$$

where $\{W_i\}$ is a partition of a bounded region W containing a probability measure which is invariant under M, and sup is the SUPREMUM.

References

Ott, E. *Chaos in Dynamical Systems.* New York: Cambridge University Press, pp. 143–144, 1993.

Topological Groupoid

A topological groupoid over B is a GROUPOID G such that B and G are TOPOLOGICAL SPACES and α, β, and multiplication are continuous maps. Here, α and β are maps from G onto $\mathbb{R}^2$ with $\alpha : (x, \gamma, y) \mapsto x$ and $\beta : (x, \gamma, y) \mapsto y$.

References

Weinstein, A. "Groupoids: Unifying Internal and External Symmetry." *Not. Amer. Math. Soc.* **43**, 744–752, 1996.

Topological Manifold

A TOPOLOGICAL SPACE M satisfying some separability (i.e., it is a HAUSDORFF SPACE) and countability (i.e., it is a PARACOMPACT SPACE) conditions such that every point $p \in M$ has a NEIGHBORHOOD homeomorphic to an OPEN SET in $\mathbb{R}^n$ for some $n \geq 0$. Every SMOOTH MANIFOLD is a topological manifold, but not necessarily vice versa. The first nonsmooth topological manifold occurs in 4-D.

Nonparacompact manifolds are of little use in mathematics, but non-Hausdorff manifolds do occasionally arise in research (Hawking and Ellis 1975). For manifolds, Hausdorff and second countable are equivalent to Hausdorff and paracompact, and both are equivalent to the manifold being embeddable in some large-dimensional Euclidean space.

see also HAUSDORFF SPACE, MANIFOLD, PARACOMPACT SPACE, SMOOTH MANIFOLD, TOPOLOGICAL SPACE

References

Hawking, S. W. and Ellis, G. F. R. *The Large Scale Structure of Space-Time.* New York: Cambridge University Press, 1975.

Topological Space

A SET X for which a TOPOLOGY T has been specified is called a topological space (Munkres 1975, p. 76).

see also KURATOWSKI'S CLOSURE-COMPONENT PROBLEM, OPEN SET, TOPOLOGICAL VECTOR SPACE

References

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Munkres, J. R. *Topology: A First Course.* Englewood Cliffs, NJ: Prentice-Hall, 1975.

Topological Vector Space

A TOPOLOGICAL SPACE such that the two algebraic operations of VECTOR SPACE are continuous in the topology.

References

Köthe, G. *Topological Vector Spaces.* New York: Springer-Verlag, 1979.

Topologically Transitive

A FUNCTION f is topologically transitive if, given any two intervals U and V, there is some POSITIVE INTEGER k such that $f^k(U) \cap V = \varnothing$. Vaguely, this means that neighborhoods of points eventually get flung out to "big" sets so that they don't necessarily stick together in one localized clump.

see also CHAOS

Topology

Topology is the mathematical study of properties of objects which are preserved through deformations, twistings, and stretchings. (Tearing, however, is not allowed.) A CIRCLE is topologically equivalent to an ELLIPSE (into which it can be deformed by stretching) and a SPHERE is equivalent to an ELLIPSOID. Continuing along these lines, the SPACE of all positions of the minute hand on a clock is topologically equivalent to a CIRCLE (where SPACE of all positions means "the collection of all positions"). Similarly, the SPACE of all positions of the minute and hour hands is equivalent to a TORUS. The SPACE of all positions of the hour, minute and second hands form a 4-D object that cannot be visualized quite as simply as the former objects since it cannot be placed in our 3-D world, although it can be visualized by other means.

There is more to topology, though. Topology began with the study of curves, surfaces, and other objects in the plane and 3-space. One of the central ideas in topology is that spatial objects like CIRCLES and SPHERES can be treated as objects in their own right, and knowledge of objects is independent of how they are "represented" or "embedded" in space. For example, the statement "if you remove a point from a CIRCLE, you get a line segment" applies just as well to the CIRCLE as to an ELLIPSE, and even to tangled or knotted CIRCLES, since the statement involves only topological properties.

Topology has to do with the study of spatial objects such as curves, surfaces, the space we call our universe, the space-time of general relativity, fractals, knots, manifolds (objects with some of the same basic spatial properties as our universe), phase spaces that are encountered in physics (such as the space of hand-positions of a clock), symmetry groups like the collection of ways of rotating a top, etc.

The "objects" of topology are often formally defined as TOPOLOGICAL SPACES. If two objects have the same topological properties, they are said to be HOMEOMORPHIC (although, strictly speaking, properties that are not destroyed by stretching and distorting an object are really properties preserved by ISOTOPY, not HOMEOMORPHISM; ISOTOPY has to do with distorting embedded objects, while HOMEOMORPHISM is intrinsic).

Topology is divided into ALGEBRAIC TOPOLOGY (also called COMBINATORIAL TOPOLOGY), DIFFERENTIAL TOPOLOGY, and LOW-DIMENSIONAL TOPOLOGY.

There is also a formal definition for a topology defined in terms of set operations. A SET X along with a collection T of SUBSETS of it is said to be a topology if the SUBSETS in T obey the following properties:

1. The (trivial) subsets X and the EMPTY SET $\varnothing$ are in T.
2. Whenever sets A and B are in T, then so is $A \cap B$.
3. Whenever two or more sets are in T, then so is their UNION

(Bishop and Goldberg 1980).

A SET X for which a topology T has been specified is called a TOPOLOGICAL SPACE (Munkres 1975, p. 76). For example, the SET $X = \{0, 1, 2, 3\}$ together with the SUBSETS $T = \{0\}, \{1, 2, 3\}, \varnothing, \{0, 1, 2, 3\}\}$ comprises a topology, and X is a TOPOLOGICAL SPACE.

Topologies can be built up from TOPOLOGICAL BASES. For the REAL NUMBERS, the topology is the UNION of OPEN INTERVALS.

see also ALGEBRAIC TOPOLOGY, DIFFERENTIAL TOPOLOGY, GENUS, KLEIN BOTTLE, KURATOWSKI REDUCTION THEOREM, LEFSHETZ TRACE FORMULA, LOW-DIMENSIONAL TOPOLOGY, POINT-SET TOPOLOGY, ZARISKI TOPOLOGY

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Topos

A CATEGORY modeled after the properties of the CATEGORY of sets.

see also CATEGORY, LOGOS

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Toric Variety

Let $m_1, m_2, \ldots, m_n$ be distinct primitive elements of a 2-D LATTICE M such that $\det(m_i, m_{i+1}) > 0$ for $i = 1, \ldots, n$. Each collection $\Gamma = \{m_1, m_2, \ldots, m_n\}$ then forms a set of rays of a unique complete fan in M, and therefore determines a 2-D toric variety X_Γ.

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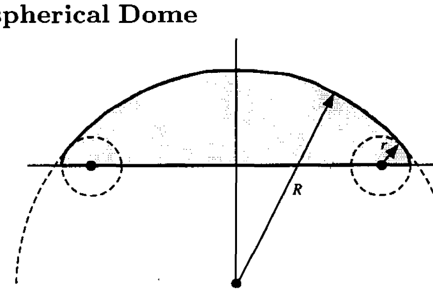
Torispherical Dome



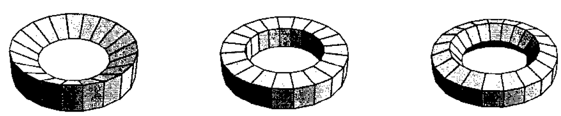
A torispherical dome is the surface obtained from the intersection of a SPHERICAL CAP with a tangent TORUS, as illustrated above. The radius of the sphere R is called the "crown radius," and the radius of the torus is called the "knuckle radius." Torispherical domes are used to construct pressure vessels.

see also DOME, SPHERICAL CAP

Torn Square Fractal

see CESÀRO FRACTAL

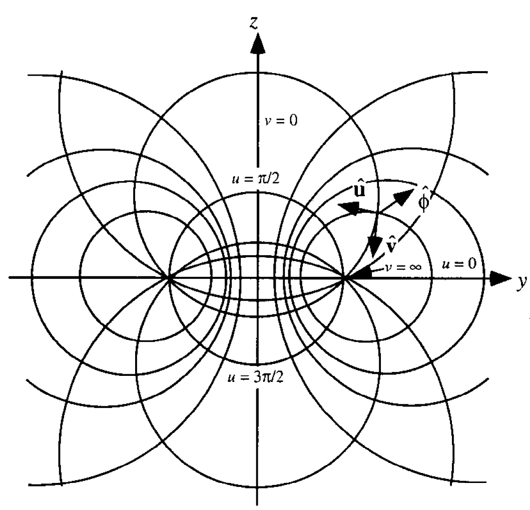
Toroid



A SURFACE OF REVOLUTION obtained by rotating a closed PLANE CURVE about an axis parallel to the plane which does not intersect the curve. The simplest toroid is the TORUS.

see also PAPPUS'S CENTROID THEOREM, SURFACE OF REVOLUTION, TORUS

Toroidal Coordinates



A system of CURVILINEAR COORDINATES for which several different notations are commonly used. In this work (u, v, ϕ) is used, whereas Arfken (1970) uses (ξ, η, φ). The toroidal coordinates are defined by

$$x = \frac{a \sinh v \cos\phi}{\cosh v - \cos u} \tag{1}$$

$$y = \frac{a \sinh v \sin\phi}{\cosh v - \cos u} \tag{2}$$

$$z = \frac{a \sin u}{\cosh v - \cos u}, \tag{3}$$

where $\sinh z$ is the HYPERBOLIC SINE and $\cosh z$ is the HYPERBOLIC COSINE. The SCALE FACTORS are

$$h_u = \frac{a}{\cosh v - \cos u} \tag{4}$$

$$h_v = \frac{a}{\cosh v - \cos u} \tag{5}$$

$$h_\phi = \frac{a \sinh v}{\cosh v - \cos u}. \tag{6}$$

The LAPLACIAN is

$$\begin{aligned}\nabla^2 f = {} & \frac{(\cosh v - \cos u)^3}{a^2} \frac{\partial}{\partial u}\left(\frac{1}{\cosh v - \cos u}\frac{\partial f}{\partial u}\right) \\ & + \frac{(\cosh v - \cos u)^3}{a^2 \sinh v} \frac{\partial}{\partial v}\left(\frac{\sinh v}{\cosh v - \cos u}\frac{\partial f}{\partial v}\right) \\ & + \frac{(\cosh v - \cos u)^2}{a^2 \sinh v}\frac{\partial^2 f}{\partial \phi^2}\end{aligned} \tag{7}$$

$$\begin{aligned} = {} & \left(\frac{-3\cos\coth^2 v + \cosh v \coth^2 v}{\cosh v - \cos u}\right. \\ & \left. + \frac{+3\cos^2 u \coth v \operatorname{csch} v - \cos^3 u \operatorname{csch}^2 v}{\cosh v - \cos u}\right)\frac{\partial^2}{\partial \phi^2} \\ & + (\cos u - \cosh v)\sin u \frac{\partial}{\partial u} + (\cosh v - \cos u)^2 \frac{\partial^2}{\partial u^2} \\ & + (\cosh v - \cos u)(\cosh v \coth v - \sinh v \\ & - \cos u \coth v)\frac{\partial}{\partial v} + (\cosh^2 v - \cos u)^2 \frac{\partial^2}{\partial v^2}.\end{aligned} \tag{8}$$

The HELMHOLTZ DIFFERENTIAL EQUATION is not separable in toroidal coordinates, but LAPLACE'S EQUATION is.

see also BISPHERICAL COORDINATES, LAPLACE'S EQUATION—TOROIDAL COORDINATES

References

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Toroidal Field

A VECTOR FIELD resembling a TORUS which is purely circular about the z-AXIS of a SPHERE (i.e., follows lines of LATITUDE). A toroidal field takes the form

$$\mathbf{T} = \begin{bmatrix} 0 \\ \frac{1}{\sin\theta}\frac{\partial T}{\partial \phi} \\ -\frac{\partial T}{\partial \theta} \end{bmatrix}.$$

see also DIVERGENCELESS FIELD, POLOIDAL FIELD

References

Stacey, F. D. *Physics of the Earth, 2nd ed.* New York: Wiley, p. 239, 1977.

Toroidal Function

A class of functions also called RING FUNCTIONS which appear in systems having toroidal symmetry. Toroidal functions can be expressed in terms of the LEGENDRE FUNCTIONS and SECOND KINDS (Abramowitz and Stegun 1972, p. 336):

$$P^{\mu}_{\nu-1/2}(\cosh\eta) = [\Gamma(1-\mu)]^{-1}2^{2\mu}(1-e^{-2\eta})^{-\mu}e^{-(\nu+1/2)\eta}$$
$$\times {}_2F_1(\tfrac{1}{2}-\mu, \tfrac{1}{2}+\nu-\mu; 1-2\mu; 1-e^{-2\eta})$$
$$P^{m}_{n-1/2}(\cosh\eta) = \frac{\Gamma(n+m+\frac{1}{2})(\sinh\eta)^m}{\Gamma(n-m+\frac{1}{2})2^m\sqrt{\pi}\,\Gamma(m+\frac{1}{2})}$$
$$\times \int_0^{\pi} \frac{sin^{2m}\phi\, d\phi}{(\cosh\eta+\cos\phi\sinh\eta)^{n+m+1/2}}$$
$$Q^{\mu}_{\nu-1/2}(\cosh\eta) = [\Gamma(1+\nu)]^{-1}\sqrt{\pi}\,e^{i\mu\pi}\Gamma(\tfrac{1}{2}+\nu+\mu)$$
$$\times (1-e^{-2\eta})^{\mu}e^{-(\nu+1/2)\eta}{}_2F_1(\tfrac{1}{2}-\mu, \tfrac{1}{2}+\nu+\mu; 1+\mu; 1-e^{-2\eta})$$
$$Q^{m}_{n-1/2}(\cosh\eta) = \frac{(-1)^m\Gamma(n+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})}$$
$$\times \int_0^{\infty} \frac{\cosh(mt)\,dt}{(\cosh\eta+\cosh t\sinh\eta)^{n+1/2}}$$

for $n > m$. Byerly (1959) identifies

$$\frac{1}{i^{n/2}}P^n_m(\coth x) = \operatorname{csch}^n x\frac{d^nP_m(\coth x)}{d(\coth x)^n}$$

as a TOROIDAL HARMONIC.

see also CONICAL FUNCTION

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Byerly, W. E. *An Elementary Treatise on Fourier's Series, and Spherical, Cylindrical, and Ellipsoidal Harmonics, with Applications to Problems in Mathematical Physics.* New York: Dover, p. 266, 1959.

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Toroidal Harmonic

see TOROIDAL FUNCTION

Toroidal Polyhedron

A toroidal polyhedron is a POLYHEDRON with GENUS $g \geq 1$ (i.e., having one or more HOLES). Examples of toroidal polyhedra include the CSÁSZÁR POLYHEDRON and SZILASSI POLYHEDRON, both of which have GENUS 1 (i.e., the TOPOLOGY of a TORUS).

The only known TOROIDAL POLYHEDRON with no DIAGONALS is the CSÁSZÁR POLYHEDRON. If another exists, it must have 12 or more VERTICES and GENUS $g \geq 6$. The smallest known single-hole toroidal POLYHEDRON made up of only EQUILATERAL TRIANGLES is composed of 48 of them.

see also CSÁSZÁR POLYHEDRON, SZILASSI POLYHEDRON

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Toronto Function

$$T(m,n,r) \equiv \frac{\Gamma(\frac{1}{2}m+\frac{1}{2})}{n!r^{-2n+m-1}}\,{}_1F_1(\tfrac{1}{2}; i+n; r^2),$$

where ${}_1F_1(a;b;z)$ is a CONFLUENT HYPERGEOMETRIC FUNCTION and $\Gamma(z)$ is the GAMMA FUNCTION (Abramowitz and Stegun 1972).

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 509, 1972.

Torricelli Point

see FERMAT POINT

Torsion (Differential Geometry)

The rate of change of the OSCULATING PLANE of a SPACE CURVE. The torsion τ is POSITIVE for a right-handed curve, and NEGATIVE for a left-handed curve. A curve with CURVATURE $\kappa \neq 0$ is planar IFF $\tau = 0$.

The torsion can be defined by

$$\tau \equiv -\mathbf{N}\cdot\mathbf{B}',$$

where $\mathbf{N}$ is the unit NORMAL VECTOR and $\mathbf{B}$ is the unit BINORMAL VECTOR. Written explicitly in terms of a parameterized VECTOR FUNCTION $\mathbf{x}$,

$$\tau = \frac{|\dot{\mathbf{x}}\,\ddot{\mathbf{x}}\,\dddot{\mathbf{x}}|}{\ddot{\mathbf{x}}\cdot\ddot{\mathbf{x}}} = \rho^2|\dot{\mathbf{x}}\,\ddot{\mathbf{x}}\,\dddot{\mathbf{x}}|,$$

where $|\mathbf{a}\,\mathbf{b}\,\mathbf{c}|$ denotes a SCALAR TRIPLE PRODUCT and ρ is the RADIUS OF CURVATURE. The quantity $1/\tau$ is called the RADIUS OF TORSION and is denoted σ or ϕ.

see also CURVATURE, RADIUS OF CURVATURE, RADIUS OF TORSION

References

Gray, A. "Drawing Space Curves with Assigned Curvature." §7.8 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 145–147, 1993.

Kreyszig, E. "Torsion." §14 in *Differential Geometry.* New York: Dover, pp. 37–40, 1991.

Torsion (Group Theory)

If G is a GROUP, then the torsion elements $\mathrm{Tor}(G)$ of G (also called the torsion of G) are defined to be the set of elements g in G such that $g^n = e$ for some NATURAL NUMBER n, where e is the IDENTITY ELEMENT of the GROUP G.

In the case that G is ABELIAN, $\mathrm{Tor}(G)$ is a SUBGROUP and is called the torsion subgroup of G. If $\mathrm{Tor}(G)$ consists only of the IDENTITY ELEMENT, the GROUP G is called torsion-free.

see also ABELIAN GROUP, GROUP, IDENTITY ELEMENT

Torsion Number

One of a set of numbers defined in terms of an invariant generated by the finite cyclic covering spaces of a KNOT complement. The torsion numbers for KNOTS up to 9 crossings were cataloged by Reidemeister (1948).

References
Reidemeister, K. *Knotentheorie.* New York: Chelsea, 1948.
Rolfsen, D. "Torsion Numbers." §6A in *Knots and Links.* Wilmington, DE: Publish or Perish Press, pp. 145–146, 1976.

Torsion Tensor

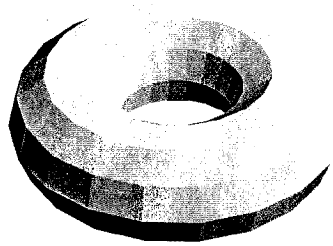
The TENSOR defined by

$$T^l{}_{jk} \equiv -(\Gamma^l{}_{jk} - \Gamma^l{}_{kj}),$$

where $\Gamma^l{}_{jk}$ are CONNECTION COEFFICIENTS.

see also CONNECTION COEFFICIENT

Torus



A torus is a surface having GENUS 1, and therefore possessing a single "HOLE." The usual torus in 3-D space is shaped like a donut, but the concept of the torus is extremely useful in higher dimensional space as well. One of the more common uses of n-D tori is in DYNAMICAL SYSTEMS. A fundamental result states that the PHASE SPACE trajectories of a HAMILTONIAN SYSTEM with n DEGREES OF FREEDOM and possessing n INTEGRALS OF MOTION lie on an n-D MANIFOLD which is topologically equivalent to an n-torus (Tabor 1989).

The usual 3-D "ring" torus is known in older literature as an "ANCHOR RING." Let the radius from the center of the hole to the center of the torus tube be c, and the radius of the tube be a. Then the equation in CARTESIAN COORDINATES is

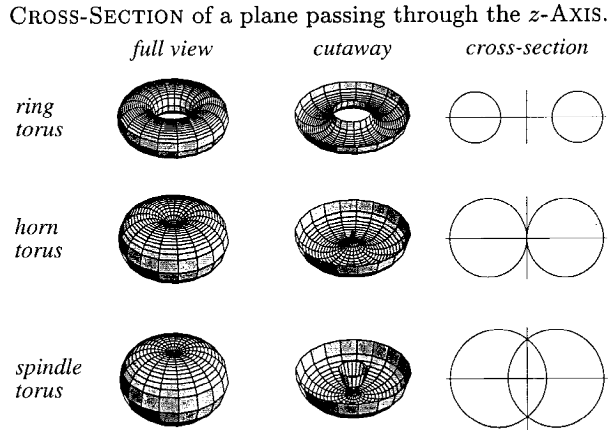
$$(c - \sqrt{x^2 + y^2}\,)^2 + z^2 = a^2. \quad (1)$$

The parametric equations of a torus are

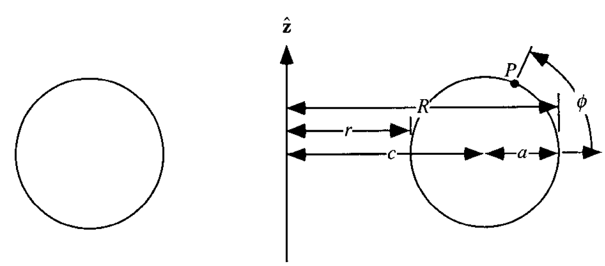
$$x = (c + a\cos v)\cos u \quad (2)$$
$$y = (c + a\cos v)\sin u \quad (3)$$
$$z = a\sin v \quad (4)$$

for $u, v \in [0, 2\pi)$. Three types of torus, known as the STANDARD TORI, are possible, depending on the relative sizes of a and c. $c > a$ corresponds to the RING TORUS (shown above), $c = a$ corresponds to a HORN TORUS which is tangent to itself at the point (0, 0, 0), and $c < a$ corresponds to a self-intersecting SPINDLE TORUS (Pinkall 1986).

If no specification is made, "torus" is taken to mean RING TORUS. The three STANDARD TORI are illustrated below, where the first image shows the full torus, the second a cut-away of the bottom half, and the third a CROSS-SECTION of a plane passing through the z-AXIS.



The STANDARD TORI and their inversions are CYCLIDES. If the coefficient of $\sin v$ in the formula for z is changed to $b \neq a$, an ELLIPTIC TORUS results.



To compute the metric properties of the ring torus, define the inner and outer radii by

$$r \equiv c - a \quad (5)$$
$$R \equiv c + a. \quad (6)$$

Solving for a and c gives

$$a = \tfrac{1}{2}(R-r) \tag{7}$$
$$c = \tfrac{1}{2}(R+r). \tag{8}$$

Then the SURFACE AREA of this torus is

$$S = (2\pi a)(2\pi c) = 4\pi^2 ac \tag{9}$$
$$= \pi^2 (R+r)(R-r), \tag{10}$$

and the VOLUME can be computed from PAPPUS'S CENTROID THEOREM

$$V = (\pi a^2)^2 \pi c = 2\pi^2 a^2 c \tag{11}$$
$$= \tfrac{1}{4}\pi^2 (R+r)(R-r)^2. \tag{12}$$

The coefficients of the first and second FUNDAMENTAL FORMS of the torus are given by

$$e = -(c + a\cos v)\cos v \tag{13}$$
$$f = 0 \tag{14}$$
$$g = -a \tag{15}$$
$$E = (c + a\cos v)^2 \tag{16}$$
$$F = 0 \tag{17}$$
$$G = a^2, \tag{18}$$

giving RIEMANNIAN METRIC

$$ds^2 = (c + a\cos v)^2\, du^2 + a^2\, dv^2, \tag{19}$$

AREA ELEMENT

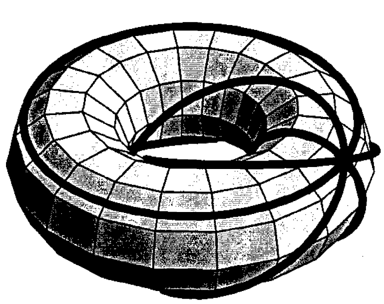
$$dA = a(c + a\cos v)\, du \wedge dv \tag{20}$$

(where $du \wedge dv$ is a WEDGE PRODUCT), and GAUSSIAN and MEAN CURVATURES as

$$K = \frac{\cos v}{a(c + a\cos v)} \tag{21}$$
$$H = -\frac{c + 2a\cos v}{2a(c + a\cos v)} \tag{22}$$

(Gray 1993, pp. 289–291).

A torus with a HOLE in *its surface* can be turned inside out to yield an identical torus. A torus can be knotted externally or internally, but not both. These two cases are AMBIENT ISOTOPIES, but not REGULAR ISOTOPIES. There are therefore three possible ways of embedding a torus with zero or one KNOT.



An arbitrary point P on a torus (not lying in the xy-plane) can have four CIRCLES drawn through it. The first circle is in the plane of the torus and the second is PERPENDICULAR to it. The third and fourth CIRCLES are called VILLARCEAU CIRCLES (Villarceau 1848, Schmidt 1950, Coxeter 1969, Melnick 1983).

To see that two additional CIRCLES exist, consider a coordinate system with origin at the center of torus, with $\hat{\mathbf{z}}$ pointing up. Specify the position of P by its ANGLE ϕ measured around the tube of the torus. Define $\phi = 0$ for the circle of points farthest away from the center of the torus (i.e., the points with $x^2 + y^2 = R^2$), and draw the x-AXIS as the intersection of a plane through the z-axis and passing through P with the xy-plane. Rotate about the y-AXIS by an ANGLE θ, where

$$\theta = \sin^{-1}\left(\frac{a}{c}\right). \tag{23}$$

In terms of the old coordinates, the new coordinates are

$$x = x_1\cos\theta - z_1\sin\theta \tag{24}$$
$$z = x_1\sin\theta + z_1\cos\theta. \tag{25}$$

So in (x_1, y_1, z_1) coordinates, equation (1) of the torus becomes

$$[\sqrt{(x_1\cos\theta - z_1\sin\theta)^2 + {y_1}^2} - c]^2 + (x_1\sin\theta + z_1\cos\theta)^2 = a^2. \tag{26}$$

Squaring both sides gives

$$(x_1\cos\theta - z_1\sin\theta)^2 + {y_1}^2 + c^2 - 2c\sqrt{(x_1\cos\theta - z_1\sin\theta)^2 + {y_1}^2} + (x_1\sin\theta + z_1\cos\theta)^2 = a^2. \tag{27}$$

But

$$(x_1\cos\theta - z_1\sin\theta)^2 + (x_1\sin\theta + z_1\cos\theta)^2 = {x_1}^2 + {z_1}^2, \tag{28}$$

so

$${x_1}^2 + {y_1}^2 + {z_1}^2 + c^2 - 2c\sqrt{(x_1\cos\theta - z_1\sin\theta)^2 + {y_1}^2} = a^2. \tag{29}$$

In the $z_1 = 0$ plane, plugging in (23) and factoring gives

$$[{x_1}^2 + (y_1 - a)^2 - c^2][{x_1}^2 + (y_1 + a) - c^2] = 0. \tag{30}$$

This gives the CIRCLES

$${x_1}^2 + (y_1 - a)^2 = c^2 \tag{31}$$

and

$${x_1}^2 + (y_1 + a)^2 = c^2 \tag{32}$$

in the z_1 plane. Written in MATRIX form with parameter $t \in [0, 2\pi)$, these are

$$C_1 = \begin{bmatrix} c\cos t \\ c\sin t + a \\ 0 \end{bmatrix} \tag{33}$$

$$C_2 = \begin{bmatrix} c\cos t \\ c\sin t - a \\ 0 \end{bmatrix}. \tag{34}$$

In the original (x, y, z) coordinates,

$$C_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} c\cos t \\ c\sin t + a \\ 0 \end{bmatrix} = \begin{bmatrix} c\cos\theta\cos t \\ c\sin t + a \\ -c\sin\theta\cos t \end{bmatrix} \tag{35}$$

$$C_2 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} c\cos t \\ c\sin t - a \\ 0 \end{bmatrix} = \begin{bmatrix} c\cos\theta\cos t \\ c\sin t - a \\ -c\sin\theta\cos t \end{bmatrix}. \tag{36}$$

The point P must satisfy

$$z = a\sin\phi = c\sin\theta\cos t, \tag{37}$$

so

$$\cos t = \frac{a\sin\phi}{c\sin\theta}. \tag{38}$$

Plugging this in for x_1 and y_1 gives the ANGLE ψ by which the CIRCLE must be rotated about the z-AXIS in order to make it pass through P,

$$\psi = \tan^{-1}\left(\frac{y}{x}\right) = \frac{c\sin t + a}{c\cos\theta\cos t} = \frac{c\sqrt{1-\cos^2 t} + a}{c\cos\theta\cos t}. \tag{39}$$

The four CIRCLES passing through P are therefore

$$C_1 = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\cos\theta\cos t \\ c\sin t + a \\ -c\sin\theta\cos t \end{bmatrix} \tag{40}$$

$$C_2 = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\cos\theta\cos t \\ c\sin t - a \\ -c\sin\theta\cos t \end{bmatrix} \tag{41}$$

$$C_3 = \begin{bmatrix} (c + a\cos\phi)\cos t \\ (c + a\cos\phi)\sin t \\ a\sin\phi \end{bmatrix} \tag{42}$$

$$C_4 = \begin{bmatrix} c + a\cos t \\ 0 \\ a\sin t \end{bmatrix}. \tag{43}$$

see also APPLE, CYCLIDE, ELLIPTIC TORUS, GENUS (SURFACE), HORN TORUS, KLEIN QUARTIC, LEMON, RING TORUS, SPINDLE TORUS, SPIRIC SECTION, STANDARD TORI, TOROID, TORUS COLORING, TORUS CUTTING

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Torus Coloring

The number of colors SUFFICIENT for MAP COLORING on a surface of GENUS g is given by the HEAWOOD CONJECTURE,

$$\chi(g) = \left\lfloor \tfrac{1}{2}(7 + \sqrt{48g+1}) \right\rfloor,$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION. The fact that $\chi(g)$ (which is called the CHROMATIC NUMBER) is also NECESSARY was proved by Ringel and Youngs (1968) with two exceptions: the SPHERE (which requires the same number of colors as the PLANE) and the KLEIN BOTTLE. A g-holed TORUS therefore requires $\chi(g)$ colors. For $g = 0, 1, \ldots$, the first few values of $\chi(g)$ are 4, 7, 8, 9, 10, 11, 12, 12, 13, 13, 14, 15, 15, 16, ... (Sloane's A000934).

see also CHROMATIC NUMBER, FOUR-COLOR THEOREM, HEAWOOD CONJECTURE, KLEIN BOTTLE, MAP COLORING

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Torus Cutting

With n cuts of a TORUS of GENUS 1, the maximum number of pieces which can be obtained is

$$N(n) = \tfrac{1}{6}(n^3 + 3n^2 + 8n).$$

The first few terms are 2, 6, 13, 24, 40, 62, 91, 128, 174, 230, ... (Sloane's A003600).

see also CAKE CUTTING, CIRCLE CUTTING, CYLINDER CUTTING, PANCAKE CUTTING, PLANE CUTTING, PIE CUTTING, SQUARE CUTTING

References

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Torus Knot

A (p,q)-torus KNOT is obtained by looping a string through the HOLE of a TORUS p times with q revolutions before joining its ends, where p and q are RELATIVELY PRIME. A (p,q)-torus knot is equivalent to a (q,p)-torus knot. The CROSSING NUMBER of a (p,q)-torus knot is

$$c = \min\{p(q-1), q(p-1)\} \tag{1}$$

(Murasugi 1991). The UNKNOTTING NUMBER of a (p,q)-torus knot is

$$u = \tfrac{1}{2}(p-1)(q-1) \tag{2}$$

(Adams 1991).

Torus knots with fewer than 11 crossings are the TREFOIL KNOT 03_{001} (3, 2), SOLOMON'S SEAL KNOT 05_{001} (5, 2), 07_{001} (7, 2), 08_{019} (4, 3), 09_{001} (9, 2), and 10_{124} (5, 3) (Adams *et al.* 1991). The only KNOTS which are not HYPERBOLIC KNOTS are torus knots and SATELLITE KNOTS (including COMPOSITE KNOTS). The $(2,q)$, $(3,4)$, and $(3,5)$-torus knots are ALMOST ALTERNATING KNOTS.

The JONES POLYNOMIAL of an (m,n)-TORUS KNOT is

$$\frac{t^{(m-1)(n-1)/2}(1-t^{m+1}-t^{n+1}+t^{m+n})}{1-t^2}. \tag{3}$$

The BRACKET POLYNOMIAL for the torus knot $K_n = (2,n)$ is given by the RECURRENCE RELATION

$$\langle K_n\rangle = A\langle K_{n-1}\rangle + (-1)^{n-1}A^{-3n+2}, \tag{4}$$

where

$$\langle K_1\rangle = -A^3. \tag{5}$$

see also ALMOST ALTERNATING KNOT, HYPERBOLIC KNOT, KNOT, SATELLITE KNOT, SOLOMON'S SEAL KNOT, TREFOIL KNOT

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Murasugi, K. "On the Braid Index of Alternating Links." *Trans. Amer. Math. Soc.* **326**, 237–260, 1991.

Total Angular Defect

see DESCARTES TOTAL ANGULAR DEFECT

Total Curvature

The total curvature of a curve is the quantity $\sqrt{\tau^2+\kappa^2}$, where τ is the TORSION and κ is the CURVATURE. The total curvature is also called the THIRD CURVATURE.

see also CURVATURE, TORSION (DIFFERENTIAL GEOMETRY)

Total Differential

see EXACT DIFFERENTIAL

Total Function

A FUNCTION defined for all possible input values.

Total Intersection Theorem

If one part of the total intersection group of a curve of order n with a curve of order $n_1 + n_2$ constitutes the total intersection with a curve of order n_1, then the other part will constitute the total intersection with a curve of order n_2.

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Total Order

A total order satisfies the conditions for a PARTIAL ORDER plus the comparability condition. A RELATION $\leq$ is a partial order on a SET S if

1. Reflexivity: $a \leq a$ for all $a \in S$
2. Antisymmetry: $a \leq b$ and $b \leq a$ implies $a = b$
3. Transitivity: $a \leq b$ and $b \leq c$ implies $a \leq c$,

and is a total order if, in addition,

4. Comparability: For any $a, b \in S$, either $a \leq b$ or $b \leq a$.

see also PARTIAL ORDER, RELATION

Total Space

The SPACE E of a FIBER BUNDLE given by the MAP $f : E \to B$, where B is the BASE SPACE of the FIBER BUNDLE.

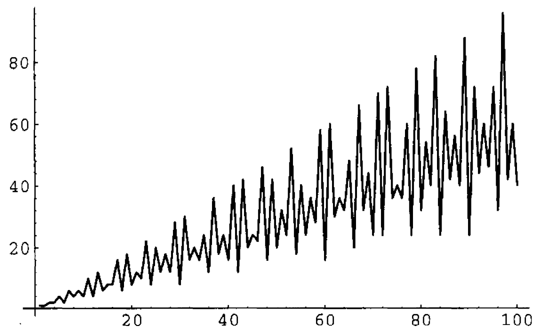
see also BASE SPACE, FIBER BUNDLE, SPACE

Totative

A POSITIVE INTEGER less than or equal to a number n which is also RELATIVELY PRIME to n, where 1 is counted as being RELATIVELY PRIME to all numbers. The number of totatives of n is the value of the TOTIENT FUNCTION $\phi(n)$.

see also RELATIVELY PRIME, TOTIENT FUNCTION

Totient Function



The totient function $\phi(n)$, also called Euler's totient function, is defined as the number of POSITIVE INTEGERS $\leq n$ which are RELATIVELY PRIME to (i.e., do not contain any factor in common with) n, where 1 is counted as being RELATIVELY PRIME to all numbers. Since a number less than or equal to and RELATIVELY PRIME to a given number is called a TOTATIVE, the totient function $\phi(n)$ can be simply defined as the number of TOTATIVES of n. For example, there are eight TOTATIVES of 24 (1, 5, 7, 11, 13, 17, 19, and 23), so $\phi(24) = 8$.

By convention, $\phi(0) = 1$. The first few values of $\phi(n)$ for $n = 1, 2, \ldots$ are 1, 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, ... (Sloane's A000010). $\phi(n)$ is plotted above for small n.

For a PRIME p,

$$\phi(p) = p - 1, \tag{1}$$

since all numbers less than p are RELATIVELY PRIME to p. If $m = p^\alpha$ is a POWER of a PRIME, then the numbers which have a common factor with m are the multiples of p: $p, 2p, \ldots, (p^{\alpha-1})p$. There are $p^{\alpha-1}$ of these multiples, so the number of factors RELATIVELY PRIME to p^α is

$$\phi(p^\alpha) = p^\alpha - p^{\alpha-1} = p^{\alpha-1}(p-1) = p^\alpha\left(1 - \frac{1}{p}\right). \tag{2}$$

Now take a general m divisible by p. Let $\phi_p(m)$ be the number of POSITIVE INTEGERS $\leq m$ not DIVISIBLE by p. As before, $p, 2p, \ldots, (m/p)p$ have common factors, so

$$\phi_p(m) = m - \frac{m}{p} = m\left(1 - \frac{1}{p}\right). \tag{3}$$

Now let q be some other PRIME dividing m. The INTEGERS divisible by q are $q, 2q, \ldots, (m/q)q$. But these duplicate $pq, 2pq, \ldots, (m/pq)pq$. So the number of terms which must be subtracted from ϕ_p to obtain ϕ_{pq} is

$$\Delta\phi_q(m) = \frac{m}{q} - \frac{m}{pq} = \frac{m}{q}\left(1 - \frac{1}{p}\right), \tag{4}$$

and

$$\begin{aligned}\phi_{pq}(m) &\equiv \phi_q(m) - \Delta\phi_q(m)\\ &= m\left(1 - \frac{1}{p}\right) - \frac{m}{q}\left(1 - \frac{1}{p}\right)\\ &= m\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right).\end{aligned} \tag{5}$$

By induction, the general case is then

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_r}\right). \tag{6}$$

An interesting identity relates $\phi(n^2)$ to $\phi(n)$,

$$\phi(n^2) = n\phi(n). \tag{7}$$

Another identity relates the DIVISORS d of n to n via

$$\sum_d \phi(d) = n. \tag{8}$$

The DIVISOR FUNCTION satisfies the CONGRUENCE

$$n\sigma(n) \equiv 2 \pmod{\phi(n)} \tag{9}$$

for all PRIMES and no COMPOSITE with the exceptions of 4, 6, and 22 (Subbarao 1974), where $\sigma(n)$ is the DIVISOR FUNCTION. No COMPOSITE solution is currently known to

$$n - 1 \equiv 0 \pmod{\phi(n)} \tag{10}$$

(Honsberger 1976, p. 35).

Walfisz (1963), building on the work of others, showed that

$$\sum_{n=1}^N \phi(n) = \frac{3N^2}{\pi^2} + \mathcal{O}[N(\ln N)^{2/3}(\ln\ln N)^{4/3}], \tag{11}$$

and Landau (1900, quoted in Dickson 1952) showed that

$$\sum_{n=1}^N \frac{1}{\phi(n)} = A\ln N + B + \mathcal{O}\left(\frac{\ln N}{N}\right), \tag{12}$$

where

$$\begin{aligned}A &= \sum_{k=1}^\infty \frac{[\mu(k)]^2}{k\phi(k)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} = \frac{315}{2\pi^4}\zeta(3)\\ &= 1.9435964368\ldots\end{aligned} \tag{13}$$

$$\begin{aligned}B &= \gamma\frac{315}{2\pi^4}\zeta(3) - \sum_{k=1}^\infty \frac{[\mu(k)]^2\ln k}{k\phi(k)}\\ &= -0.0595536246\ldots,\end{aligned} \tag{14}$$

$\mu(k)$ is the MÖBIUS FUNCTION, $\zeta(z)$ is the RIEMANN ZETA FUNCTION, and γ is the EULER-MASCHERONI CONSTANT (Dickson). A can also be written

$$A = \prod_{k=1}^{\infty} \frac{1 - {p_k}^6}{(1 - {p_k}^{-2})(1 - {p_k}^{-3})} = \prod_{k=1}^{\infty} \left[1 + \frac{1}{p_k(p_k - 1)}\right]. \quad (15)$$

Note that this constant is similar to ARTIN'S CONSTANT.

If the GOLDBACH CONJECTURE is true, then for every number m, there are PRIMES p and q such that

$$\phi(p) + \phi(q) = 2m \quad (16)$$

(Guy 1994, p. 105).

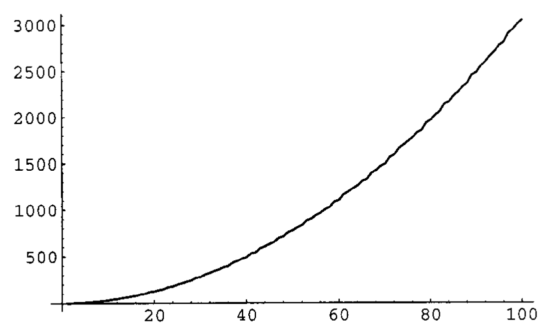
Curious equalities of consecutive values include

$$\phi(5186) = \phi(5187) = \phi(5188) = 2^5 3^4 \quad (17)$$

$$\phi(25930) = \phi(25935) = \phi(25940) = \phi(25942) = 2^7 3^4 \quad (18)$$

$$\phi(404471) = \phi(404473) = \phi(404477) = 2^8 3^2 5^2 7 \quad (19)$$

(Guy 1994, p. 91).



The SUMMATORY totient function, plotted above, is defined by

$$\Phi(n) \equiv \sum_{k=1}^{n} \phi(k) \quad (20)$$

and has the asymptotic series

$$\Phi(x) \sim \frac{1}{2\zeta(2)} x^2 + (x \ln x) \quad (21)$$

$$\sim \frac{3}{\pi^2} x^2 + \mathcal{O}(x \ln x), \quad (22)$$

where $\zeta(z)$ is the RIEMANN ZETA FUNCTION (Perrot 1881). The first values of $\Phi(n)$ are 1, 2, 4, 6, 10, 12, 18, 22, 28, ... (Sloane's A002088).

see also DEDEKIND FUNCTION, EULER'S TOTIENT RULE, FERMAT'S LITTLE THEOREM, LEHMER'S PROBLEM, LEUDESDORF THEOREM, NONCOTOTIENT, NONTOTIENT, SILVERMAN CONSTANT, TOTATIVE, TOTIENT VALENCE FUNCTION

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Totient Function Constants

see SILVERMAN CONSTANT, TOTIENT FUNCTION

Totient Valence Function

$N_\phi(m)$ is the number of INTEGERS n for which $\phi(n) = m$, also called the MULTIPLICITY of m (Guy 1994). The table below lists values for $\phi(N) \le 50$.

$\phi(N)$	m	N
1	2	1, 2
2	3	3, 4, 6
4	4	5, 8, 10, 12
6	4	7, 9, 14, 18
8	5	15, 16, 20, 24, 30
10	2	11, 22
12	6	13, 21, 26, 28, 36, 42
16	6	17, 32, 34, 40, 48, 60
18	4	19, 27, 38, 54
20	5	25, 33, 44, 50, 66
22	2	23, 46
24	10	35, 39, 45, 52, 56, 70, 72, 78, 84, 90
28	2	29, 58
30	2	31, 62
32	7	51, 64, 68, 80, 96, 102, 120
36	8	37, 57, 63, 74, 76, 108, 114, 126
40	9	41, 55, 75, 82, 88, 100, 110, 132, 150
42	4	43, 49, 86, 98
44	3	69, 92, 138
46	2	47, 94
48	11	65, 104, 105, 112, 130, 140, 144, 156, 168, 180, 210

A table listing the first value of $\phi(N)$ with multiplicities up to 100 follows (Sloane's A014573).

M	ϕ	M	ϕ	M	ϕ	M	ϕ
0	3	26	2560	51	4992	76	21840
2	1	27	384	52	17640	77	9072
3	2	28	288	53	2016	78	38640
4	4	29	1320	54	1152	79	9360
5	8	30	3696	55	6000	80	81216
6	12	31	240	56	12288	81	4032
7	32	32	768	57	4752	82	5280
8	36	33	9000	58	2688	83	4800
9	40	34	432	59	3024	84	4608
10	24	35	7128	60	13680	85	16896
11	48	36	4200	61	9984	86	3456
12	160	37	480	62	1728	87	3840
13	396	38	576	63	1920	88	10800
14	2268	39	1296	64	2400	89	9504
15	704	40	1200	65	7560	90	18000
16	312	41	15936	66	2304	91	23520
17	72	42	3312	67	22848	92	39936
18	336	43	3072	68	8400	93	5040
19	216	44	3240	69	29160	94	26208
20	936	45	864	70	5376	95	27360
21	144	46	3120	71	3360	96	6480
22	624	47	7344	72	1440	97	9216
23	1056	48	3888	73	13248	98	2880
24	1760	49	720	74	11040	99	26496
25	360	50	1680	75	27720	100	34272

It is thought that $N_\phi(m) \geq 2$ (i.e., the totient valence function never takes on the value 1), but this has not been proven. This assertion is called CARMICHAEL'S TOTIENT FUNCTION CONJECTURE and is equivalent to the statement that for all n, there exists $m \neq n$ such that $\phi(n) = \phi(m)$ (Ribenboim 1996, pp. 39–40). Any counterexample must have more than 10,000,000 DIGITS (Schlafly and Wagon 1994, Conway and Guy 1996).

see also CARMICHAEL'S TOTIENT FUNCTION CONJECTURE, TOTIENT FUNCTION

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Touchard's Congruence

$$B_{p+k} \equiv B_k + B_{k+1} \pmod{p},$$

when p is PRIME and B_n is a BELL NUMBER.

see also BELL NUMBER

Tour

A sequence of moves on a chessboard by a CHESS piece in which each square of a CHESSBOARD is visited exactly once.

see also CHESS, KNIGHT'S TOUR, MAGIC TOUR, TRAVELING SALESMAN CONSTANTS

Tournament

A COMPLETE DIRECTED GRAPH. A so-called SCORE SEQUENCE can be associated with every tournament. Every tournament contains a HAMILTONIAN PATH.

see also COMPLETE GRAPH, DIRECTED GRAPH, HAMILTONIAN PATH, SCORE SEQUENCE

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Ruskey, F. "Information on Score Sequences." http://sue.csc.uvic.ca/~cos/inf/nump/ScoreSequence.html.

Tournament Matrix

A matrix for a round-robin tournament involving n players competing in $n(n-1)/2$ matches (no ties allowed) having entries

$$a_{ij} = \begin{cases} 1 & \text{if player } i \text{ defeats player } j \\ -1 & \text{if player } i \text{ loses to player } j \\ 0 & \text{if } i = j. \end{cases}$$

The MATRIX satisfies

$$\mathsf{A} + \mathsf{A}^{\mathrm{T}} + \mathsf{I} = \mathsf{J},$$

where I is the IDENTITY MATRIX, J is an $n \times n$ MATRIX of all 1s, and A^{T} is the MATRIX TRANSPOSE of A.

The tournament matrix for n players has zero DETERMINANT IFF n is ODD (McCarthy and Benjamin 1996). The dimension of the NULLSPACE of an n-player tournament matrix is

$$\dim[\text{nullspace}] = \begin{cases} 0 & \text{for } n \text{ even} \\ 1 & \text{for } n \text{ odd} \end{cases}$$

(McCarthy 1996).

References

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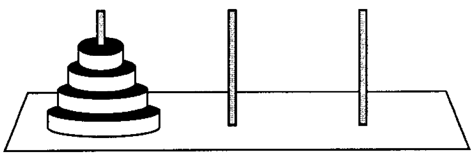
Tower of Power

see POWER TOWER

Towers of Hanoi



A PUZZLE invented by E. Lucas in 1883. Given a stack of n disks arranged from largest on the bottom to smallest on top placed on a rod, together with two empty rods, the towers of Hanoi puzzle asks for the minimum number of moves required to reverse the order of the stack (where moves are allowed only if they place smaller disks on top of larger disks). The problem is ISOMORPHIC to finding a HAMILTONIAN PATH on an n-HYPERCUBE.

For n disks, the number of moves h_n required is given by the RECURRENCE RELATION

$$h_n = 2h_{n-1} + 1.$$

Solving gives

$$h_n = 2^n - 1.$$

The number of disks moved after the kth step is the same as the element which needs to be added or deleted in the kth ADDEND of the RYSER FORMULA (Gardner 1988, Vardi 1991).

A HANOI GRAPH can be constructed whose VERTICES correspond to legal configurations of n towers of Hanoi, where the VERTICES are adjacent if the corresponding configurations can be obtained by a legal move. It can be solved using a binary GRAY CODE.

Poole (1994) gives *Mathematica*® (Wolfram Research, Champaign, IL) routines for solving an arbitrary disk configuration in the fewest possible moves. The proof of minimality is achieved using the LUCAS CORRESPONDENCE which relates PASCAL'S TRIANGLE to the HANOI GRAPH. ALGORITHMS are known for transferring disks for four pegs, but none has been proved minimal. For additional references, see Poole (1994).

see also GRAY CODE, RYSER FORMULA

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Trace (Complex)

The image of the path γ in $\mathbb{C}$ under the FUNCTION f is called the trace. This term is unrelated to that applied to MATRICES and TENSORS.

Trace (Group)

see CHARACTER (GROUP)

Trace (Map)

Let a PATCH be given by the map $\mathbf{x}: U \to \mathbb{R}^n$, where U is an open subset of $\mathbb{R}^2$, or more generally by $\mathbf{x}: A \to \mathbb{R}^n$, where A is any SUBSET of $\mathbb{R}^2$. Then $\mathbf{x}(U)$ (or more generally, $\mathbf{x}(A)$) is called the trace of $\mathbf{x}$.

see also PATCH

References

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Trace (Matrix)

The trace of an $n \times n$ SQUARE MATRIX A is defined by

$$\operatorname{Tr}(\mathsf{A}) \equiv a_{ii}, \tag{1}$$

where EINSTEIN SUMMATION is used (i.e., the a_{ii} is summed over $i = 1, \ldots, n$). For SQUARE MATRICES A and B, it is true that

$$\operatorname{Tr}(\mathsf{A}) = \operatorname{Tr}(\mathsf{A}^{\mathrm{T}}) \tag{2}$$

$$\operatorname{Tr}(\mathsf{A}+\mathsf{B}) = \operatorname{Tr}(\mathsf{A}) + \operatorname{Tr}(\mathsf{B}) \tag{3}$$

$$\operatorname{Tr}(\alpha\mathsf{A}) = \alpha \operatorname{Tr}(\mathsf{A}) \tag{4}$$

(Lange 1987, p. 40). The trace is invariant under a SIMILARITY TRANSFORMATION

$$A' \equiv BAB^{-1} \tag{5}$$

(Lange 1987, p. 64). Since

$$(bab^{-1})_{ij} = b_{il}a_{lk}b_{kj}^{-1}, \tag{6}$$

$$\begin{aligned}\mathrm{Tr}(BAB^{-1}) &= b_{il}a_{lk}b^{-1}{}_{ki}\\ &= (b^{-1}b)_{kl}a_{lk} = \delta_{kl}a_{lk}\\ &= a_{kk} = \mathrm{Tr}(A),\end{aligned} \tag{7}$$

where δ_{ij} is the KRONECKER DELTA.

The trace of a product of square matrices is independent of the order of the multiplication since

$$\begin{aligned}\mathrm{Tr}(AB) &= (ab)_{ii} = a_{ij}b_{ji} = b_{ji}a_{ij}\\ &= (ba)_{jj} = \mathrm{Tr}(BA).\end{aligned} \tag{8}$$

Therefore, the trace of the COMMUTATOR of A and B is given by

$$\mathrm{Tr}([A,B]) \equiv \mathrm{Tr}(AB) - \mathrm{Tr}(BA) = 0. \tag{9}$$

The product of a SYMMETRIC and an ANTISYMMETRIC MATRIX has zero trace,

$$\mathrm{Tr}(A_S B_A) = 0. \tag{10}$$

The value of the trace can be found using the fact that the matrix can always be transformed to a coordinate system where the z-AXIS lies along the axis of rotation. In the new coordinate system, the MATRIX is

$$A' = \begin{bmatrix} \cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}, \tag{11}$$

so the trace is

$$\mathrm{Tr}(A') = \mathrm{Tr}(A) \equiv a_{ii} = 1 + 2\cos\phi. \tag{12}$$

References

Lang, S. *Linear Algebra, 3rd ed.* New York: Springer-Verlag, pp. 40 and 64, 1987.

Trace (Tensor)

The trace of a second-RANK TENSOR T is a SCALAR given by the CONTRACTED mixed TENSOR equal to T_i^i. The trace satisfies

$$\mathrm{Tr}\left[M^{-1}(x)\frac{\partial}{\partial x^\lambda}M(x)\right] = \frac{\partial}{\partial x^\lambda}\ln[\det(x)],$$

and

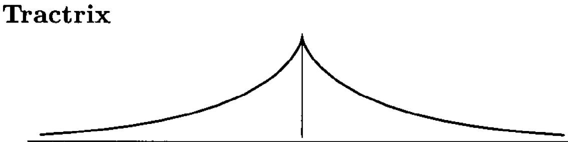
$$\begin{aligned}\delta\ln[\det M] &= \ln[\det(M+\delta M)] - \ln(\det M)\\ &= \ln\left[\frac{\det(M+\delta M)}{\det M}\right]\\ &= \ln[\det M^{-1}(M+\delta M)]\\ &= \ln[\det(1+M^{-1}\delta M)]\\ &\approx \ln[1+\mathrm{Tr}(M^{-1}\delta M)]\\ &\approx \mathrm{Tr}(M^{-1}\delta M).\end{aligned}$$

see also CONTRACTION (TENSOR)

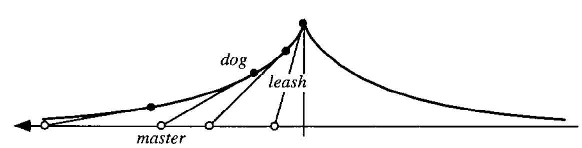
Tractory

see TRACTRIX

Tractrix



The tractrix is the CATENARY INVOLUTE described by a point initially on the vertex. It has a constant NEGATIVE CURVATURE and is sometimes called the TRACTORY or EQUITANGENTIAL CURVE. The tractrix was first studied by Huygens in 1692, who gave it the name "tractrix." Later, Leibniz, Johann Bernoulli, and others studied the curve.



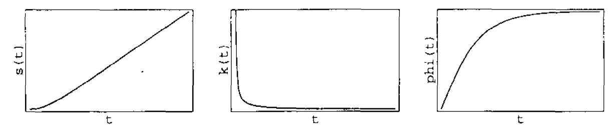
The tractrix arises from the following problem posed to Leibniz: What is the path of an object starting off with a vertical offset when it is dragged along by a string of constant length being pulled along a straight horizontal line? By associating the object with a dog, the string with a leash, and the pull along a horizontal line with the dog's master, the curve has the descriptive name HUNDKURVE (hound curve) in German. Leibniz found the curve using the fact that the axis is an asymptote to the tractrix (MacTutor Archive).

In CARTESIAN COORDINATES the tractrix has equation

$$x = a\,\mathrm{sech}^{-1}\left(\frac{y}{a}\right) - \sqrt{a^2-y^2}. \tag{1}$$

One parametric form is

$$x(t) = a(t - \tanh t) \tag{2}$$
$$y(t) = a \operatorname{sech} t. \tag{3}$$



The ARC LENGTH, CURVATURE, and TANGENTIAL ANGLE are

$$s(t) = \ln(\cosh t) \tag{4}$$
$$\kappa(t) = \operatorname{csch} t \tag{5}$$
$$\phi(t) = 2\tan^{-1}[\tanh(\tfrac{1}{2}t)]. \tag{6}$$

A second parametric form in terms of the ANGLE ϕ of the straight line tangent to the tractrix is

$$x = a\{\ln[\tan(\tfrac{1}{2}\phi)] + \cos\phi\} \tag{7}$$
$$y = a\sin\phi \tag{8}$$

(Gray 1993). This parameterization has CURVATURE

$$\kappa(\phi) = |\tan\phi|. \tag{9}$$

A parameterization which traverses the tractrix with constant speed a is given by

$$x(t) = \begin{cases} ae^{-v/a} & \text{for } v \in [0, \infty) \\ ae^{v/a} & \text{for } v \in (-\infty, 0] \end{cases} \tag{10}$$

$$y(t) = \begin{cases} a[\tanh^{-1}(\sqrt{1-e^{-2v/a}}) - \sqrt{1-e^{-2v/a}}] \\ \qquad \text{for } v \in [0, \infty) \\ a[-\tanh^{-1}(\sqrt{1-e^{2v/a}}) + \sqrt{1-e^{2v/a}}] \\ \qquad \text{for } v \in (-\infty, 0]. \end{cases} \tag{11}$$

When a tractrix is rotated around its asymptote, a PSEUDOSPHERE results. This is a surface of constant NEGATIVE CURVATURE. For a tractrix, the length of a TANGENT from its point of contact to an asymptote is constant. The AREA between the tractrix and its asymptote is finite.

see also CURVATURE, DINI'S SURFACE, MICE PROBLEM, PSEUDOSPHERE, PURSUIT CURVE, TRACTROID

References
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Lee, X. "Tractrix." `http://www.best.com/~xah/SpecialPlaneCurves_dir/Tractrix_dir/tractrix.html`.
Lockwood, E. H. "The Tractrix and Catenary." Ch. 13 in *A Book of Curves.* Cambridge, England: Cambridge University Press, pp. 118–124, 1967.
MacTutor History of Mathematics Archive. "Tractrix." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Tractrix.html`.
Yates, R. C. "Tractrix." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 221–224, 1952.

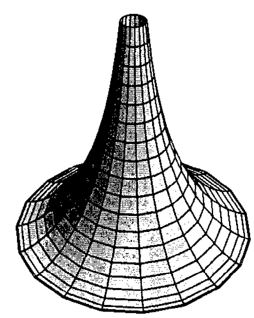
Tractrix Evolute

The EVOLUTE of the TRACTRIX is the CATENARY.

Tractrix Radial Curve

The RADIAL CURVE of the TRACTRIX is the KAPPA CURVE.

Tractroid



The SURFACE OF REVOLUTION produced by revolving the TRACTRIX

$$x = \operatorname{sech} u \tag{1}$$
$$z = u - \tanh u \tag{2}$$

about the z-AXIS is a tractroid given by

$$x = \operatorname{sech} u \cos v \tag{3}$$
$$y = \operatorname{sech} u \sin v \tag{4}$$
$$z = u - \tanh u. \tag{5}$$

see also PSEUDOSPHERE, SURFACE OF REVOLUTION, TRACTRIX

Transcendental Curve

A curve which intersects some straight line in an infinity of points (but for which not every point lies on this curve).

References
Borwein, J. M.; Borwein, P. B.; and Bailey, D. H. "Ramanujan, Modular Equations, and Approximations to Pi or How to Compute One Billion Digits of Pi." *Amer. Math. Monthly* **96**, 201–219, 1989.

Transcendental Equation

An equation or formula involving TRANSCENDENTAL FUNCTIONS.

Transcendental Function

A function which "transcends," i.e., cannot be expressed in terms of, the usual ELEMENTARY FUNCTIONS. Define

$$l_1(z) \equiv l(z) \equiv \ln(z)$$
$$e_1(z) \equiv e(z) \equiv e^z$$
$$\varsigma_1 f(z) \equiv \varsigma f(z) \equiv \int f(z)\,dz,$$

and let $l_2 \equiv l(l(z))$, etc. These are called the "elementary" transcendental functions (Watson 1966, p. 111).

see also ALGEBRAIC FUNCTION, ELEMENTARY FUNCTION

References

Watson, G. N. *A Treatise on the Theory of Bessel Functions, 2nd ed.* Cambridge, England: Cambridge University Press, 1966.

Transcendental Number

A number which is not the ROOT of *any* POLYNOMIAL equation with INTEGER COEFFICIENTS, meaning that it not an ALGEBRAIC NUMBER of any degree, is said to be transcendental. This definition guarantees that every transcendental number must also be IRRATIONAL, since a RATIONAL NUMBER is, by definition, an ALGEBRAIC NUMBER of degree one.

Transcendental numbers are important in the history of mathematics because their investigation provided the first proof that CIRCLE SQUARING, one of the GEOMETRIC PROBLEMS OF ANTIQUITY which had baffled mathematicians for more than 2000 years was, in fact, insoluble. Specifically, in order for a number to be produced by a GEOMETRIC CONSTRUCTION using the ancient Greek rules, it must be either RATIONAL or a very special kind of ALGEBRAIC NUMBER known as a EUCLIDEAN NUMBER. Because the number π is transcendental, the construction cannot be done according to the Greek rules.

Georg Cantor was the first to prove the EXISTENCE of transcendental numbers. Liouville subsequently showed how to construct special cases (such as LIOUVILLE'S CONSTANT) using LIOUVILLE'S RATIONAL APPROXIMATION THEOREM. In particular, he showed that any number which has a rapidly converging sequence of rational approximations must be transcendental. For many years, it was only known how to determine if special classes of numbers were transcendental. The determination of the status of more general numbers was considered an important enough unsolved problem that it was one of HILBERT'S PROBLEMS.

Great progress was subsequently made by GELFOND'S THEOREM, which gives a general rule for determining if special cases of numbers of the form α^β are transcendental. Baker produced a further revolution by proving the transcendence of sums of numbers of the form $\alpha \ln \beta$ for ALGEBRAIC NUMBERS α and β.

The number e was proven to be transcendental by Hermite in 1873, and PI (π) by Lindemann in 1882. e^π is transcendental by GELFOND'S THEOREM since

$$(-1)^{-i} = (e^{i\pi})^{-i} = e^\pi.$$

The GELFOND-SCHNEIDER CONSTANT $2^{\sqrt{2}}$ is also transcendental. Other known transcendentals are $\sin 1$ where $\sin x$ is the SINE function, $J_0(1)$ where $J_0(x)$ is a BESSEL FUNCTION OF THE FIRST KIND (Hardy and Wright 1985), $\ln 2$, $\ln 3/\ln 2$, the first zero $x_0 = 2.4048255\ldots$ of the BESSEL FUNCTION $J_0(x_0)$ (Le Lionnais 1983, p. 46), $\pi + \ln 2 + \sqrt{2}\ln 3$ (Borwein *et al.* 1989), the THUE-MORSE CONSTANT $P = 0.4124540336\ldots$ (Dekking 1977, Allouche and Shallit), the CHAMPERNOWNE CONSTANT $0.1234567891011\ldots$, the THUE CONSTANT

$$0.110110111110110111110110110\ldots,$$

$\Gamma(\frac{1}{3})$ (Le Lionnais 1983, p. 46), $\Gamma(\frac{1}{4})\pi^{-1/4}$ (Davis 1959), and $\Gamma(\frac{1}{4})$ (Chudnovsky, Waldschmidt), where $\Gamma(x)$ is the GAMMA FUNCTION. At least one of πe and $\pi + e$ (and probably both) are transcendental, but transcendence has not been proven for either number on its own.

It is not known if e^e, π^π, π^e, γ (the EULER-MASCHERONI CONSTANT), $I_0(2)$, or $I_1(2)$ (where $I_n(x)$ is a MODIFIED BESSEL FUNCTION OF THE FIRST KIND) are transcendental.

The "degree" of transcendence of a number can be characterized by a so-called LIOUVILLE-ROTH CONSTANT. There are still many fundamental and outstanding problems in transcendental number theory, including the CONSTANT PROBLEM and SCHANUEL'S CONJECTURE.

see also ALGEBRAIC NUMBER, CONSTANT PROBLEM, GELFOND'S THEOREM, IRRATIONAL NUMBER, LINDEMANN-WEIERSTRAß THEOREM, LIOUVILLE-ROTH CONSTANT, ROTH'S THEOREM, SCHANUEL'S CONJECTURE, THUE-SIEGEL-ROTH THEOREM

References

Allouche, J. P. and Shallit, J. In preparation.

Baker, A. "Approximations to the Logarithm of Certain Rational Numbers." *Acta Arith.* **10**, 315–323, 1964.

Baker, A. "Linear Forms in the Logarithms of Algebraic Numbers I." *Mathematika* **13**, 204–216, 1966.

Baker, A. "Linear Forms in the Logarithms of Algebraic Numbers II." *Mathematika* **14**, 102–107, 1966.

Baker, A. "Linear Forms in the Logarithms of Algebraic Numbers III." *Mathematika* **14**, 220–228, 1966.

Baker, A. "Linear Forms in the Logarithms of Algebraic Numbers IV." *Mathematika* **15**, 204–216, 1966.

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Dekking, F. M. "Transcendence du nombre de Thue-Morse." *Comptes Rendus de l'Academie des Sciences de Paris* **285**, 157–160, 1977.
Gray, R. "Georg Cantor and Transcendental Numbers." *Amer. Math. Monthly* **101**, 819–832, 1994.
Hardy, G. H. and Wright, E. M. *An Introduction to the Theory of Numbers, 5th ed.* Oxford, England: Oxford University Press, 1985.
Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 46, 1983.
Siegel, C. L. *Transcendental Numbers.* New York: Chelsea, 1965.

Transcritical Bifurcation

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a one-parameter family of C^2 maps satisfying

$$f(0,\mu) = 0 \tag{1}$$
$$\left[\frac{\partial f}{\partial x}\right]_{\mu=0,x=0} = 1 \tag{2}$$
$$\left[\frac{\partial f}{\partial x}\right]_{\mu,x} = \left[\frac{\partial f}{\partial x}\right]_{\mu=0,x=\mu} \tag{3}$$
$$\left[\frac{\partial^2 f}{\partial x \partial \mu}\right]_{0,0} > 0 \tag{4}$$
$$\left[\frac{\partial^2 f}{\partial \mu^2}\right]_{\mu=0,x=0} > 0. \tag{5}$$

Then there are two branches, one stable and one unstable. This BIFURCATION is called a transcritical bifurcation. An example of an equation displaying a transcritical bifurcation is

$$\dot{x} = \mu x - x^2. \tag{6}$$

(Guckenheimer and Holmes 1997, p. 145).

see also BIFURCATION

References

Guckenheimer, J. and Holmes, P. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, 3rd ed.* New York: Springer-Verlag, pp. 145 and 149–150, 1997.
Rasband, S. N. *Chaotic Dynamics of Nonlinear Systems.* New York: Wiley, pp. 27–28, 1990.

Transfer Function

The engineering terminology for one use of FOURIER TRANSFORMS. By breaking up a wave pulse into its frequency spectrum

$$f_\nu = F(\nu)e^{2\pi i\nu t}, \tag{1}$$

the entire signal can be written as a sum of contributions from each frequency,

$$f(t) = \int_{-\infty}^{\infty} f_\nu \, d\nu = \int_{-\infty}^{\infty} F(\nu)e^{2\pi i\nu t}\, d\nu. \tag{2}$$

If the signal is modified in some way, it will become

$$g_\nu(t) = \phi(\nu) f_\nu(t) = \phi(\nu)F(\nu)e^{2\pi i\nu t} \tag{3}$$
$$g(t) = \int_{-\infty}^{\infty} g_\nu(t)\, dt = \int_{-\infty}^{\infty} \phi(\nu)F(\nu)e^{2\pi i\nu t}\, d\nu, \tag{4}$$

where $\phi(\nu)$ is known as the "transfer function." FOURIER TRANSFORMING ϕ and F,

$$\phi(\nu) = \int_{-\infty}^{\infty} \Phi(t)e^{-2\pi i\nu t}\, dt \tag{5}$$

$$F(\nu) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i\nu t}\, dt. \tag{6}$$

From the CONVOLUTION THEOREM,

$$g(t) = f(t) * \Phi(t) = \int_{-\infty}^{\infty} f(t)\Phi(t-\tau)\, d\tau. \tag{7}$$

see also CONVOLUTION THEOREM, FOURIER TRANSFORM

Transfinite Diameter

Let

$$\phi(z) = cz + c_0 + c_1 z^{-1} + c_2 z^{-2} + \ldots$$

be an ANALYTIC FUNCTION, REGULAR and UNIVALENT for $|z| > 1$, which maps $|z| > 1$ CONFORMALLY onto the region T preserving the POINT AT INFINITY and its direction. Then the function $\phi(z)$ is uniquely determined and c is called the transfinite diameter, sometimes also known as ROBIN'S CONSTANT or the CAPACITY of $\phi(z)$.

see also ANALYTIC FUNCTION, REGULAR FUNCTION, UNIVALENT FUNCTION

Transfinite Number

One of Cantor's ORDINAL NUMBERS $\omega, \omega+1, \omega+2, \ldots, \omega+\omega, \omega+\omega+1, \ldots$ which is "larger" than any WHOLE NUMBER.

see also $\aleph_0$, $\aleph_1$, CARDINAL NUMBER, CONTINUUM, ORDINAL NUMBER, WHOLE NUMBER

References

Pappas, T. "Transfinite Numbers." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, pp. 156–158, 1989.

Transform

A shortened term for INTEGRAL TRANSFORM.

Geometrically, if S and T are two transformations, then the SIMILARITY TRANSFORMATION TST^{-1} is sometimes called the transform (Woods 1961).

see also ABEL TRANSFORM, BOUSTROPHEDON TRANSFORM, DISCRETE FOURIER TRANSFORM, FAST FOURIER TRANSFORM, FOURIER TRANSFORM, FRACTIONAL FOURIER TRANSFORM, HANKEL TRANSFORM, HARTLEY TRANSFORM, HILBERT TRANSFORM, LAPLACE-STIELTJES TRANSFORM, LAPLACE TRANSFORM, MELLIN TRANSFORM, NUMBER THEORETIC TRANSFORM, PONCELET TRANSFORM, RADON TRANSFORM, WAVELET TRANSFORM, z-TRANSFORM, Z-TRANSFORM

References

Woods, F. S. *Higher Geometry: An Introduction to Advanced Methods in Analytic Geometry.* New York: Dover, p. 5, 1961.

Transformation

see FUNCTION, MAP

Transitive

A RELATION R on a SET S is transitive provided that for all x, y and z in S such that xRy and yRz, we also have xRz.

see also ASSOCIATIVE, COMMUTATIVE, RELATION

Transitive Closure

The transitive closure of a binary RELATION R on a SET X is the minimal TRANSITIVE relation R' on X that contains R. Thus $aR'b$ for any elements a and b of X, provided either that aRb or that there exists some element c of X such that aRc and cRb.

see also REFLEXIVE CLOSURE, TRANSITIVE REDUCTION

Transitive Reduction

The transitive reduction of a binary RELATION R on a SET X is the minimum relation R' on X with the same TRANSITIVE CLOSURE as R. Thus $aR'b$ for any elements a and b of X, provided that aRb and there exists no element c of X such that aRc and cRb.

see also REFLEXIVE REDUCTION, TRANSITIVE CLOSURE

Transitivity Class

Let $S(T)$ be the group of symmetries which map a MONOHEDRAL TILING T onto itself. The TRANSITIVITY CLASS of a given tile T is then the collection of all tiles to which T can be mapped by one of the symmetries of $S(T)$.

see also MONOHEDRAL TILING

References

Berglund, J. "Is There a k-Anisohedral Tile for $k \geq 5$?" *Amer. Math. Monthly* **100**, 585–588, 1993.

Translation

A transformation consisting of a constant offset with no ROTATION or distortion. In n-D EUCLIDEAN SPACE, a translation may be specified simply as a VECTOR giving the offset in each of the n coordinates.

see also AFFINE GROUP, DILATION, EUCLIDEAN GROUP, EXPANSION, GLIDE, IMPROPER ROTATION, INVERSION OPERATION, MIRROR IMAGE, REFLECTION, ROTATION

References

Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 211, 1987.

Translation Relation

A mathematical relationship transforming a function $f(x)$ to the form $f(x+a)$.

see also ARGUMENT ADDITION RELATION, ARGUMENT MULTIPLICATION RELATION, RECURRENCE RELATION, REFLECTION RELATION

Transpose

The object obtained by replacing all elements a_{ij} with a_{ji}. For a second-RANK TENSOR a_{ij}, the tensor transpose is simply a_{ji}. The matrix transpose, written A^{T}, is the MATRIX obtained by exchanging A's rows and columns, and satisfies the identity

$$(\mathsf{A}^{\mathrm{T}})^{-1} = (\mathsf{A}^{-1})^{\mathrm{T}}.$$

The product of two transposes satisfies

$$\begin{aligned}(\mathsf{B}^{\mathrm{T}}\mathsf{A}^{\mathrm{T}})_{ij} &= (b^{\mathrm{T}})_{ik}(a^{\mathrm{T}})_{kj} = b_{ki}a_{jk} = a_{jk}b_{ki} = (\mathsf{AB})_{ji} \\ &= (\mathsf{AB})^{\mathrm{T}}_{ij}.\end{aligned}$$

Therefore,

$$(\mathsf{AB})^{\mathrm{T}} = \mathsf{B}^{\mathrm{T}}\mathsf{A}^{\mathrm{T}}.$$

Transpose Map

see PULLBACK MAP

Transposition

An exchange of two elements of a SET with all others staying the same. A transposition is therefore a PERMUTATION of two elements. For example, the swapping of 2 and 5 to take the list 123456 to 153426 is a transposition.

see also PERMUTATION, TRANSPOSITION ORDER

Transposition Group

A PERMUTATION GROUP in which the PERMUTATIONS are limited to TRANSPOSITIONS.

see also PERMUTATION GROUP

Transposition Order

An ordering of PERMUTATIONS in which each two adjacent permutations differ by the TRANSPOSITION of two elements. For the permutations of $\{1,2,3\}$ there are two listings which are in transposition order. One is 123, 132, 312, 321, 231, 213, and the other is 123, 321, 312, 213, 231, 132.

see also LEXICOGRAPHIC ORDER, PERMUTATION

References

Ruskey, F. "Information on Combinations of a Set." http://sue.csc.uvic.ca/~cos/inf/comb/CombinationsInfo.html.

Transversal Array

A set of n cells in an $n \times n$ SQUARE such that no two come from the same row and no two come from the same column. The number of transversals of an $n \times n$ SQUARE is $n!$ (n FACTORIAL).

Transversal Design

A transversal design $\mathrm{TD}_\lambda(k,n)$ of order n, block size k, and index λ is a triple (V, G, B) such that

1. V is a set of kn elements,
2. G is a partition of V into k classes, each of size n (the "groups"),
3. B is a collection of k-subsets of V (the "blocks"), and
4. Every unordered pair of elements from V is contained in either exactly one group or in exactly λ blocks, but not both.

References

Colbourn, C. J. and Dinitz, J. H. (Eds.) *CRC Handbook of Combinatorial Designs.* Boca Raton, FL: CRC Press, p. 112, 1996.

Transversal Line

A transversal line is a LINE which intersects each of a given set of other lines. It is also called a SEMISECANT.

see also LINE

Transylvania Lottery

A lottery in which three numbers are picked at random from the INTEGERS 1–14.

see also FANO PLANE

Trapdoor Function

An easily computed function whose inverse is extremely difficult to compute. An example is the multiplication of two large PRIMES. Finding and verifying two large PRIMES is easy, as is their multiplication. But factorization of the resultant product is very difficult.

see also RSA ENCRYPTION

References

Gardner, M. Chs. 13–14 in *Penrose Tiles and Trapdoor Ciphers... and the Return of Dr. Matrix, reissue ed.* New York: W. H. Freeman, pp. 299–300, 1989.

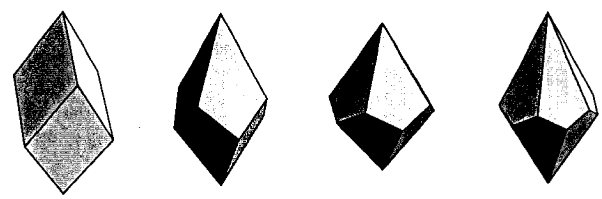
Trapezium

There are two common definitions of the trapezium. The American definition is a QUADRILATERAL with no PARALLEL sides. The British definition for a trapezium is a QUADRILATERAL *with* two sides PARALLEL. Such a trapezium is equivalent to a TRAPEZOID and therefore has AREA

$$A = \tfrac{1}{2}(a+b)h.$$

see also DIAMOND, LOZENGE, PARALLELOGRAM, QUADRILATERAL, RHOMBOID, RHOMBUS, SKEW QUADRILATERAL, TRAPEZOID

Trapezohedron



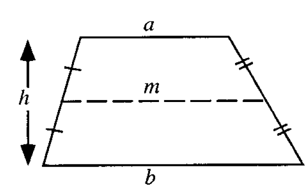
The trapezohedra are the DUAL POLYHEDRA of the Archimedean ANTIPRISMS. However, their faces are not TRAPEZOIDS.

see also ANTIPRISM, DIPYRAMID, HEXAGONAL SCALENOHEDRON, PRISM, TRAPEZOID

References

Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., p. 117, 1989.

Trapezoid



A QUADRILATERAL with two sides PARALLEL. The trapezoid depicted above satisfies

$$m = \tfrac{1}{2}(a+b)$$

and has AREA

$$A = \tfrac{1}{2}(a+b)h = mh.$$

The trapezoid is equivalent to the British definition of TRAPEZIUM.

see also PYRAMIDAL FRUSTUM, TRAPEZIUM

References

Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 123, 1987.

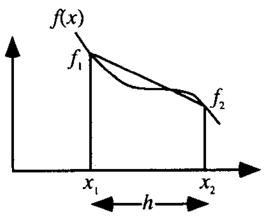
Trapezoidal Hexecontahedron

see DELTOIDAL HEXECONTAHEDRON

Trapezoidal Icositetrahedron

see DELTOIDAL ICOSITETRAHEDRON

Trapezoidal Rule



The 2-point NEWTON-COTES FORMULA

$$\int_{x_1}^{x_2} f(x)\,dx = \tfrac{1}{2}h(f_1+f_2) - \tfrac{1}{2}h^3 f''(\xi),$$

where $f_i \equiv f(x_i)$, h is the separation between the points, and ξ is a point satisfying $x_1 \leq \xi \leq x_2$. Picking ξ to maximize $f''(\xi)$ gives an upper bound for the error in the trapezoidal approximation to the INTEGRAL.

see also BODE'S RULE, HARDY'S RULE, NEWTON-COTES FORMULAS, SIMPSON'S 3/8 RULE, SIMPSON'S RULE, WEDDLE'S RULE

References

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Traveling Salesman Constants

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let $L(n,d)$ be the smallest TOUR length for n points in a d-D HYPERCUBE. Then there exists a smallest constant $\alpha(d)$ such that for all optimal TOURS in the HYPERCUBE,

$$\limsup_{n\to\infty} \frac{L(n,d)}{n^{(d-1)/d}\sqrt{d}} \leq \alpha(d), \qquad (1)$$

and a constant $\beta(d)$ such that for *almost* all optimal tours in the HYPERCUBE,

$$\lim_{n\to\infty} \frac{L(n,d)}{n^{(d-1)/d}\sqrt{d}} = \beta(d). \qquad (2)$$

These constants satisfy the inequalities

$$0.44194 < \gamma_2 = \tfrac{5}{16}\sqrt{2} \leq \beta(2) \leq \delta < 0.6508 < 0.75983 < 3^{-1/4} \leq \alpha(2) \leq \phi < 0.98398 \qquad (3)$$

$$0.37313 < \gamma_3 \leq \beta(3) \leq 12^{1/6}6^{-1/2} < 0.61772 < 0.64805 < 2^{1/6}3^{-1/2} \leq \alpha(3) \leq 0.90422 \qquad (4)$$

$$0.34207 < \gamma_4 \leq \beta(4) \leq 12^{1/8}6^{-1/2} < 0.55696 < 0.59460 < 2^{-3/4} \leq \alpha(4) \leq 0.8364 \qquad (5)$$

(Fejes Tóth 1940, Verblunsky 1951, Few 1955, Beardwood *et al.* 1959), where

$$\gamma_d \equiv \frac{\Gamma\left(3+\frac{1}{d}\right)\left[\Gamma\left(\frac{1}{2}d+1\right)\right]^{1/d}}{2\sqrt{\pi}(d^{1/2}+d^{-1/2})}, \qquad (6)$$

$\Gamma(z)$ is the GAMMA FUNCTION, δ is an expression involving STRUVE FUNCTIONS and NEUMANN FUNCTIONS,

$$\phi \equiv \frac{280(3-\sqrt{3})}{840-280\sqrt{3}+4\sqrt{5}-\sqrt{10}} \qquad (7)$$

(Karloff 1989), and

$$\psi \equiv \tfrac{1}{2}3^{-2/3}(4+\ln 3)^{2/3} \qquad (8)$$

(Goddyn 1990). In the LIMIT $d \to \infty$,

$$0.24197 < \lim_{d\to\infty} \gamma_d = \frac{1}{\sqrt{2\pi e}} \leq \liminf_{d\to\infty} \beta(d) \leq \limsup_{d\to\infty} \beta(d) \leq \lim_{d\to\infty} 12^{1/(2d)}6^{-1/2} = \frac{1}{\sqrt{6}} < 0.40825 \qquad (9)$$

and

$$0.24197 < \frac{1}{\sqrt{2\pi e}} \leq \lim_{d\to\infty} \alpha(d) \leq \frac{2(3-\sqrt{3})\theta}{\sqrt{2\pi e}} < 0.4052, \qquad (10)$$

where

$$\tfrac{1}{2} \leq \theta = \lim_{d\to\infty} [\theta(d)]^{1/d} \leq 0.6602, \qquad (11)$$

and $\theta(d)$ is the best SPHERE PACKING density in d-D space (Goddyn 1990, Moran 1984, Kabatyanskii and Levenshtein 1978). Steele and Snyder (1989) proved that the limit $\alpha(d)$ exists.

Now consider the constant

$$\kappa \equiv \lim_{n\to\infty} \frac{L(n,2)}{\sqrt{n}} = \beta(2)\sqrt{2}, \qquad (12)$$

so

$$\tfrac{5}{8} = \gamma_2\sqrt{2} \leq \kappa \leq \delta\sqrt{2} < 0.9204. \qquad (13)$$

The best current estimate is $\kappa \approx 0.7124$.

A certain self-avoiding SPACE-FILLING CURVE is an optimal TOUR through a set of n points, where n can be arbitrarily large. It has length

$$\lambda \equiv \lim_{m\to\infty} \frac{L_m}{\sqrt{n_m}} = \frac{4(1+2\sqrt{2})\sqrt{51}}{153} = 0.7147827\ldots, \qquad (14)$$

where L_m is the length of the curve at the mth iteration and n_m is the point-set size (Moscato and Norman).

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Traveling Salesman Problem

A problem in GRAPH THEORY requiring the most efficient (i.e., least total distance) TOUR (i.e., closed path) a salesman can take through each of n cities. No general method of solution is known, and the problem is NP-HARD.

see also TRAVELING SALESMAN CONSTANTS

References
Platzman, L. K. and Bartholdi, J. J. "Spacefilling Curves and the Planar Travelling Salesman Problem." *J. Assoc. Comput. Mach.* **46**, 719–737, 1989.

Trawler Problem

A fast boat is overtaking a slower one when fog suddenly sets in. At this point, the boat being pursued changes course, but not speed. How should the pursuing vessel proceed in order to be sure of catching the other boat?

The amazing answer is that the pursuing boat should continue to the point where the slow boat would be if it had set its course directly for the pursuing boat when the fog set in. If the boat is not there, it should proceed in a SPIRAL whose origin is the point where the slow boat was when the fog set in. The SPIRAL can be constructed in such a way that the two boats will intersect before a complete turn is made.

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Trebly Magic Square

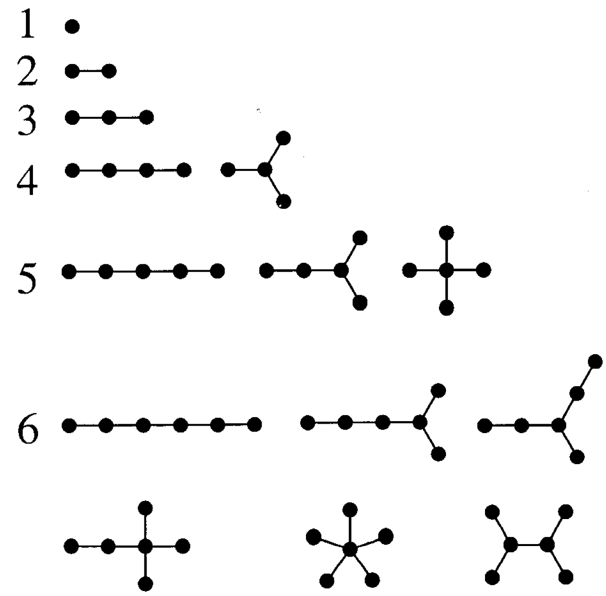
see TRIMAGIC SQUARE

Tredecillion

In the American system, 10^{42}.

see also LARGE NUMBER

Tree



A tree is a mathematical structure which can be viewed as either a GRAPH or as a DATA STRUCTURE. The two views are equivalent, since a tree DATA STRUCTURE contains not only a set of elements, but also connections between elements, giving a tree graph.

A tree graph is a set of straight line segments connected at their ends containing no closed loops (cycles). A tree with n nodes has $n-1$ EDGES. The points of connection are known as FORKS and the segments as BRANCHES. Final segments and the nodes at their ends are called LEAVES. A tree with two BRANCHES at each FORK and with one or two LEAVES at the end of each branch is called a BINARY TREE.

When a special node is designated to turn a tree into a ROOTED TREE, it is called the ROOT (or sometimes "EVE.") In such a tree, each of the nodes which is one EDGE further away from a given EDGE is called a CHILD, and nodes connected to the same node are then called SIBLINGS.

Note that two BRANCHES placed end-to-end are equivalent to a single BRANCH which means, for example, that there is only *one* tree of order 3. The number $t(n)$ of nonisomorphic trees of order $n = 1, 2, \ldots$ (where trees of orders 1, 2, ..., 6 are illustrated above), are 1, 1, 1, 2, 3, 6, 11, 23, 47, 106, 235, ... (Sloane's A000055).

Otter showed that

$$\lim_{n\to\infty} \frac{t(n)n^{5/2}}{\alpha^n} = \beta, \tag{1}$$

(Otter 1948, Harary and Palmer 1973, Knuth 1969), where the constants α and β are sometimes called OTTER'S TREE ENUMERATION CONSTANTS. Write the GENERATING FUNCTION for ROOTED TREES as

$$f(z) = \sum_{i=0}^{\infty} f_i z^i, \tag{2}$$

where the COEFFICIENTS are

$$f_{i+1} = \frac{1}{i} \sum_{j=1}^{i} \left(\sum_{d|j} d f_d \right) f_{i-j+1}, \tag{3}$$

with $f_0 = 0$ and $f_1 = 1$. Then

$$\alpha = 2.955765\ldots \tag{4}$$

is the unique POSITIVE ROOT of

$$f\left(\frac{1}{x}\right) = 1, \tag{5}$$

and

$$\beta = \frac{1}{\sqrt{2\pi}} \left[1 + \sum_{k=2}^{\infty} f'\left(\frac{1}{\alpha_k}\right) \frac{1}{\alpha_k} \right]^{3/2} = 0.5349485\ldots. \tag{6}$$

see also B-TREE, BINARY TREE, CATERPILLAR GRAPH, CAYLEY TREE, CHILD, DIJKSTRA TREE, EVE, FOREST, KRUSKAL'S ALGORITHM, KRUSKAL'S TREE THEOREM, LEAF (TREE), ORCHARD-PLANTING PROBLEM, ORDERED TREE, PATH GRAPH, PLANTED PLANAR TREE, PÓLYA ENUMERATION THEOREM, QUADTREE, RED-BLACK TREE, ROOT (TREE), ROOTED TREE, SIBLING, STAR GRAPH, STERN-BROCOT TREE, WEAKLY BINARY TREE, WEIGHTED TREE

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Tree-Planting Problem

see ORCHARD-PLANTING PROBLEM

Tree Searching

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

In database structures, two quantities are generally of interest: the average number of comparisons required to

1. Find an existing random record, and
2. Insert a new random record into a data structure.

Some constants which arise in the theory of digital tree searching are

$$\alpha \equiv \sum_{k=1}^{\infty} \frac{1}{2^k - 1} = 1.6066951524\ldots \tag{1}$$

$$\beta \equiv \sum_{n=1}^{\infty} \frac{1}{(2^n - 1)^2} = 1.1373387363\ldots. \tag{2}$$

Erdős (1948) proved that α is IRRATIONAL. The expected number of comparisons for a successful search is

$$E = \frac{\ln n}{\ln 2} + \frac{\gamma - 1}{\ln 2} - \alpha + \tfrac{3}{2} + \delta(n) + \mathcal{O}(n^{-1/2}) \tag{3}$$

$$\sim \lg n - 0.716644\ldots + \delta(n), \tag{4}$$

and for an unsuccessful search is

$$E = \frac{\ln n}{\ln 2} + \frac{\gamma}{\ln 2} - \alpha + \tfrac{1}{2} + \delta(n) + \mathcal{O}(n^{-1/2}) \tag{5}$$

$$\sim \lg n - 0.273948\ldots + \delta(n). \tag{6}$$

Here $\delta(n)$, $\epsilon(s)$, and $\rho(n)$ are small-amplitude periodic functions, and LG is the base 2 LOGARITHM. The VARIANCE for searching is

$$V \sim \frac{1}{12} + \frac{\pi^2 + 6}{6(\ln 2)^2} - \alpha - \beta + \epsilon(s) \sim 2.844383\ldots + \epsilon(s) \tag{7}$$

and for inserting is

$$V \sim \frac{1}{12} + \frac{\pi^2}{6(\ln 2)^2} - \alpha - \beta + \epsilon(s) \sim 0.763014\ldots + \epsilon(s). \tag{8}$$

The expected number of pairs of twin vacancies in a digital search tree is

$$\langle A_n \rangle = \left[\theta + 1 - \frac{1}{Q}\left(\frac{1}{\ln 2} + \alpha^2 - \alpha\right) + \rho(n)\right] n + \mathcal{O}(\sqrt{n}), \qquad (9)$$

where

$$Q \equiv \prod_{k=1}^{\infty}\left(1 - \frac{1}{2^k}\right) = 0.2887880950\ldots \qquad (10)$$

$$= \frac{1}{3} - \frac{1}{3\cdot 7} + \frac{1}{3\cdot 5\cdot 15} - \frac{1}{3\cdot 5\cdot 15\cdot 21} + \ldots \qquad (11)$$

$$= \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n(2^n-1)}\right] \qquad (12)$$

$$= \sqrt{\frac{2\pi}{\ln 2}}\exp\left(\frac{\ln 2}{24} - \frac{\pi^2}{6\ln 2}\right) \times \prod_{n=1}^{\infty}\left[1 - \exp\left(-\frac{4\pi^2 n}{\ln 2}\right)\right] \qquad (13)$$

and

$$\theta = \sum_{k=1}^{\infty}\frac{k2^{k+1}}{1\cdot 3\cdot 7\cdot 16\cdots(2^k-1)}\sum_{j=1}^{k}\frac{1}{2^j-1} = 7.7431319855\ldots. \qquad (14)$$

(Flajolet and Sedgewick 1986). The linear COEFFICIENT of $\langle A_n \rangle$ fluctuates around

$$c = \theta + 1 - \frac{1}{Q}\left(\frac{1}{\ln 2} + \alpha^2 - \alpha\right) = 0.3720486812\ldots, \qquad (15)$$

which can also be written

$$c = \frac{1}{\ln 2}\int_0^{\infty}\frac{x}{1+x} \times \frac{dx}{(1+x)(1+\frac{1}{2}x)(1+\frac{1}{4}x)(1+\frac{1}{8}x)\cdots}. \qquad (16)$$

(Flajolet and Richmond 1992).

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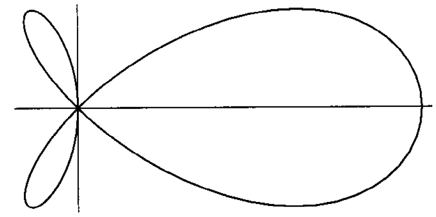
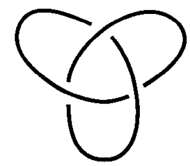
Trefoil Curve



The plane curve given by the equation

$$x^4 + x^2y^2 + y^4 = x(x^2 - y^2).$$

Trefoil Knot



The knot 03_{001}, also called the THREEFOIL KNOT, which is the unique PRIME KNOT of three crossings. It has BRAID WORD ${\sigma_1}^3$. The trefoil and its MIRROR IMAGE are not equivalent. The trefoil has ALEXANDER POLYNOMIAL $-x^2 + x - 1$ and is a (3, 2)-TORUS KNOT. The BRACKET POLYNOMIAL can be computed as follows.

$$\begin{aligned}\langle L \rangle &= A^3d^{2-1} + A^2Bd^{1-1} + A^2Bd^{1-1} + AB^2d^{2-1} \\ &\quad + A^2Bd^{1-1} + AB^2d^{2-1} + AB^2d^{2-1} + B^3d^{3-1} \\ &= A^3d^1 + 3A^2Bd^0 + 3AB^2d^1 + B^3d^2.\end{aligned}$$

Plugging in

$$B = A^{-1}$$
$$d = -A^2 - A^{-2}$$

gives

$$\langle L \rangle = A^{-7} - A^{-3} - A^5.$$

The normalized one-variable KAUFFMAN POLYNOMIAL X is then given by

$$\begin{aligned}X_L &= (-A^3)^{-w(L)}\langle L \rangle = (-A^3)^{-3}(A^{-7} - A^{-3} - A^5) \\ &= A^{-4} + A^{-12} - A^{-16},\end{aligned}$$

where the WRITHE $w(L) = 3$. The JONES POLYNOMIAL is therefore

$$V(t) = L(A = t^{-1/4}) = t + t^3 - t^4 = t(1 + t^2 - t^3).$$

Since $V(t^{-1}) \neq V(t)$, we have shown that the mirror images are not equivalent.

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Trench Diggers' Constant
see BEAM DETECTOR

Triabolo
A 3-POLYABOLO.

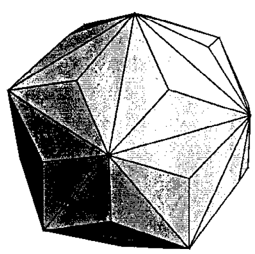
Triacontagon
A 30-sided POLYGON.

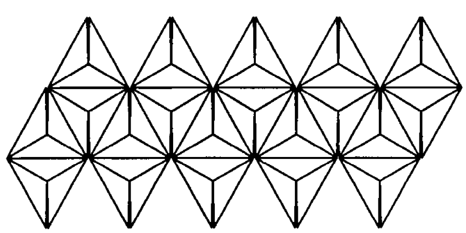
Triacontahedron
A 30-sided POLYHEDRON such as the RHOMBIC TRIACONTAHEDRON.

Triad
A SET with three elements.

see also HEXAD, MONAD, QUARTET, QUINTET, TETRAD

Triakis Icosahedron





The DUAL POLYHEDRON of the TRUNCATED DODECAHEDRON ARCHIMEDEAN SOLID. The triakis icosahedron is also ICOSAHEDRON STELLATION #2.

References

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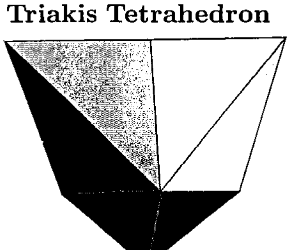
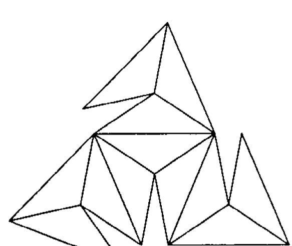
Triakis Octahedron
see GREAT TRIAKIS OCTAHEDRON, SMALL TRIAKIS OCTAHEDRON

Triakis Tetrahedron





The DUAL POLYHEDRON of the TRUNCATED TETRAHEDRON ARCHIMEDEAN SOLID.

Trial
In statistics, a trial is a single measurable random event, such as the flipping of a COIN, the generation of a RANDOM NUMBER, the dropping of a ball down the apex of a triangular lattice and having it fall into a single bin at the bottom, etc.

see also BERNOULLI TRIAL, LEXIS TRIALS, POISSON TRIALS

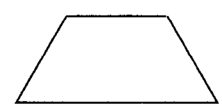
Trial Division
A brute-force method of finding a DIVISOR of an INTEGER n by simply plugging in one or a set of INTEGERS and seeing if they DIVIDE n. Repeated application of trial division to obtain the complete PRIME FACTORIZATION of a number is called DIRECT SEARCH FACTORIZATION. An individual integer being tested is called a TRIAL DIVISOR.

see also DIRECT SEARCH FACTORIZATION, DIVISION, PRIME FACTORIZATION

Trial Divisor
An INTEGER n which is tested to see if it divides a given number.

see also TRIAL DIVISION

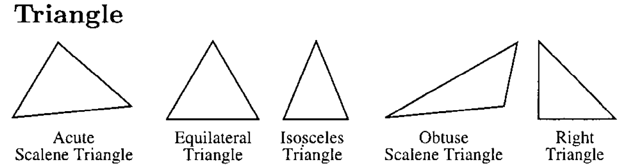
Triamond



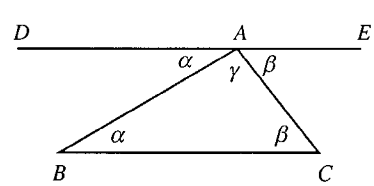
The unique 3-POLYIAMOND, illustrated above.

see also POLYIAMOND, TRAPEZOID

Triangle



A triangle is a 3-sided POLYGON sometimes (but not very commonly) called the TRIGON. All triangles are convex. An ACUTE TRIANGLE is a triangle whose three angles are all ACUTE. A triangle with all sides equal is called EQUILATERAL. A triangle with two sides equal is called ISOSCELES. A triangle having an OBTUSE ANGLE is called an OBTUSE TRIANGLE. A triangle with a RIGHT ANGLE is called RIGHT. A triangle with all sides a different length is called SCALENE.



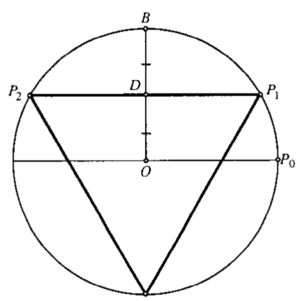
The sum of ANGLES in a triangle is 180°. This can be established as follows. Let $DAE \| BC$ (DAE be PARALLEL to BC) in the above diagram, then the angles α and β

satisfy $\alpha = \angle DAB = \angle ABC$ and $\beta = \angle EAC = \angle BCE$, as indicated. Adding γ, it follows that

$$\alpha + \beta + \gamma = 180°, \tag{1}$$

since the sum of angles for the line segment must equal two RIGHT ANGLES. Therefore, the sum of angles in the triangle is also 180°.

Let S stand for a triangle side and A for an angle, and let a set of Ss and As be concatenated such that adjacent letters correspond to adjacent sides and angles in a triangle. Triangles are uniquely determined by specifying three sides (SSS THEOREM), two angles and a side (AAS THEOREM), or two sides with an adjacent angle (SAS THEOREM). In each of these cases, the unknown three quantities (there are three sides and three angles total) can be uniquely determined. Other combinations of sides and angles do not uniquely determine a triangle: three angles specify a triangle only modulo a scale size (AAA THEOREM), and one angle and two sides not containing it may specify one, two, or no triangles (ASS THEOREM).



Equilateral Triangle

The RULER and COMPASS construction of the triangle can be accomplished as follows. In the above figure, take OP_0 as a RADIUS and draw $OB \perp OP_0$. Then bisect OB and construct $P_2P_1 \| OP_0$. Extending BO to locate P_3 then gives the EQUILATERAL TRIANGLE $\Delta P_1P_2P_3$.

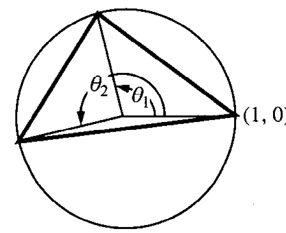
In Proposition IV.4 of the *Elements*, Euclid showed how to inscribe a CIRCLE (the INCIRCLE) in a given triangle by locating the CENTER as the point of intersection of ANGLE BISECTORS. In Proposition IV.5, he showed how to circumscribe a CIRCLE (the CIRCUMCIRCLE) about a given triangle by locating the CENTER as the point of intersection of the perpendicular bisectors.

If the coordinates of the triangle VERTICES are given by (x_i, y_i) where $i = 1, 2, 3$, then the AREA Δ is given by the DETERMINANT

$$\Delta = \frac{1}{2!}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}. \tag{2}$$

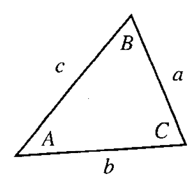
If the coordinates of the triangle VERTICES are given in 3-D by (x_i, y_i, z_i) where $i = 1, 2, 3$, then

$$\Delta = \frac{1}{2}\sqrt{\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2}. \tag{3}$$



In the above figure, let the CIRCUMCIRCLE passing through a triangle's VERTICES have RADIUS r, and denote the CENTRAL ANGLES from the first point to the second θ_1, and to the third point by θ_2. Then the AREA of the triangle is given by

$$\Delta = 2r^2 \left|\sin(\tfrac{1}{2}\theta_1)\sin(\tfrac{1}{2}\theta_2)\sin[\tfrac{1}{2}(\theta_1 - \theta_2)]\right|. \tag{4}$$



If a triangle has sides a, b, c, call the angles opposite these sides A, B, and C, respectively. Also define the SEMIPERIMETER s as HALF the PERIMETER:

$$s \equiv \tfrac{1}{2}p = \tfrac{1}{2}(a + b + c). \tag{5}$$

The AREA of a triangle is then given by HERON'S FORMULA

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}, \tag{6}$$

as well by the FORMULAS

$$\Delta = \tfrac{1}{4}\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)} \tag{7}$$

$$= \tfrac{1}{4}\sqrt{2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)} \tag{8}$$

$$= \tfrac{1}{4}\sqrt{[(a+b)^2 - c^2][c^2 - (a-b)^2]} \tag{9}$$

$$= \tfrac{1}{4}\sqrt{p(p-2a)(p-2b)(p-2c)}, \tag{10}$$

$$= 2R^2 \sin A \sin B \sin C \tag{11}$$

$$= \frac{abc}{4R} = rs \tag{12}$$

$$= \tfrac{1}{2}ah_a \tag{13}$$

$$= \tfrac{1}{2}bc\sin A. \tag{14}$$

In the above formulas, h_i is the ALTITUDE on side i, R is the CIRCUMRADIUS, and r is the INRADIUS (Johnson 1929, p. 11). Expressing the side lengths a, b, and c in terms of the radii a', b', and c' of the mutually tangent circles centered on the TRIANGLE vertices (which define the SODDY CIRCLES),

$$a = b' + c' \tag{15}$$

$$b = a' + c' \tag{16}$$

$$c = a' + b', \tag{17}$$

gives the particularly pretty form

$$\Delta = \sqrt{a'b'c'(a' + b' + c')}. \tag{18}$$

For additional FORMULAS, see Beyer (1987) and Baker (1884), who gives 110 FORMULAS for the AREA of a triangle.

The ANGLES of a triangle satisfy

$$\cot A = \frac{b^2 + c^2 - a^2}{4\Delta} \tag{19}$$

where Δ is the AREA (Johnson 1929, p. 11, with missing squared symbol added). This gives the pretty identity

$$\cot A + \cot B + \cot C = \frac{a^2 + b^2 + c^2}{4\Delta}. \tag{20}$$

Let a triangle have ANGLES A, B, and C. Then

$$\sin A \sin B \sin C \leq kABC, \tag{21}$$

where

$$k = \left(\frac{3\sqrt{3}}{2\pi}\right)^3 \tag{22}$$

(Abi-Khuzam 1974, Le Lionnais 1983). This can be used to prove that

$$8\omega^3 < ABC, \tag{23}$$

where ω is the BROCARD ANGLE.

TRIGONOMETRIC FUNCTIONS of half angles can be expressed in terms of the triangle sides:

$$\cos(\tfrac{1}{2}A) = \sqrt{\frac{s(s-a)}{bc}} \tag{24}$$

$$\sin(\tfrac{1}{2}A) = \sqrt{\frac{(s-b)(s-c)}{bc}} \tag{25}$$

$$\tan(\tfrac{1}{2}A) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \tag{26}$$

where s is the SEMIPERIMETER.

The number of different triangles which have INTEGRAL sides and PERIMETER n is

$$\begin{aligned} T(n) &= P_3(n) - \sum_{1 \leq j \leq \lfloor n/2 \rfloor} P_2(j) \\ &= \left[\frac{n^2}{12}\right] - \left\lfloor\frac{n}{4}\right\rfloor \left\lfloor\frac{n+2}{4}\right\rfloor \\ &= \begin{cases} \left[\frac{n^2}{48}\right] & \text{for } n \text{ even} \\ \left[\frac{(n+3)^2}{48}\right] & \text{for } n \text{ odd,} \end{cases} \end{aligned} \tag{27}$$

where P_2 and P_3 are PARTITION FUNCTIONS P, $[x]$ is the NINT function, and $\lfloor x \rfloor$ is the FLOOR FUNCTION (Jordan *et al.* 1979, Andrews 1979, Honsberger 1985). The values of $T(n)$ for $n = 1, 2, \ldots$ are 0, 0, 1, 0, 1, 1, 2, 1, 3, 2, 4, 3, 5, 4, 7, 5, 8, 7, 10, 8, 12, 10, 14, 12, 16, ... (Sloane's A005044), which is also ALCUIN'S SEQUENCE padded with two initial 0s. $T(n)$ also satisfies

$$T(2n) = T(2n-3) = P_3(n). \tag{28}$$

It is not known if a triangle with INTEGER sides, MEDIANS, and AREA exists (although there are incorrect PROOFS of the impossibility in the literature). However, R. L. Rathbun, A. Kemnitz, and R. H. Buchholz have shown that there are infinitely many triangles with RATIONAL sides (HERONIAN TRIANGLES) with *two* RATIONAL MEDIANS (Guy 1994).

In the following paragraph, assume the specified sides and angles are adjacent to each other. Specifying three ANGLES does not uniquely define a triangle, but any two triangles with the same ANGLES are similar (the AAA THEOREM). Specifying two ANGLES A and B and a side a uniquely determines a triangle with AREA

$$\Delta = \frac{a^2 \sin B \sin C}{2 \sin A} = \frac{a^2 \sin B \sin(\pi - A - B)}{2 \sin A} \tag{29}$$

(the AAS THEOREM). Specifying an ANGLE A, a side c, and an ANGLE B uniquely specifies a triangle with AREA

$$\Delta = \frac{c^2}{2(\cot A + \cot B)} \tag{30}$$

(the ASA THEOREM). Given a triangle with two sides, a the smaller and c the larger, and one known ANGLE A, ACUTE and opposite a, if $\sin A < a/c$, there are two possible triangles. If $\sin A = a/c$, there is one possible triangle. If $\sin A > a/c$, there are no possible triangles. This is the ASS THEOREM. Let a be the base length and h be the height. Then

$$\Delta = \tfrac{1}{2}ah = \tfrac{1}{2}ac \sin B \tag{31}$$

(the SAS THEOREM). Finally, if all three sides are specified, a unique triangle is determined with AREA given by HERON'S FORMULA or by

$$\Delta = \frac{abc}{4R}, \tag{32}$$

where R is the CIRCUMRADIUS. This is the SSS THEOREM.

There are four CIRCLES which are tangent to the sides of a triangle, one internal and the rest external. Their centers are the points of intersection of the ANGLE BISECTORS of the triangle.

Any triangle can be positioned such that its shadow under an orthogonal projection is EQUILATERAL.

see also AAA THEOREM, AAS THEOREM, ACUTE TRIANGLE, ALCUIN'S SEQUENCE, ALTITUDE, ANGLE BISECTOR, ANTICEVIAN TRIANGLE, ANTICOMPLEMENTARY TRIANGLE, ANTIPEDAL TRIANGLE, ASS THEOREM, BELL TRIANGLE, BRIANCHON POINT, BROCARD ANGLE, BROCARD CIRCLE, BROCARD MIDPOINT, BROCARD POINTS, BUTTERFLY THEOREM, CENTROID (TRIANGLE), CEVA'S THEOREM, CEVIAN, CEVIAN TRIANGLE, CHASLES'S THEOREM, CIRCUMCENTER, CIRCUMCIRCLE, CIRCUMRADIUS, CONTACT TRIANGLE, CROSSED LADDERS PROBLEM, CRUCIAL POINT, D-TRIANGLE, DE LONGCHAMPS POINT, DESARGUES' THEOREM, DISSECTION, ELKIES POINT, EQUAL DETOUR POINT, EQUILATERAL TRIANGLE, EULER LINE, EULER'S TRIANGLE, EULER TRIANGLE FORMULA, EXCENTER, EXCENTRAL TRIANGLE, EXCIRCLE, EXETER POINT, EXMEDIAN, EXMEDIAN POINT, EXRADIUS, EXTERIOR ANGLE THEOREM, FAGNANO'S PROBLEM, FAR-OUT POINT, FERMAT POINT, FERMAT'S PROBLEM, FEUERBACH POINT, FEUERBACH'S THEOREM, FUHRMANN TRIANGLE, GERGONNE POINT, GREBE POINT, GRIFFITHS POINTS, GRIFFITHS' THEOREM, HARMONIC CONJUGATE POINTS, HEILBRONN TRIANGLE PROBLEM, HERON'S FORMULA, HERONIAN TRIANGLE, HOFSTADTER TRIANGLE, HOMOTHETIC TRIANGLES, INCENTER, INCIRCLE, INRADIUS, ISODYNAMIC POINTS, ISOGONAL CONJUGATE, ISOGONIC CENTERS, ISOPERIMETRIC POINT, ISOSCELES TRIANGLE, KABON TRIANGLES, KANIZSA TRIANGLE, KIEPERT'S HYPERBOLA, KIEPERT'S PARABOLA, LAW OF COSINES, LAW OF SINES, LAW OF TANGENTS, LEIBNIZ HARMONIC TRIANGLE, LEMOINE CIRCLE, LEMOINE POINT, LINE AT INFINITY, MALFATTI POINTS, MEDIAL TRIANGLE, MEDIAN (TRIANGLE), MEDIAN TRIANGLE, MENELAUS' THEOREM, MID-ARC POINTS, MITTENPUNKT, MOLLWEIDE'S FORMULAS, MORLEY CENTERS, MORLEY'S THEOREM, NAGEL POINT, NAPOLEON'S THEOREM, NAPOLEON TRIANGLES, NEWTON'S FORMULAS, NINE-POINT CIRCLE, NUMBER TRIANGLE, OBTUSE TRIANGLE, ORTHIC TRIANGLE, ORTHOCENTER, ORTHOLOGIC, PARALOGIC TRIANGLES, PASCAL'S TRIANGLE, PASCH'S AXIOM, PEDAL TRIANGLE, PERPENDICULAR BISECTOR, PERSPECTIVE TRIANGLES, PETERSEN-SHOUTE THEOREM, PIVOT THEOREM, POWER POINT, POWER (TRIANGLE), PRIME TRIANGLE, PURSER'S THEOREM, QUADRILATERAL, RATIONAL TRIANGLE, ROUTH'S THEOREM, SAS THEOREM, SCALENE TRIANGLE, SCHIFFLER POINT, SCHWARZ TRIANGLE, SCHWARZ'S TRIANGLE PROBLEM, SEIDEL-ENTRINGER-ARNOLD TRIANGLE, SEYDEWITZ'S THEOREM, SIMSON LINE, SPIEKER CENTER, SSS THEOREM, STEINER-LEHMUS THEOREM, STEINER POINTS, STEWART'S THEOREM, SYMMEDIAN POINT, TANGENTIAL TRIANGLE, TANGENTIAL TRIANGLE CIRCUMCENTER, TARRY POINT, THOMSEN'S FIGURE, TORRICELLI POINT, TRIANGLE TILING, TRIANGLE TRANSFORMATION PRINCIPLE, YFF POINTS, YFF TRIANGLES

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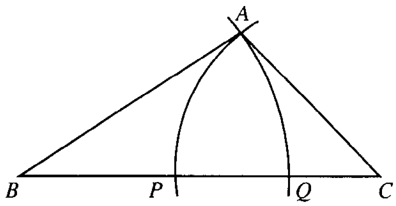
Triangle Arcs



In the above figure, the curves are arcs of a CIRCLE and

$$a = BC \quad (1)$$
$$b = CA = CP \quad (2)$$
$$c = BA = BQ. \quad (3)$$

Then

$$PQ^2 = 2BP \cdot QC. \quad (4)$$

The figure also yields the algebraic identity

$$(b + c - \sqrt{b^2 + c^2})^2 = 2(\sqrt{b^2 + c^2} - b)(\sqrt{b^2 + c^2} - c). \quad (5)$$

see also ARC

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Triangle Center

A triangle center is a point whose TRILINEAR COORDINATES are defined in terms of the side lengths and angles of a TRIANGLE. The function giving the coordinates $\alpha : \beta : \gamma$ is called the TRIANGLE CENTER FUNCTION. The four ancient centers are the CENTROID, INCENTER, CIRCUMCENTER, and ORTHOCENTER. For a listing of these and other triangle centers, see Kimberling (1994).

A triangle center is said to be REGULAR IFF there is a TRIANGLE CENTER FUNCTION which is a POLYNOMIAL in Δ, a, b, and c (where Δ is the AREA of the TRIANGLE) such that the TRILINEAR COORDINATES of the center are

$$f(a,b,c) : f(b,c,a) : f(c,a,b).$$

A triangle center is said to be a MAJOR TRIANGLE CENTER if the TRIANGLE CENTER FUNCTION α is a function of ANGLE A alone, and therefore β and γ of B and C alone, respectively.

see also MAJOR TRIANGLE CENTER, REGULAR TRIANGLE CENTER, TRIANGLE, TRIANGLE CENTER FUNCTION, TRILINEAR COORDINATES, TRILINEAR POLAR

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Triangle Center Function

A HOMOGENEOUS FUNCTION $f(a,b,c)$, i.e., a function f such that

$$f(ta,tb,tc) = t^n f(a,b,c),$$

which gives the TRILINEAR COORDINATES of a TRIANGLE CENTER as

$$\alpha : \beta : \gamma = f(a,b,c) : f(b,c,a) : f(c,a,b).$$

The variables may correspond to angles (A, B, C) or side lengths (a, b, c), since these can be interconverted using the LAW OF COSINES.

see also MAJOR TRIANGLE CENTER, REGULAR TRIANGLE CENTER, TRIANGLE CENTER, TRILINEAR COORDINATES

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Triangle Coefficient

A function of three variables written $\Delta(abc) \equiv \Delta(a,b,c)$ and defined by

$$\Delta(abc) \equiv \sqrt{\frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!}}.$$

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Triangle Condition

The condition that j takes on the values

$$j = j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2|,$$

denoted $\Delta(j_1 j_2 j)$.

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Triangle Counting

Given rods of length 1, 2, ..., n, how many distinct triangles $T(n)$ can be made? Lengths for which

$$l_i = l_j + l_k$$

obviously do not give triangles, but all other combinations of three rods do. The answer is

$$T(n) = \begin{cases} \frac{1}{24}n(n-2)(2n-5) & \text{for } n \text{ even} \\ \frac{1}{24}(n-1)(n-3)(2n-1) & \text{for } n \text{ odd.} \end{cases}$$

The values for $n = 1, 2, \ldots$ are 0, 0, 0, 1, 3, 7, 13, 22, 34, 50, ... (Sloane's A002623). Somewhat surprisingly, this sequence is also given by the GENERATING FUNCTION

$$f(x) = \frac{x^4}{(1-x)^3(1-x^2)} = x^4 + 3x^5 + 7x^6 + 13x^7 + \ldots.$$

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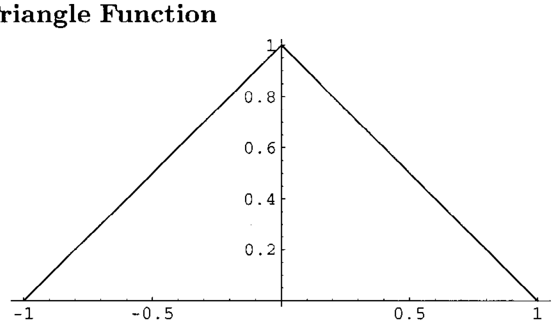
Triangle of Figurate Numbers

see FIGURATE NUMBER TRIANGLE

Triangle Function



$$\Lambda(x) \equiv \begin{cases} 0 & |x| > 1 \\ 1 - |x| & |x| < 1 \end{cases} \quad (1)$$

$$= \Pi(x) * \Pi(x) \quad (2)$$

$$= \Pi(x) * H(x + \tfrac{1}{2}) - \Pi(x) * H(x - \tfrac{1}{2}), \quad (3)$$

where Π is the RECTANGLE FUNCTION and H is the HEAVISIDE STEP FUNCTION. An obvious generalization used as an APODIZATION FUNCTION goes by the name of the BARTLETT FUNCTION.

There is also a three-argument function known as the triangle function:

$$\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \quad (4)$$

It follows that

$$\lambda(a^2, b^2, c^2) = (a+b+c)(a+b-c)(a-b+c)(a-b-c). \quad (5)$$

see also ABSOLUTE VALUE, BARTLETT FUNCTION, HEAVISIDE STEP FUNCTION, RAMP FUNCTION, SGN, TRIANGLE COEFFICIENT

Triangle Inequality

Let $\mathbf{x}$ and $\mathbf{y}$ be vectors

$$|\mathbf{x}| - |\mathbf{y}| \leq |\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|. \quad (1)$$

Equivalently, for COMPLEX NUMBERS z_1 and z_2,

$$|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2|. \quad (2)$$

A generalization is

$$\left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{n} |a_k|. \quad (3)$$

see also p-ADIC NUMBER, STRONG TRIANGLE INEQUALITY

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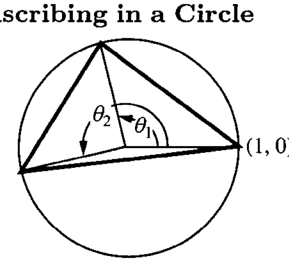
Triangle Inscribing in a Circle



Select three points at random on a unit CIRCLE. Find the distribution of possible areas. The first point can be assigned coordinates (1, 0) without loss of generality. Call the central angles from the first point to the second and third θ_1 and θ_2. The range of θ_1 can be restricted to $[0, \pi]$ because of symmetry, but θ_2 can range from $[0, 2\pi)$. Then

$$A(\theta_1, \theta_2) = 2\left|\sin(\tfrac{1}{2}\theta_1)\sin(\tfrac{1}{2}\theta_2)\sin[\tfrac{1}{2}(\theta_1 - \theta_2)]\right|, \quad (1)$$

so

$$\bar{A} = \frac{\int_0^\pi \int_0^{2\pi} A(\theta_1, \theta_2)\, d\theta_2\, d\theta_1}{C}, \quad (2)$$

where

$$C \equiv \int_0^\pi \int_0^{2\pi} d\theta_2\, d\theta_1 = 2\pi^2. \quad (3)$$

Therefore,

$$\bar{A} = \frac{2}{2\pi^2}\int_0^\pi \int_0^{2\pi} \left|\sin(\tfrac{1}{2}\theta_1)\sin(\tfrac{1}{2}\theta_2)\sin[\tfrac{1}{2}(\theta_1-\theta_2)]\right| d\theta_2\, d\theta_1$$
$$= \frac{1}{\pi^2}\int_0^\pi \sin(\tfrac{1}{2}\theta_1)\left[\int_0^{2\pi} \sin(\tfrac{1}{2}\theta_2)\left|\sin[\tfrac{1}{2}(\theta_2-\theta_1)]\right| d\theta_2\right] d\theta_1$$
$$= \frac{1}{\pi^2}\underset{\theta_2-\theta_1>0}{\int_0^\pi \int_0^{2\pi}} \sin(\tfrac{1}{2}\theta_1)\sin(\tfrac{1}{2}\theta_2)\sin[\tfrac{1}{2}(\theta_1-\theta_2)]\, d\theta_2\, d\theta_1$$
$$+ \frac{1}{\pi^2}\underset{\theta_2-\theta_1<0}{\int_0^\pi \int_0^{2\pi}} \sin(\tfrac{1}{2}\theta_1)\sin(\tfrac{1}{2}\theta_2)\sin[\tfrac{1}{2}(\theta_1-\theta_2)]\, d\theta_2\, d\theta_1$$
$$= \frac{1}{\pi^2}\int_0^\pi \sin(\tfrac{1}{2}\theta_1)\left[\int_{\theta_1}^{2\pi} \sin(\tfrac{1}{2}\theta_2)\sin[\tfrac{1}{2}(\theta_2-\theta_1)]\, d\theta_2\right] d\theta_1$$
$$+ \frac{1}{\pi^2}\int_0^\pi \sin(\tfrac{1}{2}\theta_1)\left[\int_0^{\theta_1} \sin(\tfrac{1}{2}\theta_2)\sin[\tfrac{1}{2}(\theta_2-\theta_1)]\, d\theta_2\right] d\theta_1. \tag{4}$$

But

$$\int (\tfrac{1}{2}\theta_2)\sin[\tfrac{1}{2}(\theta_2-\theta_1)]\, d\theta_2$$
$$= \int \sin(\tfrac{1}{2}\theta_2)\left[\sin(\tfrac{1}{2}\theta_2)\cos(\tfrac{1}{2}\theta_2) - \sin(\tfrac{1}{2}\theta_1)\cos(\tfrac{1}{2}\theta_2)\right] d\theta_2$$
$$= \cos(\tfrac{1}{2}\theta_1)\int \sin^2(\tfrac{1}{2}\theta_2)\, d\theta_2 - \sin(\tfrac{1}{2}\theta_1)\int \sin(\tfrac{1}{2}\theta_1)\cos(\tfrac{1}{2}\theta_2)\, d\theta_2$$
$$= \tfrac{1}{2}\cos(\tfrac{1}{2}\theta_1)\int (1-\cos\theta_2)\, d\theta_2 - \tfrac{1}{2}\sin(\tfrac{1}{2}\theta_2)\int \sin\theta_2\, d\theta_2$$
$$= \tfrac{1}{2}\cos(\tfrac{1}{2}\theta_1)(\theta_2 - \sin\theta_2) + \tfrac{1}{2}\sin(\tfrac{1}{2}\theta_1)\cos(\theta_2). \tag{5}$$

Write (4) as

$$\bar{A} = \frac{1}{\pi^2}\left[\int_0^\pi \sin(\tfrac{1}{2}\theta_1) I_1\, d\theta_1 + \int_0^\pi \sin(\tfrac{1}{2}\theta_1) I_2\, d\theta_1\right], \tag{6}$$

then

$$I_1 \equiv \int_{\theta_1}^{2\pi} \sin(\tfrac{1}{2}\theta_2)\sin[\tfrac{1}{2}(\theta_2-\theta_1)]\, d\theta_2, \tag{7}$$

and

$$I_2 \equiv \int_0^{\theta_1} \sin(\tfrac{1}{2}\theta_2)\sin[\tfrac{1}{2}(\theta_1-\theta_2)]\, d\theta_2. \tag{8}$$

From (6),

$$I_1 = \tfrac{1}{2}\cos(\tfrac{1}{2}\theta_2)[\theta_2 - \sin\theta_2]_{\theta_1}^{2\pi} + \tfrac{1}{2}\sin(\tfrac{1}{2}\theta_1)[\cos\theta_2]_{\theta_1}^{2\pi}$$
$$= \tfrac{1}{2}\cos(\tfrac{1}{2}\theta_1)(2\pi - \theta_1 + \sin\theta_1)$$
$$+ \tfrac{1}{2}\sin(\tfrac{1}{2}\theta_1)(1-\cos\theta_1)$$
$$= \pi\cos(\tfrac{1}{2}\theta_1) - \tfrac{1}{2}\theta_1\cos(\tfrac{1}{2}\theta_1) + \tfrac{1}{2}[\cos(\tfrac{1}{2}\theta_1)\sin\theta_1$$
$$- \cos\theta_1\sin(\tfrac{1}{2}\theta_1)] + \tfrac{1}{2}\sin(\tfrac{1}{2}\theta_1)$$
$$= \pi\cos(\tfrac{1}{2}\theta_1) - \tfrac{1}{2}\theta_1\cos(\tfrac{1}{2}\theta_1) + \tfrac{1}{2} + \tfrac{1}{2}\sin(\theta_1 - \tfrac{1}{2}\theta_1)$$
$$+ \tfrac{1}{2}\sin(\tfrac{1}{2}\theta_1)$$
$$= \pi\cos(\tfrac{1}{2}\theta_1) - \tfrac{1}{2}\theta_1\cos(\tfrac{1}{2}\theta_1) + \sin(\tfrac{1}{2}\theta_1), \tag{9}$$

so

$$\int_0^\pi I_1\sin(\tfrac{1}{2}\theta_1)\, d\theta_1 = \tfrac{5}{4}\pi. \tag{10}$$

Also,

$$I_2 = \tfrac{1}{2}\cos(\tfrac{1}{2}\theta_1)[\sin\theta_2 - \theta_2]_0^{\theta_1} - \tfrac{1}{2}\sin(\tfrac{1}{2}\theta_1)[\cos\theta_2]_0^{\theta_1}$$
$$= \tfrac{1}{2}\cos(\tfrac{1}{2}\theta_2)(\sin\theta_1 - \theta_1) - \tfrac{1}{2}\sin(\tfrac{1}{2}\theta_1)(\cos\theta_1 - 1)$$
$$= -\tfrac{1}{2}\theta_1\cos(\tfrac{1}{2}\theta_1)$$
$$+ \tfrac{1}{2}[\sin\theta_1\cos(\tfrac{1}{2}\theta_1) - \cos\theta_1\sin(\tfrac{1}{2}\theta_2)]$$
$$+ \tfrac{1}{2}\sin(\tfrac{1}{2}\theta_1)$$
$$= -\tfrac{1}{2}\theta_1\cos(\tfrac{1}{2}\theta_1) + \sin(\tfrac{1}{2}\theta_1), \tag{11}$$

so

$$\int_0^\pi I_2\sin(\tfrac{1}{2}\theta_1)\, d\theta_1 = \tfrac{1}{4}\pi. \tag{12}$$

Combining (10) and (12) gives

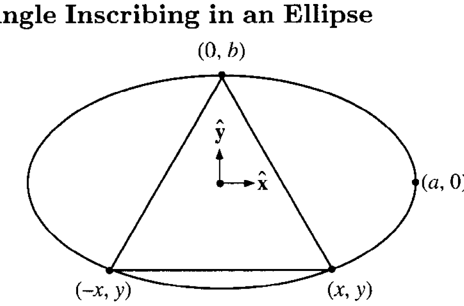
$$\bar{A} = \frac{1}{\pi^2}\left(\frac{5\pi}{4} + \frac{\pi}{4}\right) = \frac{3}{2\pi} \approx 0.4775. \tag{13}$$

The VARIANCE is

$$\sigma_A{}^2 = \frac{1}{2\pi^2}\int_0^\pi \int_0^{2\pi} [A(\theta_1,\theta_2) - \tfrac{3}{2\pi}]^2\, d\theta_2\, d\theta_1$$
$$= \frac{1}{2\pi^2}\int_0^\pi \int_0^{2\pi} \Big[2\left|\sin(\tfrac{1}{2}\theta_1)\sin(\tfrac{1}{2}\theta_2)\sin[\tfrac{1}{2}(\theta_1-\theta_2)]\right|$$
$$- \frac{3}{2\pi}\Big]^2\, d\theta_2\, d\theta_1$$
$$= \frac{1}{2\pi^2}\int_0^\pi \int_0^{2\pi} \Big\{4\sin^2(\tfrac{1}{2}\theta_1)\sin^2(\tfrac{1}{2}\theta_2)\sin^2[\tfrac{1}{2}(\theta_2-\theta_1)]$$
$$- \frac{6}{\pi}\left|\sin(\tfrac{1}{2}\theta_1)\sin(\tfrac{1}{2}\theta_2)\sin[\tfrac{1}{2}(\theta_1-\theta_2)]\right| + \frac{9}{4\pi^2}\Big\}\, d\theta_2\, d\theta_1$$
$$= \frac{1}{2\pi^2}\Big[\int_0^\pi \pi(2+\theta_1)\sin^2(\tfrac{1}{2}\theta_1)\, d\theta_1$$
$$- \frac{6}{\pi}\left(\frac{5\pi}{4} + \frac{\pi}{4}\right) + \frac{9}{4\pi^2}(2\pi^2)\Big]$$
$$= \frac{1}{2\pi^2}\left(\frac{3\pi^2}{4} - 9 + \frac{9}{2}\right) = \frac{1}{2\pi^2}\left(\frac{3\pi^2}{4} - \frac{9}{2}\right)$$
$$= \frac{3(\pi^2-6)}{8\pi^2} \approx 0.1470. \tag{14}$$

see also POINT-POINT DISTANCE—1-D, TETRAHEDRON INSCRIBING

Triangle Inscribing in an Ellipse



To inscribe an EQUILATERAL TRIANGLE in an ELLIPSE, place the top VERTEX at $(0, b)$, then solve to find the (x, y) coordinate of the other two VERTICES.

$$\sqrt{x^2 + (b-y)^2} = 2x \tag{1}$$

$$x^2 + (b-y)^2 = 4x^2 \tag{2}$$

$$3x^2 = (b-y)^2. \tag{3}$$

Now plugging in the equation of the ELLIPSE

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{4}$$

gives

$$3a^2\left(1 - \frac{y^2}{b^2}\right) = b^2 - 2by + y^2 \tag{5}$$

$$y^2\left(1 + 3\frac{a^2}{b^2}\right) - 2by + (b^2 - 3a^2) = 0 \tag{6}$$

$$\begin{aligned} y &= \frac{2b - \sqrt{4b^2 - 4(b^2 - 3a^2)\left(1 + 3\frac{a^2}{b^2}\right)}}{2\left(1 + 3\frac{a^2}{b^2}\right)} \\ &= \frac{1 - \sqrt{1 - \left(1 - 3\frac{a^2}{b^2}\right)\left(1 + 3\frac{a^2}{b^2}\right)}}{1 + 3\frac{a^2}{b^2}}\, b, \end{aligned} \tag{7}$$

and

$$x = \pm a\sqrt{1 - \frac{y^2}{b^2}}. \tag{8}$$

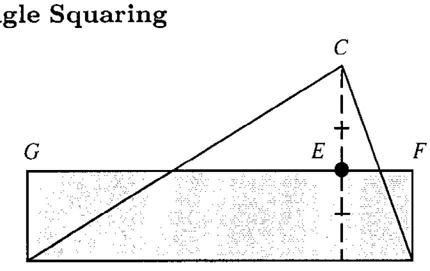
Triangle Postulate

The sum of the ANGLES of a TRIANGLE is two RIGHT ANGLES. This POSTULATE is equivalent to the PARALLEL AXIOM.

References

Dunham, W. "Hippocrates' Quadrature of the Lune." Ch. 1 in *Journey Through Genius: The Great Theorems of Mathematics.* New York: Wiley, p. 54, 1990.

Triangle Squaring



Let CD be the ALTITUDE of a TRIANGLE ΔABC and let E be its MIDPOINT. Then

$$\text{area}(\Delta ABC) = \tfrac{1}{2}AB \cdot CD = AB \cdot DE,$$

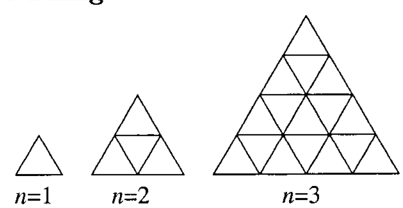
and $\square ABFG$ can be squared by RECTANGLE SQUARING. The general POLYGON can be treated by drawing diagonals, squaring the constituent triangles, and then combining the squares together using the PYTHAGOREAN THEOREM.

see also PYTHAGOREAN THEOREM, RECTANGLE SQUARING

References

Dunham, W. "Hippocrates' Quadrature of the Lune." Ch. 1 in *Journey Through Genius: The Great Theorems of Mathematics.* New York: Wiley, pp. 14–15, 1990.

Triangle Tiling



The total number of triangle (including inverted ones) in the above figures is given by

$$N(n) = \begin{cases} \frac{1}{8}n(n+2)(2n+1) & \text{for } n \text{ even} \\ \frac{1}{8}[n(n+2)(2n+1) - 1] & \text{for } n \text{ odd.} \end{cases}$$

The first few values are 1, 5, 13, 27, 48, 78, 118, 170, 235, 315, 411, 525, 658, 812, 988, 1188, 1413, 1665, ... (Sloane's A002717).

References

Conway, J. H. and Guy, R. K. "How Many Triangles." In *The Book of Numbers.* New York: Springer-Verlag, pp. 83–84, 1996.

Sloane, N. J. A. Sequence A002717/M3827 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Triangle Transformation Principle

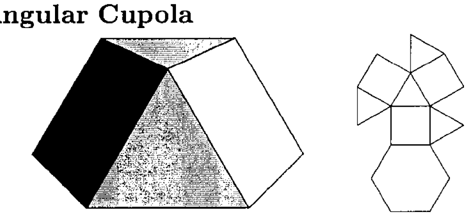
The triangle transformation principle gives rules for transforming equations involving an INCIRCLE to equations about EXCIRCLES.

see also EXCIRCLE, INCIRCLE

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 191–192, 1929.

Triangular Cupola



JOHNSON SOLID J_3. The bottom six VERTICES are

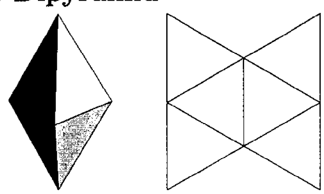
$$(\pm\tfrac{1}{2}\sqrt{3}, \pm\tfrac{1}{2}, 0), (0, \pm 1, 0),$$

and the top three VERTICES are

$$\left(\frac{1}{\sqrt{3}}, 0, \sqrt{\frac{2}{3}}\right), -\left(\frac{1}{2\sqrt{3}}, \pm\frac{1}{2}, \sqrt{\frac{2}{3}}\right).$$

see also JOHNSON SOLID

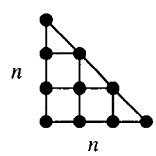
Triangular Dipyramid



The triangular (or TRIGONAL) dipyramid is one of the convex DELTAHEDRA, and JOHNSON SOLID J_{12}.

see also DELTAHEDRON, DIPYRAMID, JOHNSON SOLID, PENTAGONAL DIPYRAMID

Triangular Graph



The triangular graph with n nodes on a side is denoted $T(n)$. Tutte (1970) showed that the CHROMATIC POLYNOMIALS of planar triangular graphs possess a ROOT close to $\phi^2 = 2.618033\ldots$, where ϕ is the GOLDEN MEAN. More precisely, if n is the number of VERTICES of G, then

$$P_G(\phi^2) \leq \phi^{5-n}$$

(Le Lionnais 1983, p. 46). Every planar triangular graph possesses a VERTEX of degree 3, 4, or 5 (Le Lionnais 1983, pp. 49 and 53).

see also LATTICE GRAPH

References

Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, 1983.

Tutte, W. T. "On Chromatic Polynomials and the Golden Ratio." *J. Combin. Theory* **9**, 289–296, 1970.

Triangular Hebesphenorotunda

see JOHNSON SOLID

Triangular Matrix

An upper triangular MATRIX U is defined by

$$U_{ij} = \begin{cases} a_{ij} & \text{for } i \leq j \\ 0 & \text{for } i > j. \end{cases} \tag{1}$$

Written explicitly,

$$\mathsf{U} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}. \tag{2}$$

A lower triangular MATRIX L is defined by

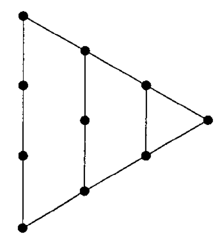
$$L_{ij} = \begin{cases} a_{ij} & \text{for } i \geq j \\ 0 & \text{for } i < j. \end{cases} \tag{3}$$

Written explicitly,

$$\mathsf{L} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}. \tag{4}$$

see also HESSENBERG MATRIX, HILBERT MATRIX, MATRIX, VANDERMONDE MATRIX

Triangular Number



A FIGURATE NUMBER of the form $T_n \equiv n(n+1)/2$ obtained by building up regular triangles out of dots. The first few triangle numbers are 1, 3, 6, 10, 15, 21, ... (Sloane's A000217). $T_4 = 10$ gives the number and arrangement of BOWLING pins, while $T_5 = 15$ gives the number and arrangement of balls in BILLIARDS. Triangular numbers satisfy the RECURRENCE RELATION

$$T_{n+1}{}^2 - T_n{}^2 = (n+1)^3, \tag{1}$$

as well as

$$3T_n + T_{n-1} = T_{2n} \tag{2}$$

$$3T_n + T_{n+1} = T_{2n+1} \tag{3}$$

$$1 + 3 + 5 + \ldots + (2n-1) = T_n + T_{n-1} \tag{4}$$

and

$$(2n+1)^2 = 8T + 1 = T_{n-1} + 6T_n + T_{n+1} \tag{5}$$

(Conway and Guy 1996). They have the simple GENERATING FUNCTION

$$f(x) = \frac{x}{(1-x)^3} = x + 3x^2 + 6x^3 + 10x^4 + 15x^5 + \ldots. \tag{6}$$

Every triangular number is also a HEXAGONAL NUMBER, since

$$\tfrac{1}{2}r(r+1) = \begin{cases} \left(\frac{r+1}{2}\right)\left[2\left(\frac{r+1}{2}\right) - 1\right] & \text{for } r \text{ odd} \\ \left(-\frac{r}{2}\right)\left[2\left(-\frac{r}{2}\right) - 1\right] & \text{for } r \text{ even.} \end{cases} \tag{7}$$

Also, every PENTAGONAL NUMBER is 1/3 of a triangular number. The sum of consecutive triangular numbers is a SQUARE NUMBER, since

$$\begin{aligned} T_r + T_{r-1} &= \tfrac{1}{2}r(r+1) + \tfrac{1}{2}(r-1)r \\ &= \tfrac{1}{2}r[(r+1)+(r-1)] = r^2. \end{aligned} \tag{8}$$

Interesting identities involving triangular numbers and SQUARE NUMBERS are

$$\sum_{k=1}^{2n-1} (-1)^{k+1} T_k = n^2 \tag{9}$$

$$T_n{}^2 = \sum_{k=1}^{n} k^3 = \tfrac{1}{4}n^2(n+1)^2 \tag{10}$$

$$\sum_{k=1,3,\ldots,q} k^3 = T_n \tag{11}$$

for q ODD and

$$n = \tfrac{1}{2}(q^2 + 2q - 1). \tag{12}$$

All EVEN PERFECT NUMBERS are triangular T_p with PRIME p. Furthermore, every EVEN PERFECT NUMBER $P > 6$ is of the form

$$P = 1 + 9T_n = T_{3n+1}, \tag{13}$$

where T_n is a triangular number with $n = 8j + 2$ (Eaton 1995, 1996). Therefore, the nested expression

$$9(9\cdots(9(9(9(9T_n+1)+1)+1)+1)\ldots+1)+1 \tag{14}$$

generates triangular numbers for any T_n. An INTEGER k is a triangular number IFF $8k+1$ is a SQUARE NUMBER > 1.

The numbers 1, 36, 1225, 41616, 1413721, 48024900, ... (Sloane's A001110) are SQUARE TRIANGULAR NUMBERS, i.e., numbers which are simultaneously triangular and SQUARE (Pietenpol 1962). Numbers which are simultaneously triangular and TETRAHEDRAL satisfy the BINOMIAL COEFFICIENT equation

$$\binom{n}{2} = \binom{m}{3}, \tag{15}$$

the only solutions of which are $(m, n) = (10, 16)$, (22, 56), and (36, 120) (Guy 1994, p. 147).

The smallest of two INTEGERS for which $n^3 - 13$ is four times a triangular number is 5 (Cesaro 1886; Le Lionnais 1983, p. 56). The only FIBONACCI NUMBERS which are triangular are 1, 3, 21, and 55 (Ming 1989), and the only PELL NUMBER which is triangular is 1 (McDaniel 1996). The BEAST NUMBER 666 is triangular, since

$$T_{6\cdot 6} = T_{36} = 666. \tag{16}$$

In fact, it is the largest REPDIGIT triangular number (Bellew and Weger 1975–76).

FERMAT'S POLYGONAL NUMBER THEOREM states that every POSITIVE INTEGER is a sum of *most* three TRIANGULAR NUMBERS, four SQUARE NUMBERS, five PENTAGONAL NUMBERS, and n n-POLYGONAL NUMBERS. Gauss proved the triangular case, and noted the event in his diary on July 10, 1796, with the notation

$$**\,E\Upsilon RHKA \qquad num = \Delta + \Delta + \Delta. \tag{17}$$

This case is equivalent to the statement that every number of the form $8m + 3$ is a sum of three ODD SQUARES (Duke 1997). Dirichlet derived the number of ways in which an INTEGER m can be expressed as the sum of three triangular numbers (Duke 1997). The result is particularly simple for a PRIME of the form $8m + 3$, in which case it is the number of squares mod $8m+3$ minus the number of nonsquares mod $8m + 3$ in the INTERVAL $4m + 1$ (Deligne 1973).

The only triangular numbers which are the PRODUCT of three consecutive INTEGERS are 6, 120, 210, 990, 185136, 258474216 (Guy 1994, p. 148).

see also FIGURATE NUMBER, PRONIC NUMBER, SQUARE TRIANGULAR NUMBER

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Triangular Orthobicupola

see JOHNSON SOLID

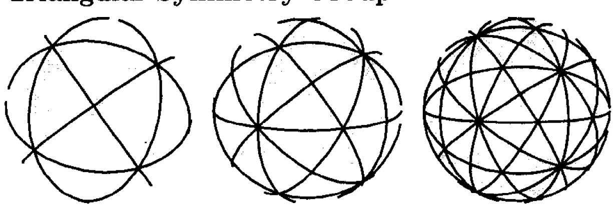
Triangular Pyramid

see TETRAHEDRON

Triangular Square Number

see SQUARE TRIANGULAR NUMBER

Triangular Symmetry Group



(2, 3, 3) (2, 3, 4) (2, 3, 5)

Given a TRIANGLE with angles $(\pi/p,\ \pi/q,\ \pi/r)$, the resulting symmetry GROUP is called a (p,q,r) triangle group (also known as a SPHERICAL TESSELLATION). In 3-D, such GROUPS must satisfy

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1,$$

and so the only solutions are $(2,2,n)$, $(2,3,3)$, $(2,3,4)$, and $(2,3,5)$ (Ball and Coxeter 1987). The group $(2,3,6)$ gives rise to the semiregular planar TESSELLATIONS of types 1, 2, 5, and 7. The group $(2,3,7)$ gives hyperbolic tessellations.

see also GEODESIC DOME

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Triangulation

Triangulation is the division of a surface into a set of TRIANGLES, usually with the restriction that each TRIANGLE side is entirely shared by two adjacent TRIANGLES. It was proved in 1930 that every surface has a triangulation, but it might require an infinite number of TRIANGLES. A surface with a finite number of triangles in its triangulation is called COMPACT. B. Chazelle showed that an arbitrary SIMPLE POLYGON can be triangulated in linear time.

see also COMPACT SURFACE, DELAUNAY TRIANGULATION, JAPANESE TRIANGULATION THEOREM, SIMPLE POLYGON

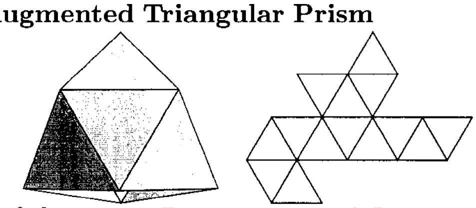
Triaugmented Dodecahedron

see JOHNSON SOLID

Triaugmented Hexagonal Prism

see JOHNSON SOLID

Triaugmented Triangular Prism



One of the convex DELTAHEDRA and JOHNSON SOLID J_{51}. The VERTICES are $(\pm1,\pm1,0)$, $(0,0,\sqrt{2})$, $(0,\pm1,-\sqrt{3})$, $(\pm(1+\sqrt{6})/2,0,-(\sqrt{2}+\sqrt{3})/2)$, where the x and z coordinates of the last are found by solving

$$x^2+1^2+(z+\sqrt{3})^2=2^2$$
$$(x-1)^2+1^2+z^2=2^2.$$

see also DELTAHEDRON, JOHNSON SOLID

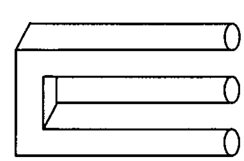
Triaugmented Truncated Dodecahedron

see JOHNSON SOLID

Triaxial Ellipsoid

see ELLIPSOID

Tribar



An IMPOSSIBLE FIGURE published by R. Penrose (1958). It also exists as a TRIBOX.

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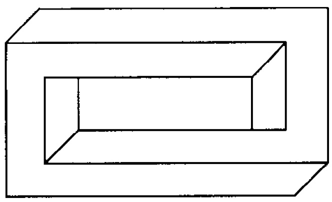
Penrose, R. "Impossible Objects: A Special Type of Visual Illusion." *Brit. J. Psychology* **49**, 31–33, 1958.

Tribox



An IMPOSSIBLE FIGURE.

see also IMPOSSIBLE FIGURE, TRIBAR

References

Jablan, S. "Are Impossible Figures Possible?" http://members.tripod.com/~modularity/kulpa.htm.

Tribonacci Number

The tribonacci numbers are a generalization of the FIBONACCI NUMBERS defined by $T_1 = 1$, $T_2 = 1$, $T_3 = 2$, and the RECURRENCE RELATION

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \tag{1}$$

for $n \geq 4$. The represent the $n = 3$ case of the FIBONACCI n-STEP NUMBERS. The first few terms are 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, ... (Sloane's A000073). The ratio of adjacent terms tends to 1.83929, which is the REAL ROOT of $x^4 - 2x^3 + 1 = 0$. The Tribonacci numbers can also be computed using the GENERATING FUNCTION

$$\frac{1}{1-z-z^2-z^3} = 1 + z + 2z^2 + 4z^3 + 7z^4 + 13z^5 + 24z^6 + 44z^7 + 81z^8 + 149z^9 + \ldots. \tag{2}$$

An explicit FORMULA for T_n is also given by

$$\left[3\frac{\{\frac{1}{3}(19+3\sqrt{33})^{1/3}+\frac{1}{3}(19-3\sqrt{33})^{1/3}+\frac{1}{3}\}^n(586+102\sqrt{33})^{1/3}}{(586+102\sqrt{33})^{2/3}+4-2(586+102\sqrt{33})^{1/3}}\right], \tag{3}$$

where $[x]$ denotes the NINT function (Plouffe). The first part of a NUMERATOR is related to the REAL root of $x^3 - x^2 - x - 1$, but determination of the DENOMINATOR requires an application of the LLL ALGORITHM. The numbers increase asymptotically to

$$T_n \sim c^n, \tag{4}$$

where

$$\begin{aligned} c &= (\tfrac{19}{27} + \tfrac{1}{9}\sqrt{33})^{1/3} + \tfrac{4}{9}(\tfrac{19}{27} + \tfrac{1}{9}\sqrt{33})^{-1/3} + \tfrac{1}{3} \\ &= 1.83928675521\ldots \end{aligned} \tag{5}$$

(Plouffe).

see also FIBONACCI n-STEP NUMBER, FIBONACCI NUMBER, TETRANACCI NUMBER

References

Plouffe, S. "Tribonacci Constant." http://lacim.uqam.ca/piDATA/tribo.txt.
Sloane, N. J. A. Sequence A000073/M1074 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Trichotomy Law

Every REAL NUMBER is NEGATIVE, 0, or POSITIVE.

Tricolorable

A projection of a LINK is tricolorable if each of the strands in the projection can be colored in one of three different colors such that, at each crossing, all three colors come together or only one does and at least two different colors are used. The TREFOIL KNOT and trivial 2-link are tricolorable, but the UNKNOT, WHITEHEAD LINK, and FIGURE-OF-EIGHT KNOT are not.

If the projection of a knot is tricolorable, then REIDEMEISTER MOVES on the knot preserve tricolorability, so either every projection of a knot is tricolorable or none is.

Tricomi Function

see CONFLUENT HYPERGEOMETRIC FUNCTION OF THE SECOND KIND, GORDON FUNCTION

Tricuspoid

see DELTOID

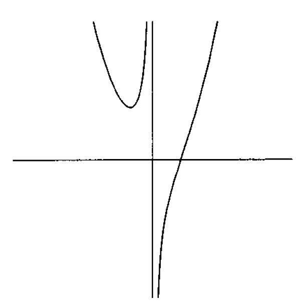
Tricylinder

see STEINMETZ SOLID

Tridecagon

A 13-sided POLYGON, sometimes also called the TRISKAIDECAGON.

Trident

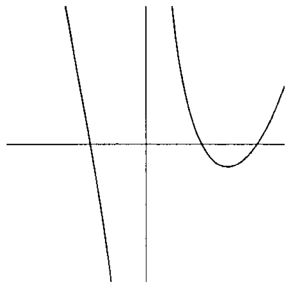


The plane curve given by the equation

$$xy = x^3 - a^3.$$

see also TRIDENT OF DESCARTES, TRIDENT OF NEWTON

Trident of Descartes



The plane curve given by the equation

$$(a+x)(a-x)(2a-x) = x^3 - 2ax^2 - a^2x + 2a^3 = axy$$

$$y = \frac{(a+x)(a-x)(2a-x)}{ax}$$

The above plot has $a = 2$.

Trident of Newton

The CUBIC CURVE defined by

$$ax^3 + bx^2 + cx + d = xy$$

with $a \neq 0$. The curve cuts the axis in either one or three points. It was the 66th curve in Newton's classification of CUBICS. Newton stated that the curve has four infinite legs and that the y-axis is an ASYMPTOTE to two tending toward contrary parts.

References
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 109–110, 1972.
MacTutor History of Mathematics Archive. "Trident of Newton." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Trident.html`.

Tridiagonal Matrix

A MATRIX with NONZERO elements only on the diagonal and slots horizontally or vertically adjacent the diagonal. A general 4×4 tridiagonal MATRIX has the form

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}.$$

Inversion of such a matrix requires only n (as opposed to n^3) arithmetic operations (Acton 1990).

see also DIAGONAL MATRIX, JACOBI ALGORITHM

References
Acton, F. S. *Numerical Methods That Work, 2nd printing.* Washington, DC: Math. Assoc. Amer., p. 103, 1990.
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Tridiagonal and Band Diagonal Systems of Equations." §2.4 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 42–47, 1992.

Tridiminished Icosahedron

see JOHNSON SOLID

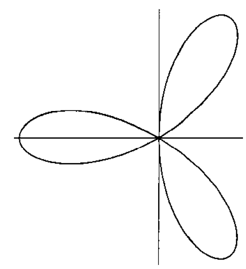
Tridiminished Rhombicosidodecahedron

see JOHNSON SOLID

Tridyakis Icosahedron

The DUAL POLYHEDRON of the ICOSITRUNCATED DODECADODECAHEDRON.

Trifolium



Lawrence (1972) defines a trifolium as a FOLIUM with $b \in (0, 4a)$. However, the term "the" trifolium is sometimes applied to the FOLIUM with $b = a$, which is then the 3-petalled ROSE with Cartesian equation

$$(x^2 + y^2)[y^2 + x(x+a)] = 4axy^2$$

and polar equation

$$r = a\cos\theta(4\sin^2\theta - 1) = -a\cos(3\theta).$$

The trifolium with $b = a$ is the RADIAL CURVE of the DELTOID.

see also BIFOLIUM, FOLIUM, QUADRIFOLIUM

References
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 152–153, 1972.
MacTutor History of Mathematics Archive. "Trifolium." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Trifolium.html`.

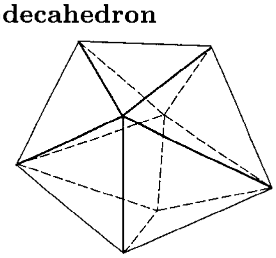
Trigon

see TRIANGLE

Trigonal Dipyramid

see TRIANGULAR DIPYRAMID

Trigonal Dodecahedron



An irregular DODECAHEDRON.

see also DODECAHEDRON, PYRITOHEDRON, RHOMBIC DODECAHEDRON

References
Cotton, F. A. *Chemical Applications of Group Theory, 3rd ed.* New York: Wiley, p. 62, 1990.

Trigonometric Functions

see TRIGONOMETRY

Trigonometric Series

$$\begin{aligned}A\sin(2\phi)+B\sin(4\phi)+C\sin(6\phi)+D\sin(8\phi)\\ =\sin(2\phi)(A'+\cos(2\phi)(B'+\cos(2\phi)(C'+D'\cos(2\phi)))),\end{aligned}$$

where

$$\begin{aligned}A'&\equiv A-C\\ B'&\equiv 2B-4D\\ C'&\equiv 4C\\ D'&\equiv 8D.\end{aligned}$$

$$\begin{aligned}A\sin\phi+B\sin(3\phi)+C\sin(5\phi)+D\sin(7\phi)\\ =\sin\phi(A'+\sin^2\phi(B'+\sin^2\phi(C'+D'\sin^2\phi))),\end{aligned}$$

where

$$\begin{aligned}A'&\equiv A+3B+5C+7D\\ B'&\equiv -4B-20C-56D\\ C'&\equiv 16C+112D\\ D'&\equiv -64D.\end{aligned}$$

$$\begin{aligned}A+B\cos(2\phi)+C\cos(4\phi)+D\cos(6\phi)+E\cos(8\phi)\\ =A'+\cos(2\phi)(B'+\cos(2\phi)(C'+\cos(2\phi)\\ \times(D'+E'\cos(2\phi)))),\end{aligned}$$

where

$$\begin{aligned}A'&\equiv A-C+E\\ B'&\equiv B-3D\\ C'&\equiv 2C-8E\\ D'&\equiv 4D\\ E'&\equiv 8E.\end{aligned}$$

References
Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, p. 19, 1987.

Trigonometric Substitution

INTEGRALS of the form

$$\int f(\cos\theta,\sin\theta)\,d\theta \tag{1}$$

can be solved by making the substitution $z=e^{i\theta}$ so that $dz=ie^{i\theta}\,d\theta$ and expressing

$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}=\frac{z+z^{-1}}{2} \tag{2}$$

$$\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}=\frac{z-z^{-1}}{2i}. \tag{3}$$

The integral can then be solved by CONTOUR INTEGRATION.

Alternatively, making the substitution $t\equiv\tan(\theta/2)$ transforms (1) into

$$\int f\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)\frac{2\,dt}{1+t^2}. \tag{4}$$

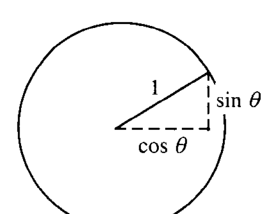
The following table gives trigonometric substitutions which can be used to transform integrals involving square roots.

Form	Substitution
$\sqrt{a^2-x^2}$	$x=a\sin\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$

see also HYPERBOLIC SUBSTITUTION

Trigonometry

The study of ANGLES and of the angular relationships of planar and 3-D figures is known as trigonometry. The trigonometric functions (also called the CIRCULAR FUNCTIONS) comprising trigonometry are the COSECANT $\csc x$, COSINE $\cos x$, COTANGENT $\cot x$, SECANT $\sec x$, SINE $\sin x$, and TANGENT $\tan x$. The inverses of these functions are denoted $\csc^{-1}x$, $\cos^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\sin^{-1}x$, and $\tan^{-1}x$. Note that the f^{-1} NOTATION here means INVERSE FUNCTION, *not* f to the -1 POWER.



The trigonometric functions are most simply defined using the UNIT CIRCLE. Let θ be an ANGLE measured counterclockwise from the x-AXIS along an ARC of the CIRCLE. Then $\cos\theta$ is the horizontal coordinate of the ARC endpoint, and $\sin\theta$ is the vertical component. The RATIO $\sin\theta/\cos\theta$ is defined as $\tan\theta$. As a result of this definition, the trigonometric functions are periodic with period 2π, so

$$\text{func}(2\pi n+\theta)=\text{func}(\theta), \qquad (1)$$

where n is an INTEGER and func is a trigonometric function.

From the PYTHAGOREAN THEOREM,

$$\sin^2\theta+\cos^2\theta=1. \qquad (2)$$

Therefore, it is also true that

$$\tan^2\theta+1=\sec^2\theta \qquad (3)$$

$$1+\cot^2\theta=\csc^2\theta. \qquad (4)$$

The trigonometric functions can be defined algebraically in terms of COMPLEX EXPONENTIALS (i.e., using the EULER FORMULA) as

$$\sin z\equiv\frac{e^{iz}-e^{-iz}}{2i} \qquad (5)$$

$$\csc z\equiv\frac{1}{\sin z}=\frac{2i}{e^{iz}-e^{-iz}} \qquad (6)$$

$$\cos z\equiv\frac{e^{iz}+e^{-iz}}{2} \qquad (7)$$

$$\sec z\equiv\frac{1}{\cos z}=\frac{2}{e^{iz}+e^{-iz}} \qquad (8)$$

$$\tan z\equiv\frac{\sin z}{\cos z}=\frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})} \qquad (9)$$

$$\cot z\equiv\frac{1}{\tan z}=\frac{i(e^{iz}+e^{-iz})}{e^{iz}-e^{-iz}}=\frac{i(1+e^{-2iz})}{1-e^{-2iz}}. \qquad (10)$$

OSBORNE'S RULE gives a prescription for converting trigonometric identities to analogous identities for HYPERBOLIC FUNCTIONS.

The ANGLES $n\pi/m$ (with m,n integers) for which the trigonometric function may be expressed in terms of finite root extraction of *real numbers* are limited to values of m which are precisely those which produce constructible POLYGONS. Gauss showed these to be of the form

$$m=2^k p_1p_2\cdots p_s, \qquad (11)$$

where k is an INTEGER ≥ 0 and the p_i are distinct FERMAT PRIMES. The first few values are m = 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, ... (Sloane's A003401). Although formulas for trigonometric functions may be found analytically for other m as well, the expressions involve ROOTS of COMPLEX NUMBERS obtained by solving a CUBIC, QUARTIC, or higher order equation. The cases $m=7$ and $m=9$ involve the CUBIC EQUATION and QUARTIC EQUATION, respectively. A partial table of the analytic values of SINE, COSINE, and TANGENT for arguments π/m is given below. Derivations of these formulas appear in the following entries.

°	rad	$\sin x$	$\cos x$	$\tan x$
0.0	0	0	1	0
15.0	$\frac{1}{12}\pi$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$2-\sqrt{3}$
18.0	$\frac{1}{10}\pi$	$\frac{1}{4}(\sqrt{5}-1)$	$\frac{1}{4}\sqrt{10+2\sqrt{5}}$	$\frac{1}{5}\sqrt{25-10\sqrt{5}}$
22.5	$\frac{1}{8}\pi$	$\frac{1}{2}\sqrt{2-\sqrt{2}}$	$\frac{1}{2}\sqrt{2+\sqrt{2}}$	$\sqrt{2}-1$
30.0	$\frac{1}{6}\pi$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$
36.0	$\frac{1}{5}\pi$	$\frac{1}{4}\sqrt{10-2\sqrt{5}}$	$\frac{1}{4}(1+\sqrt{5})$	$\sqrt{5-2\sqrt{5}}$
45.0	$\frac{1}{4}\pi$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1
60.0	$\frac{1}{3}\pi$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$
90.0	$\frac{1}{2}\pi$	1	0	∞
180.0	π	0	-1	0

The INVERSE TRIGONOMETRIC FUNCTIONS are generally defined on the following domains.

Function	Domain
$\sin^{-1}x$	$-\frac{1}{2}\pi\leq y\leq\frac{1}{2}\pi$
$\cos^{-1}x$	$0\leq y\leq\pi$
$\tan^{-1}x$	$-\frac{1}{2}\pi<y<\frac{1}{2}\pi$
$\csc^{-1}x$	$0\leq y\leq\frac{1}{2}\pi$ or $\pi\leq y\leq\frac{3\pi}{2}$
$\sec^{-1}x$	$0\leq y\leq\pi$
$\cot^{-1}x$	$0\leq y\leq\frac{1}{2}\pi$ or $-\pi\leq y\leq-\frac{1}{2}\pi$

Inverse-forward identities are

$$\tan^{-1}(\cot x)=\tfrac{1}{2}\pi-x \qquad (12)$$

$$\sin^{-1}(\cos x)=\tfrac{1}{2}\pi-x \qquad (13)$$

$$\sec^{-1}(\csc x)=\tfrac{1}{2}\pi-x, \qquad (14)$$

and forward-inverse identities are

$$\cos(\sin^{-1}x)=\sqrt{1-x^2} \qquad (15)$$

$$\cos(\tan^{-1}x)=\frac{1}{\sqrt{1+x^2}} \qquad (16)$$

$$\sin(\cos^{-1}x)=\sqrt{1-x^2}$$

$$\sin(\tan^{-1}x)=\frac{x}{\sqrt{1+x^2}} \qquad (17)$$

$$\tan(\cos^{-1}x)=\frac{\sqrt{1-x^2}}{x} \qquad (18)$$

$$\tan(\sin^{-1}x)=\frac{x}{\sqrt{1-x^2}}. \qquad (19)$$

Inverse sum identities include

$$\sin^{-1}x+\cos^{-1}x=\tfrac{1}{2}\pi \qquad (20)$$

$$\tan^{-1}x+\cot^{-1}x=\tfrac{1}{2}\pi \qquad (21)$$

$$\sec^{-1}x+\csc^{-1}x=\tfrac{1}{2}\pi, \qquad (22)$$

where (20) follows from

$$x=\sin(\sin^{-1}x)=\cos(\tfrac{1}{2}\pi-\sin^{-1}x). \qquad (23)$$

Complex inverse identities in terms of LOGARITHMS include

$$\sin^{-1}(z) = -i\ln(iz \pm \sqrt{1-z^2}\,) \tag{24}$$

$$\cos^{-1}(z) = -i\ln(z \pm i\sqrt{1-z^2}\,) \tag{25}$$

$$\tan^{-1}(z) = -i\ln\left(\frac{1+iz}{\sqrt{1+z^2}}\right) \tag{26}$$

$$= \tfrac{1}{2}i\ln\left(\frac{1-iz}{1+iz}\right). \tag{27}$$

For IMAGINARY arguments,

$$\sin(iz) = i\sinh z \tag{28}$$

$$\cos(iz) = \cosh z. \tag{29}$$

For COMPLEX arguments,

$$\sin(x+iy) = \sin x\cosh y + i\cos x\sinh y \tag{30}$$

$$\cos(x+iy) = \cos x\cosh y - i\sin x\sinh y. \tag{31}$$

For the ABSOLUTE SQUARE of COMPLEX arguments $z = x+iy$,

$$|\sin(x+iy)|^2 = \sin^2 x + \sinh^2 y \tag{32}$$

$$|\cos(x+iy)|^2 = \cos^2 x + \sinh^2 y. \tag{33}$$

The MODULUS also satisfies the curious identity

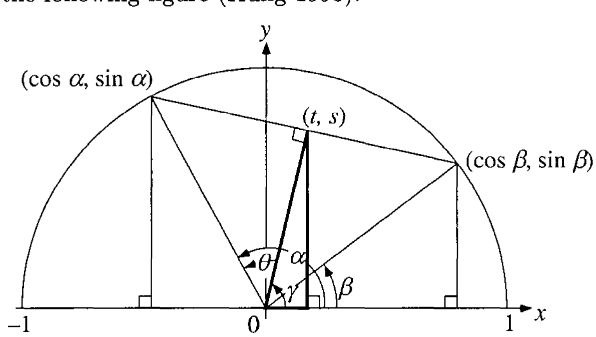
$$|\sin(x+iy)| = |\sin x + \sin(iy)|. \tag{34}$$

The only functions satisfying identities of this form,

$$|f(x+iy)| = |f(x) + f(iy)| \tag{35}$$

are $f(z) = Az$, $f(z) = A\sin(bz)$, and $f(z) = A\sinh(bz)$ (Robinson 1957).

Trigonometric product formulas can be derived using the following figure (Kung 1996).



In the figure,

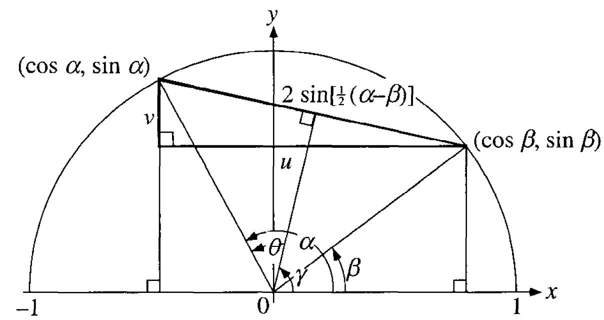
$$\theta = \tfrac{1}{2}(\alpha - \beta) \tag{36}$$

$$\gamma = \tfrac{1}{2}(\alpha + \beta), \tag{37}$$

so

$$s = \tfrac{1}{2}(\sin\alpha + \sin\beta) = \cos[\tfrac{1}{2}(\alpha-\beta)]\sin[\tfrac{1}{2}(\alpha+\beta)] \tag{38}$$

$$t = \tfrac{1}{2}(\cos\alpha + \cos\beta) = \cos[\tfrac{1}{2}(\alpha-\beta)]\cos[\tfrac{1}{2}(\alpha+\beta)]. \tag{39}$$



With θ and γ as previously defined, the above figure (Kung 1996) gives

$$u = \cos\beta - \cos\alpha = 2\sin[\tfrac{1}{2}(\alpha-\beta)]\sin[\tfrac{1}{2}(\alpha+\beta)] \tag{40}$$

$$v = \sin\alpha - \sin\beta = 2\sin[\tfrac{1}{2}(\alpha-\beta)]\cos[\tfrac{1}{2}(\alpha+\beta)]. \tag{41}$$

Angle addition FORMULAS express trigonometric functions of sums of angles $\alpha \pm \beta$ in terms of functions of α and β. They can be simply derived used COMPLEX exponentials and the EULER FORMULA,

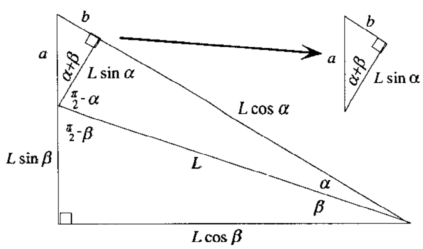
$$\begin{aligned}\sin(\alpha+\beta) &= \frac{e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}}{2i} = \frac{e^{i\alpha}e^{i\beta} - e^{-i\alpha}e^{-i\beta}}{2i}\\ &= \frac{(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)}{2i}\\ &\quad - \frac{(\cos\alpha - i\sin\alpha)(\cos\beta - i\sin\beta)}{2i}\\ &= \frac{\cos\alpha\cos\beta + i\sin\beta\cos\alpha + i\sin\alpha\cos\beta - \sin\alpha\sin\beta}{2i}\\ &\quad + \frac{-\cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta + \sin\alpha\sin\beta}{2i}\\ &= \sin\alpha\cos\beta + \sin\beta\cos\alpha\end{aligned} \tag{42}$$

$$\begin{aligned}\cos(\alpha+\beta) &= \frac{e^{i(\alpha+\beta)} + e^{-i(\alpha+\beta)}}{2} = \frac{e^{i\alpha}e^{i\beta} + e^{-i\alpha}e^{-i\beta}}{2}\\ &= \frac{(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)}{2}\\ &\quad + \frac{(\cos\alpha - i\sin\alpha)(\cos\beta - i\sin\beta)}{2}\\ &= \frac{\cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta - \sin\alpha\sin\beta}{2}\\ &\quad + \frac{\cos\alpha\cos\beta - i\cos\alpha\sin\beta - i\sin\alpha\cos\beta - \sin\alpha\sin\beta}{2}\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta.\end{aligned} \tag{43}$$

Taking the ratio gives the tangent angle addition FORMULA

$$\tan(\alpha+\beta) \equiv \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta+\sin\beta\cos\alpha}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$
$$= \frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}{1-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\alpha\beta}} = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}. \quad (44)$$

The angle addition FORMULAS can also be derived purely algebraically without the use of COMPLEX NUMBERS. Consider the following figure.



From the large RIGHT TRIANGLE,

$$\sin(\alpha+\beta) = \frac{L\sin\beta+a}{L\cos\alpha+b} \quad (45)$$
$$\cos(\alpha+\beta) = \frac{L\cos\beta}{L\cos\alpha+b}. \quad (46)$$

But, from the small triangle (inset at upper right),

$$a = \frac{L\sin\alpha}{\cos(\alpha+\beta)} \quad (47)$$
$$b = L\sin\alpha\tan(\alpha+\beta). \quad (48)$$

Plugging a and b from (47) and (48) into (45) and (46) gives

$$\sin(\alpha+\beta) = \frac{L\sin\beta+\frac{L\sin\alpha}{\cos(\alpha+\beta)}}{L\cos\alpha+\frac{L\sin\alpha\sin(\alpha+\beta)}{\cos(\alpha+\beta)}}$$
$$= \frac{\sin\beta\cos(\alpha+\beta)+\sin\alpha}{\cos\alpha\cos(\alpha+\beta)+\sin\alpha\sin(\alpha+\beta)}, \quad (49)$$

and

$$\cos(\alpha+\beta) = \frac{L\cos\beta}{L\cos\alpha+\frac{L\sin\alpha\sin(\alpha+\beta)}{\cos(\alpha+\beta)}}$$
$$= \frac{\cos\beta}{\cos\alpha+\frac{\sin\alpha\sin(\alpha+\beta)}{\cos(\alpha+\beta)}}. \quad (50)$$

Now solve (50) for $\cos(\alpha+\beta)$,

$$\cos(\alpha+\beta)\cos\alpha+\sin\alpha\sin(\alpha+\beta) = \cos\beta \quad (51)$$

to obtain

$$\cos(\alpha+\beta) = \frac{\cos\beta-\sin\alpha\sin(\alpha+\beta)}{\cos\alpha}. \quad (52)$$

Plugging (52) into (49) gives

$$\sin(\alpha+\beta) = \frac{\sin\beta\left[\frac{\cos\beta-\sin\alpha\sin(\alpha+\beta)}{\cos\alpha}\right]+\sin\alpha}{\cos\alpha\left[\frac{\cos\beta-\sin\alpha\sin(\alpha+\beta)}{\cos\alpha}\right]+\sin\alpha\sin(\alpha+\beta)}$$
$$= \frac{\sin\beta\cos\beta-\sin\alpha\sin\beta\sin(\alpha+\beta)+\sin\alpha\cos\alpha}{\cos\alpha\cos\beta-\sin\alpha\cos\alpha\sin(\alpha+\beta)+\sin\alpha\cos\alpha\sin(\alpha+\beta)}$$
$$= \frac{\sin\beta\cos\beta-\sin\alpha\sin\beta\sin(\alpha+\beta)+\sin\alpha\cos\alpha}{\cos\alpha\cos\beta}$$
$$= \frac{\sin\alpha\cos\alpha+\sin\beta\cos\beta}{\cos\alpha\cos\beta} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}\sin(\alpha+\beta), \quad (53)$$

so

$$\sin(\alpha+\beta)\left(1+\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}\right) = \frac{\sin\alpha\cos\alpha+\sin\beta\cos\beta}{\cos\alpha\cos\beta} \quad (54)$$

$$\sin(\alpha+\beta)(\cos\alpha\cos\beta+\sin\alpha\sin\beta)$$
$$= \sin\alpha\cos\alpha+\sin\beta\cos\beta, \quad (55)$$

and

$$\sin(\alpha+\beta) = \frac{\sin\alpha\cos\alpha+\sin\beta\cos\beta}{\sin\alpha\sin\beta+\cos\alpha\cos\beta}$$
$$= \frac{\sin\alpha\cos\alpha+\sin\beta\cos\beta}{\sin\alpha\sin\beta+\cos\alpha\cos\beta}\,\frac{\sin\alpha\cos\beta+\sin\beta\cos\alpha}{\sin\alpha\cos\beta+\sin\beta\cos\alpha}. \quad (56)$$

Multiplying out the DENOMINATOR gives

$$(\cos\alpha\cos\beta+\sin\alpha\sin\beta)(\sin\alpha\cos\beta+\sin\beta\cos\alpha)$$
$$= \sin\alpha\cos\alpha\cos^2\beta+\cos^2\alpha\sin\beta\cos\beta$$
$$+\sin^2\alpha\sin\beta\cos\beta+\sin\alpha\cos\alpha\sin^2\beta$$
$$= \sin\alpha\cos\alpha+\sin\beta\cos\beta, \quad (57)$$

so

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta+\sin\beta\cos\alpha. \quad (58)$$

Multiplying out (50),

$$\cos(\alpha+\beta)\cos\alpha+\sin\alpha\sin(\alpha+\beta) = \cos\beta \quad (59)$$

$$\cos(\alpha+\beta) = \frac{\cos\beta-\sin\alpha\sin(\alpha+\beta)}{\cos\alpha}$$
$$= \frac{\cos\beta-\sin\alpha(\sin\alpha\cos\beta+\sin\beta\cos\alpha)}{\cos\alpha}$$
$$= \frac{\cos\beta(1-\sin^2\alpha)+\sin\alpha\cos\alpha\sin\beta}{\cos\alpha}$$
$$= \frac{\cos^2\alpha\cos\beta+\sin\alpha\cos\alpha\sin\beta}{\cos\alpha}$$
$$= \cos\alpha\cos\beta+\sin\alpha\sin\beta. \quad (60)$$

Summarizing,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha \quad (61)$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha \quad (62)$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad (63)$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \quad (64)$$
$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} \quad (65)$$
$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}. \quad (66)$$

The sine and cosine angle addition identities can be summarized by the MATRIX EQUATION

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{bmatrix} = \begin{bmatrix} \cos(x+y) & \sin(x+y) \\ -\sin(x+y) & \cos(x+y) \end{bmatrix}. \quad (67)$$

The double angle formulas are

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha \quad (68)$$
$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha \quad (69)$$
$$= 2\cos^2\alpha - 1 \quad (70)$$
$$= 1 - 2\sin^2\alpha \quad (71)$$
$$\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}. \quad (72)$$

General multiple angle formulas are

$$\sin(n\alpha) = 2\sin[(n-1)\alpha]\cos\alpha - \sin[(n-2)\alpha] \quad (73)$$
$$\sin(nx) = n\cos^{n-1}x\sin x - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cos^{n-3}x\sin^3 x + \ldots \quad (74)$$
$$\cos(n\alpha) = 2\cos[(n-1)\alpha]\cos\alpha - \cos[(n-2)\alpha] \quad (75)$$
$$\cos(nx) = \cos^n x - \frac{n(n-1)}{1\cdot 2}\cos^{n-2}x\sin^2 x + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}\cos^{n-4}x\sin^4 x - \ldots \quad (76)$$
$$\tan(n\alpha) = \frac{\tan[(n-1)\alpha] + \tan\alpha}{1-\tan[(n-1)\alpha]\tan\alpha}. \quad (77)$$

Therefore, any trigonometric function of a sum can be broken up into a sum of trigonometric functions with $\sin\alpha\cos\alpha$ cross terms. Particular cases for multiple angle formulas up to $n = 4$ are given below.

$$\sin(3\alpha) = 3\sin\alpha - 4\sin^3\alpha \quad (78)$$
$$\cos(3\alpha) = 4\cos^3\alpha - 3\cos\alpha \quad (79)$$
$$\tan(3\alpha) = \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha} \quad (80)$$
$$\sin(4\alpha) = 4\sin\alpha\cos\alpha - 8\sin^3\alpha\cos\alpha \quad (81)$$
$$\cos(4\alpha) = 8\cos^4\alpha - 8\cos^2\alpha + 1 \quad (82)$$
$$\tan(4\alpha) = \frac{4\tan\alpha - 4\tan^3\alpha}{1-6\tan^2\alpha + \tan^4\alpha}. \quad (83)$$

Beyer (1987, p. 139) gives formulas up to $n = 6$.

Sum identities include

$$\frac{\tan(\alpha-\beta)}{\tan(\alpha+\beta)} = \frac{\sin(\alpha-\beta)\cos(\alpha+\beta)}{\cos(\alpha-\beta)\sin(\alpha+\beta)} = \frac{(\sin\alpha\cos\beta - \sin\beta\cos\alpha)(\cos\alpha\cos\beta - \sin\alpha\sin\beta)}{(\cos\alpha\cos\beta + \sin\alpha\sin\beta)(\sin\alpha\cos\beta + \sin\beta\cos\alpha)} = \frac{\sin\alpha\cos\alpha - \sin\beta\cos\beta}{\sin\alpha\cos\alpha + \sin\beta\cos\beta}. \quad (84)$$

Infinite sum identities include

$$\sum_{k=1,\,3,\,5,\ldots}^{\infty} \frac{e^{-kx}\sin(ky)}{k} = \frac{1}{2}\tan^{-1}\left(\frac{\sin y}{\sinh x}\right). \quad (85)$$

Trigonometric half-angle formulas include

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos\alpha}{2}} \quad (86)$$
$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos\alpha}{2}} \quad (87)$$
$$\tan\left(\frac{\alpha}{2}\right) = \frac{\sin\alpha}{1+\cos\alpha} \quad (88)$$
$$= \frac{1-\cos\alpha}{\sin\alpha} \quad (89)$$
$$= \frac{1\pm\sqrt{1+\tan^2\alpha}}{\tan\alpha} \quad (90)$$
$$= \frac{\tan\alpha\sin\alpha}{\tan\alpha + \sin\alpha}. \quad (91)$$

The PROSTHAPHAERESIS FORMULAS are

$$\sin\alpha + \sin\beta = 2\sin[\tfrac{1}{2}(\alpha+\beta)]\cos[\tfrac{1}{2}(\alpha-\beta)] \quad (92)$$
$$\sin\alpha - \sin\beta = 2\cos[\tfrac{1}{2}(\alpha+\beta)]\sin[\tfrac{1}{2}(\alpha-\beta)] \quad (93)$$
$$\sin\alpha + \cos\beta = 2\cos[\tfrac{1}{2}(\alpha+\beta)]\cos[\tfrac{1}{2}(\alpha-\beta)] \quad (94)$$
$$\cos\alpha - \cos\beta = -2\sin[\tfrac{1}{2}(\alpha+\beta)]\sin[\tfrac{1}{2}(\alpha-\beta)]. \quad (95)$$

Related formulas are

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)] \quad (96)$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \quad (97)$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)] \quad (98)$$
$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]. \quad (99)$$

Multiplying both sides by 2 gives the equations sometimes known as the WERNER FORMULAS.

Trigonometric product/sum formulas are

$$\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta = \cos^2\beta - \cos^2\alpha \quad (100)$$

$$\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha. \quad (101)$$

Power formulas include

$$\sin^2 x = \tfrac{1}{2}[1-\cos(2x)] \quad (102)$$
$$\sin^3 x = \tfrac{1}{4}[3\sin x - \sin(3x)] \quad (103)$$
$$\sin^4 x = \tfrac{1}{8}[3-4\cos(2x)+\cos(4x)] \quad (104)$$

and

$$\cos^2 x = \tfrac{1}{2}[1+\cos(2x)] \quad (105)$$
$$\cos^3 x = \tfrac{1}{4}[3\cos x + \cos(3x)] \quad (106)$$
$$\cos^4 x = \tfrac{1}{8}[3+4\cos(2x)+\cos(4x)] \quad (107)$$

(Beyer 1987, p. 140). Formulas of these types can also be given analytically as

$$\sin^{2n} x = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}}\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\cos[2(n-k)x] \quad (108)$$

$$\sin^{2n+1} = \frac{(-1)^n}{4^n}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}\sin[(2n+1-2k)x] \quad (109)$$

$$\cos^{2n} x = \frac{(-1)^n}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}\binom{2n}{k}\cos[2(n-k)x] \quad (110)$$

$$\cos^{2n+1} x = \frac{1}{4^n}\sum_{k=0}^{n}\binom{2n+1}{k}\cos[(2n+1-2k)x] \quad (111)$$

(Kogan), where $\binom{n}{m}$ is a BINOMIAL COEFFICIENT.

see also COSECANT, COSINE, COTANGENT, EUCLIDEAN NUMBER, INVERSE COSECANT, INVERSE COSINE, INVERSE COTANGENT, INVERSE SECANT, INVERSE SINE, INVERSE TANGENT, INVERSE TRIGONOMETRIC FUNCTIONS, OSBORNE'S RULE, POLYGON, SECANT, SINE, TANGENT, TRIGONOMETRY VALUES: π, $\pi/2$, $\pi/3$, $\pi/4$, $\pi/5$, $\pi/6$, $\pi/7$, $\pi/8$, $\pi/9$, $\pi/10$, $\pi/11$, $\pi/12$, $\pi/15$, $\pi/16$, $\pi/17$, $\pi/18$, $\pi/20$, 0, WERNER FORMULAS

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Trigonometry Values—π

By the definition of the trigonometric functions,

$$\sin\pi = 0 \quad (1)$$
$$\cos\pi = -1 \quad (2)$$
$$\tan\pi = 0 \quad (3)$$
$$\csc\pi = \infty \quad (4)$$
$$\sec\pi = -1 \quad (5)$$
$$\cot\pi = \infty. \quad (6)$$

Trigonometry Values—$\pi/2$

By the definition of the trigonometric functions,

$$\sin\left(\frac{\pi}{2}\right) = 1 \quad (1)$$
$$\cos\left(\frac{\pi}{2}\right) = 0 \quad (2)$$
$$\tan\left(\frac{\pi}{2}\right) = \infty \quad (3)$$
$$\csc\left(\frac{\pi}{2}\right) = 1 \quad (4)$$
$$\sec\left(\frac{\pi}{2}\right) = \infty \quad (5)$$
$$\cot\left(\frac{\pi}{2}\right) = 0. \quad (6)$$

see also DIGON

Trigonometry Values—$\pi/3$

From TRIGONOMETRY VALUES: $\pi/6$

$$\sin\left(\frac{\pi}{6}\right) = \tfrac{1}{2} \quad (1)$$

$$\cos\left(\frac{\pi}{6}\right) = \tfrac{1}{2}\sqrt{3} \quad (2)$$

together with the trigonometric identity

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha, \quad (3)$$

the identity

$$\sin\left(\frac{\pi}{3}\right) = 2\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) = 2(\tfrac{1}{2})(\tfrac{1}{2}\sqrt{3}) = \tfrac{1}{2}\sqrt{3} \quad (4)$$

is obtained. Using the identity

$$\cos(2\alpha) = 1 - 2\sin^2\alpha, \quad (5)$$

then gives

$$\cos\left(\frac{\pi}{3}\right) = 1 - 2\sin^2\left(\frac{\pi}{6}\right) = 1 - 2(\tfrac{1}{2})^2 = \tfrac{1}{2}. \quad (6)$$

Summarizing,

$$\sin\left(\frac{\pi}{3}\right) = \tfrac{1}{2}\sqrt{3} \quad (7)$$

$$\cos\left(\frac{\pi}{3}\right) = \tfrac{1}{2} \quad (8)$$

$$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}. \quad (9)$$

see also EQUILATERAL TRIANGLE

Trigonometry Values—$\pi/4$

For a RIGHT ISOSCELES TRIANGLE, symmetry requires that the angle at each VERTEX be given by

$$\tfrac{1}{2}\pi + 2\alpha = \pi, \quad (1)$$

so $\alpha = \pi/4$. The sides are equal, so

$$\sin^2\alpha + \cos^2\alpha = 2\sin^2\alpha = 1. \quad (2)$$

Solving,

$$\sin\left(\frac{\pi}{4}\right) = \tfrac{1}{2}\sqrt{2} \quad (3)$$

$$\cos\left(\frac{\pi}{4}\right) = \tfrac{1}{2}\sqrt{2} \quad (4)$$

$$\tan\left(\frac{\pi}{4}\right) = 1. \quad (5)$$

see also SQUARE

Trigonometry Values—$\pi/5$

Use the identity

$$\sin(5\alpha) = 5\sin\alpha - 20\sin^3\alpha + 16\sin^5\alpha. \quad (1)$$

Now, let $\alpha \equiv \pi/5$ and $x \equiv \sin\alpha$. Then

$$\sin\pi = 0 = 5x - 20x^3 + 16x^5 \quad (2)$$

$$16x^4 - 20x^2 + 5 = 0. \quad (3)$$

Solving the QUADRATIC EQUATION for x^2 gives

$$\sin^2\left(\frac{\pi}{5}\right) = x^2 = \frac{20 \pm \sqrt{(-20)^2 - 4\cdot 16\cdot 5}}{2\cdot 16} = \frac{20 \pm \sqrt{80}}{32} = \tfrac{1}{8}(5 \pm \sqrt{5}). \quad (4)$$

Now, $\sin(\pi/5)$ must be less than

$$\sin\left(\frac{\pi}{4}\right) = \tfrac{1}{2}\sqrt{2}, \quad (5)$$

so taking the MINUS SIGN and simplifying gives

$$\sin\left(\frac{\pi}{5}\right) = \sqrt{\frac{5-\sqrt{5}}{8}} = \tfrac{1}{4}\sqrt{10 - 2\sqrt{5}}. \quad (6)$$

$\cos(\pi/5)$ can be computed from

$$\cos\left(\frac{\pi}{5}\right) = \sqrt{1 - \sin^2\left(\frac{\pi}{5}\right)} = \tfrac{1}{4}(1+\sqrt{5}). \quad (7)$$

Summarizing,

$$\sin\left(\frac{\pi}{5}\right) = \tfrac{1}{4}\sqrt{10 - 2\sqrt{5}} \quad (8)$$

$$\sin\left(\frac{2\pi}{5}\right) = \tfrac{1}{4}\sqrt{10 + 2\sqrt{5}} \quad (9)$$

$$\sin\left(\frac{3\pi}{5}\right) = \tfrac{1}{4}\sqrt{10 + 2\sqrt{5}} \quad (10)$$

$$\sin\left(\frac{4\pi}{5}\right) = \tfrac{1}{4}\sqrt{10 - 2\sqrt{5}} \quad (11)$$

$$\cos\left(\frac{\pi}{5}\right) = \tfrac{1}{4}(1 + \sqrt{5}) \quad (12)$$

$$\cos\left(\frac{2\pi}{5}\right) = \tfrac{1}{4}(-1 + \sqrt{5}) \quad (13)$$

$$\cos\left(\frac{3\pi}{5}\right) = \tfrac{1}{4}(1 - \sqrt{5}) \quad (14)$$

$$\cos\left(\frac{4\pi}{5}\right) = -\tfrac{1}{4}(1 + \sqrt{5}) \quad (15)$$

$$\tan\left(\frac{\pi}{5}\right) = \sqrt{5 - 2\sqrt{5}} \quad (16)$$

$$\tan\left(\frac{2\pi}{5}\right) = \sqrt{5 + 2\sqrt{5}} \quad (17)$$

$$\tan\left(\frac{3\pi}{5}\right) = -\sqrt{5 + 2\sqrt{5}} \quad (18)$$

$$\tan\left(\frac{4\pi}{5}\right) = -\sqrt{5 - 2\sqrt{5}}. \quad (19)$$

see also DODECAHEDRON, ICOSAHEDRON, PENTAGON, PENTAGRAM

Trigonometry Values—$\pi/6$

Given a RIGHT TRIANGLE with angles defined to be α and 2α, it must be true that

$$\alpha + 2\alpha + \tfrac{1}{2}\pi = \pi, \qquad (1)$$

so $\alpha = \pi/6$. Define the hypotenuse to have length 1 and the side opposite α to have length x, then the side opposite 2α has length $\sqrt{1-x^2}$. This gives $\sin\alpha \equiv x$ and

$$\sin(2\alpha) = \sqrt{1-x^2}. \qquad (2)$$

But

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha = 2x\sqrt{1-x^2}, \qquad (3)$$

so we have

$$\sqrt{1-x^2} = 2x\sqrt{1-x^2}. \qquad (4)$$

This gives $2x = 1$, or

$$\sin\left(\frac{\pi}{6}\right) = \tfrac{1}{2}. \qquad (5)$$

$\cos(\pi/6)$ is then computed from

$$\cos\left(\frac{\pi}{6}\right) = \sqrt{1-\sin^2\left(\frac{\pi}{6}\right)} = \sqrt{1-(\tfrac{1}{2})^2} = \tfrac{1}{2}\sqrt{3}. \quad (6)$$

Summarizing,

$$\sin\left(\frac{\pi}{6}\right) = \tfrac{1}{2} \qquad (7)$$

$$\cos\left(\frac{\pi}{6}\right) = \tfrac{1}{2}\sqrt{3} \qquad (8)$$

$$\tan\left(\frac{\pi}{6}\right) = \tfrac{1}{3}\sqrt{3}. \qquad (9)$$

see also HEXAGON, HEXAGRAM

Trigonometry Values—$\pi/7$

Trigonometric functions of $n\pi/7$ for n an integer cannot be expressed in terms of sums, products, and finite root extractions on *real* rational numbers because 7 is not a FERMAT PRIME. This also means that the HEPTAGON is not a CONSTRUCTIBLE POLYGON.

However, exact expressions involving roots of *complex* numbers can still be derived using the trigonometric identity

$$\sin(n\alpha) = 2\sin[(n-1)\alpha]\cos\alpha - \sin[(n-2)\alpha]. \qquad (1)$$

The case $n = 7$ gives

$$\begin{aligned}\sin(7\alpha) &= 2\sin(6\alpha)\cos\alpha - \sin(5\alpha)\\ &= 2(32\cos^5\alpha\sin\alpha - 32\cos^3\alpha\sin\alpha + 6\cos\alpha\sin\alpha)\cos\alpha\\ &\quad -(5\sin\alpha - 20\sin^3\alpha + 16\sin^5\alpha)\\ &= 64\cos^6\alpha\sin\alpha - 64\cos^4\alpha\sin\alpha + 12\cos^2\alpha\sin\alpha\\ &\quad -5\sin\alpha + 20(1-\cos^2\alpha)\sin\alpha\\ &\quad -16(1-2\cos^2\alpha+\cos^4\alpha)\sin\alpha\\ &= \sin\alpha(64\cos^6\alpha - 80\cos^4\alpha + 24\cos^2\alpha - 1). \qquad (2)\end{aligned}$$

Rewrite this using the identity $\cos^2\alpha = 1 - \sin^2\alpha$,

$$\begin{aligned}\sin\left(\frac{\pi}{7}\right) &= \sin\alpha(7 - 56\sin^2\alpha + 112\sin^4\alpha - 64\sin^6\alpha)\\ &= -64\sin\alpha(\sin^6\alpha - \tfrac{112}{64}\sin^4\alpha + \tfrac{56}{64}\sin^2\alpha - \tfrac{7}{64}). \quad (3)\end{aligned}$$

Now, let $\alpha \equiv \pi/7$ and $x \equiv \sin^2\alpha$, then

$$\sin(\pi) = 0 = x^3 - \tfrac{7}{4}x^2 + \tfrac{7}{8}x - \tfrac{7}{64}, \qquad (4)$$

which is a CUBIC EQUATION in x. The ROOTS are numerically found to be $x \approx 0.188255, 0.611260\ldots, 0.950484\ldots$. But $\sin\alpha = \sqrt{x}$, so these ROOTS correspond to $\sin\alpha \approx 0.4338$, $\sin(2\alpha) \approx 0.7817$, $\sin(3\alpha) \approx 0.9749$. By NEWTON'S RELATION

$$\prod_i r_i = -a_0, \qquad (5)$$

we have

$$x_1x_2x_3 = \tfrac{7}{64}, \qquad (6)$$

or

$$\sin\left(\frac{\pi}{7}\right)\sin\left(\frac{2\pi}{7}\right)\sin\left(\frac{3\pi}{7}\right) = \sqrt{\frac{7}{64}} = \tfrac{1}{8}\sqrt{7}. \qquad (7)$$

Similarly,

$$\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{3\pi}{7}\right) = \frac{1}{8}. \qquad (8)$$

The constants of the CUBIC EQUATION are given by

$$Q \equiv \tfrac{1}{9}(3a_1 - a_2{}^2) = \tfrac{1}{9}[3\cdot\tfrac{7}{8} - (-\tfrac{7}{4})^2] = -\tfrac{7}{144} \qquad (9)$$

$$\begin{aligned}R &\equiv \tfrac{1}{54}(9a_2a_1 - 2a_2^3 - 27a_0)\\ &= \tfrac{1}{54}[9(-\tfrac{7}{4})(\tfrac{1}{7}8) - 2(-\tfrac{7}{4})^3 - 27(-\tfrac{7}{64})]\\ &= -\tfrac{7}{3456}. \qquad (10)\end{aligned}$$

The DISCRIMINANT is then

$$\begin{aligned}D \equiv Q^3 + R^2 &= -\tfrac{343}{2{,}985{,}984} + \tfrac{49}{11{,}943{,}936}\\ &= -\tfrac{49}{442{,}368} < 0, \qquad (11)\end{aligned}$$

so there are three distinct REAL ROOTS. Finding the first one,

$$x = \sqrt[3]{R+\sqrt{D}} + \sqrt[3]{R-\sqrt{D}} - \tfrac{1}{3}a_2. \qquad (12)$$

Writing

$$\sqrt{D} = 3^{-3/2}\tfrac{7}{128}i, \qquad (13)$$

plugging in from above, and anticipating that the solution we have picked corresponds to $\sin(3\pi/7)$,

$$\sin\left(\frac{3\pi}{7}\right) = \sqrt{x} =$$

$$\sqrt{\sqrt[3]{-\frac{7}{3456}+3^{-3/2}\frac{7}{128}i}+\sqrt[3]{-\frac{7}{3456}-3^{-3/2}\frac{7}{128}i}-\frac{1}{3}\left(-\frac{7}{4}\right)}$$
$$= \sqrt{\sqrt[3]{-\frac{7}{3456}+3^{-3/2}\frac{7}{128}i}+\sqrt[3]{-\frac{7}{3456}-3^{-3/2}\frac{7}{128}i}+\frac{7}{12}}$$
$$= \sqrt{\sqrt[3]{\frac{7}{3456}(-1+3^{3/2}i)}-\sqrt[3]{\frac{7}{3456}(1+3^{3/2}i)}+\frac{7}{12}}$$
$$= \sqrt{\frac{1}{12}\left[\sqrt[3]{\frac{7}{2}(-1+3^{3/2}i)}-\sqrt[3]{\frac{7}{2}(1+3^{3/2}i)}+7\right]}. \qquad (14)$$

see also HEPTAGON

Trigonometry Values—$\pi/8$

$$\sin\left(\frac{\pi}{8}\right) = \sin\left(\frac{1}{2}\cdot\frac{\pi}{4}\right) = \sqrt{\frac{1}{2}\left(1-\cos\frac{\pi}{4}\right)}$$
$$= \sqrt{\tfrac{1}{2}(1-\tfrac{1}{2}\sqrt{2})} = \tfrac{1}{2}\sqrt{2-\sqrt{2}}. \qquad (1)$$

Now, checking to see if the SQUARE ROOT can be simplified gives

$$a^2 - b^2c = 2^2 - 1^2\cdot 2 = 4-2 = 2, \qquad (2)$$

which is not a PERFECT SQUARE, so the above expression cannot be simplified. Similarly,

$$\cos\left(\frac{\pi}{8}\right) = \cos\left(\frac{1}{2}\frac{\pi}{4}\right) = \sqrt{\frac{1}{2}\left(1+\cos\frac{\pi}{4}\right)}$$
$$= \sqrt{\frac{1}{2}\left(1+\frac{\sqrt{2}}{2}\right)} = \tfrac{1}{2}\sqrt{2+\sqrt{2}} \qquad (3)$$

$$\tan\left(\frac{\pi}{8}\right) = \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} = \sqrt{\frac{(2-\sqrt{2})^2}{4-2}} = \sqrt{\frac{4+2-4\sqrt{2}}{2}}$$
$$= \sqrt{\frac{6-4\sqrt{2}}{2}} = \sqrt{3-2\sqrt{2}}. \qquad (4)$$

But

$$a^2 - b^2c = 3^2 - 2^2 2 = 9-8 = 1 \qquad (5)$$

is a PERFECT SQUARE, so we can find

$$d = \tfrac{1}{2}(3\pm 1) = 1, 2.$$

Rewrite the above as

$$\tan\left(\frac{\pi}{8}\right) = \sqrt{2}-1 \qquad (6)$$
$$\cot\left(\frac{\pi}{8}\right) = \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{2-1} = \sqrt{2}+1. \qquad (7)$$

Summarizing,

$$\sin\left(\frac{\pi}{8}\right) = \tfrac{1}{2}\sqrt{2-\sqrt{2}} \qquad (8)$$
$$\sin\left(\frac{3\pi}{8}\right) = \tfrac{1}{2}\sqrt{2+\sqrt{2}} \qquad (9)$$
$$\cos\left(\frac{\pi}{8}\right) = \tfrac{1}{2}\sqrt{2+\sqrt{2}} \qquad (10)$$
$$\cos\left(\frac{3\pi}{8}\right) = \tfrac{1}{2}\sqrt{2-\sqrt{2}} \qquad (11)$$
$$\tan\left(\frac{\pi}{8}\right) = \sqrt{2}-1 \qquad (12)$$
$$\tan\left(\frac{3\pi}{8}\right) = \sqrt{2}+1. \qquad (13)$$

see also OCTAGON

Trigonometry Values—$\pi/9$

Trigonometric functions of $n\pi/9$ radians for n an integer not divisible by 3 (e.g., 40° and 80°) cannot be expressed in terms of sums, products, and finite root extractions on *real* rational numbers because 9 is not a product of distinct FERMAT PRIMES. This also means that the NONAGON is not a CONSTRUCTIBLE POLYGON.

However, exact expressions involving roots of *complex* numbers can still be derived using the trigonometric identity

$$\sin(3\alpha) = 3\sin\alpha - 4\sin^3\alpha. \qquad (1)$$

Let $\alpha \equiv \pi/9$ and $x \equiv \sin\alpha$. Then the above identity gives the CUBIC EQUATION

$$4x^3 - 3x + \tfrac{1}{2}\sqrt{3} = 0 \qquad (2)$$

$$x^3 - \tfrac{3}{4}x = -\tfrac{1}{8}\sqrt{3}. \qquad (3)$$

This cubic is of the form

$$x^3 + px = q, \qquad (4)$$

where

$$p = -\tfrac{3}{4} \qquad (5)$$
$$q = -\tfrac{1}{8}\sqrt{3}. \qquad (6)$$

The DISCRIMINANT is then

$$D \equiv \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$$
$$= \left(-\frac{1}{4}\right)^3 + \left(\frac{\sqrt{3}}{16}\right)^2 = -\frac{1}{16\cdot 4} + \frac{3}{16\cdot 16} = \frac{-4+3}{256}$$
$$= -\frac{1}{256} < 0. \qquad (7)$$

There are therefore three REAL distinct roots, which are approximately -0.9848, 0.3240, and 0.6428. We want the one in the first QUADRANT, which is 0.3240.

$$\begin{aligned}\sin\left(\frac{\pi}{9}\right) &= \sqrt[3]{-\frac{\sqrt{3}}{16}+\sqrt{-\frac{1}{256}}}+\sqrt[3]{-\frac{\sqrt{3}}{16}-\sqrt{-\frac{1}{256}}}\\ &= \sqrt[3]{-\frac{\sqrt{3}}{16}+\frac{1}{16}i}-\sqrt[3]{\frac{\sqrt{3}}{16}+\frac{1}{16}i}\\ &= 2^{-4/3}(\sqrt[3]{i-\sqrt{3}}-\sqrt[3]{i+\sqrt{3}})\\ &\approx 0.3240\ldots.\end{aligned}\tag{8}$$

Similarly,

$$\begin{aligned}\cos\left(\frac{\pi}{9}\right) &= 2^{-4/3}(\sqrt[3]{1+i\sqrt{3}}+\sqrt[3]{1-i\sqrt{3}})\\ &\approx 0.7660\ldots.\end{aligned}\tag{9}$$

Because of the NEWTON'S RELATIONS, we have the identities

$$\sin\left(\frac{\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)\sin\left(\frac{4\pi}{9}\right)=\tfrac{1}{8}\tag{10}$$

$$\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right)=\tfrac{1}{8}\sqrt{3}\tag{11}$$

$$\tan\left(\frac{\pi}{9}\right)\tan\left(\frac{2\pi}{9}\right)\tan\left(\frac{4\pi}{9}\right)=\sqrt{3}.\tag{12}$$

see also NONAGON, STAR OF GOLIATH

Trigonometry Values—$\pi/10$

$$\begin{aligned}\sin\left(\frac{\pi}{10}\right) &= \sin\left(\frac{1}{2}\cdot\frac{\pi}{5}\right)=\sqrt{\tfrac{1}{2}\left[1-\cos\left(\frac{\pi}{5}\right)\right]}\\ &= \sqrt{\tfrac{1}{2}[1-\tfrac{1}{4}(1+\sqrt{5})]}=\tfrac{1}{4}(\sqrt{5}-1).\end{aligned}\tag{1}$$

So we have

$$\begin{aligned}\cos\left(\frac{\pi}{10}\right) &= \cos\left(\frac{1}{2}\cdot\frac{\pi}{5}\right)=\sqrt{\frac{1}{2}\left[1+\cos\left(\frac{\pi}{5}\right)\right]}\\ &= \sqrt{\tfrac{1}{2}[1+\tfrac{1}{4}(1+\sqrt{5})]}\\ &= \tfrac{1}{4}\sqrt{10+2\sqrt{5}},\end{aligned}\tag{2}$$

and

$$\tan\left(\frac{\pi}{10}\right)=\sqrt{\frac{3-\sqrt{5}}{5+\sqrt{5}}}=\tfrac{1}{5}\sqrt{25-10\sqrt{5}}.\tag{3}$$

Summarizing,

$$\sin\left(\frac{\pi}{10}\right)=\tfrac{1}{4}(\sqrt{5}-1)\tag{4}$$

$$\cos\left(\frac{\pi}{10}\right)=\tfrac{1}{4}\sqrt{10+2\sqrt{5}}\tag{5}$$

$$\tan\left(\frac{\pi}{10}\right)=\tfrac{1}{5}\sqrt{25-10\sqrt{5}}\tag{6}$$

$$\sin\left(\frac{3\pi}{10}\right)=\tfrac{1}{4}(1+\sqrt{5})\tag{7}$$

$$\cos\left(\frac{3\pi}{10}\right)=\tfrac{1}{4}(10-2\sqrt{5})\tag{8}$$

$$\tan\left(\frac{3\pi}{10}\right)=\tfrac{1}{5}\sqrt{25+10\sqrt{5}}.\tag{9}$$

An interesting near-identity is given by

$$\frac{1}{4}\left[\cos(\tfrac{1}{10})+\cosh(\tfrac{1}{10})+2\cos(\tfrac{1}{20}\sqrt{2})\cosh(\tfrac{1}{20}\sqrt{2})\right]\approx 1.\tag{10}$$

In fact, the left-hand side is approximately equal to $1+2.480\times 10^{-13}$.

see also DECAGON, DECAGRAM

Trigonometry Values—$\pi/11$

Trigonometric functions of $n\pi/11$ for n an integer cannot be expressed in terms of sums, products, and finite root extractions on *real* rational numbers because 11 is not a FERMAT PRIME. This also means that the UNDECAGON is not a CONSTRUCTIBLE POLYGON.

However, exact expressions involving roots of *complex* numbers can still be derived using the trigonometric identity

$$\begin{aligned}\sin(11\alpha) &= \sin(12\alpha-\alpha)\cos\alpha-\cos(12\alpha)\sin\alpha\\ &= 2\sin(6\alpha)\cos(6\alpha)\cos\alpha-[1-2\sin^2(6\alpha)]\sin\alpha.\end{aligned}\tag{1}$$

Using the identities from Beyer (1987, p. 139),

$$\sin(6\alpha)=\sin\alpha\cos\alpha[32\cos^4\alpha-32\cos^2\alpha+6]\tag{2}$$

$$\cos(6\alpha)=32\cos^6\alpha-48\cos^4\alpha+18\cos^2\alpha-1\tag{3}$$

gives

$$\begin{aligned}\sin(11\alpha) &= 2\cos^2\alpha\sin\alpha(32\cos^4\alpha-32\cos^2\alpha+6)\\ &\quad\times(32\cos^6\alpha-48\cos^4\alpha+18\cos^2\alpha-1)\\ &\quad-\sin\alpha[1-2\sin^2\alpha\cos^2\alpha(32\cos^4\alpha-32\cos^2\alpha+6)^2]\\ &= \sin\alpha(11-220\sin^2\alpha+1232\sin^4\alpha\alpha\\ &\quad-2816\sin^6\alpha+2816\sin^8-1024\sin^{10}\alpha).\end{aligned}\tag{4}$$

Now, let $\alpha\equiv\pi/11$ and $x\equiv\sin^2\alpha$, then

$$\begin{aligned}\sin\pi &= 0\\ &= 11-220x+1232x^2-2816x^3+2816x^4-1024x^5.\end{aligned}\tag{5}$$

This equation is an irreducible QUINTIC EQUATION, so an analytic solution involving FINITE ROOT extractions does not exist. The numerical ROOTS are $x = 0.07937$, 0.29229, 0.57115, 0.82743, 0.97974. So $\sin\alpha = 0.2817$, $\sin(2\alpha) = 0.5406$, $\sin(3\alpha) = 0.7557$, $\sin(4\alpha) = 0.9096$, $\sin(5\alpha) = 0.9898$. From one of NEWTON'S IDENTITIES,

$$\sin\left(\frac{\pi}{11}\right)\sin\left(\frac{2\pi}{11}\right)\sin\left(\frac{3\pi}{11}\right)\sin\left(\frac{4\pi}{11}\right)\sin\left(\frac{5\pi}{11}\right) = \sqrt{\frac{11}{1024}} = \frac{\sqrt{11}}{32} \quad (6)$$

$$\cos\left(\frac{\pi}{11}\right)\cos\left(\frac{2\pi}{11}\right)\cos\left(\frac{3\pi}{11}\right)\cos\left(\frac{4\pi}{11}\right)\cos\left(\frac{5\pi}{11}\right) = \frac{1}{32} \quad (7)$$

$$\tan\left(\frac{\pi}{11}\right)\tan\left(\frac{2\pi}{11}\right)\tan\left(\frac{3\pi}{11}\right)\tan\left(\frac{4\pi}{11}\right)\tan\left(\frac{5\pi}{11}\right) = \sqrt{11}. \quad (8)$$

The trigonometric functions of $\pi/11$ also obey the identity

$$\tan\left(\frac{3\pi}{11}\right) + 4\sin\left(\frac{2\pi}{11}\right) = \sqrt{11}. \quad (9)$$

see also UNDECAGON

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Trigonometry Values—$\pi/12$

$$\begin{aligned}\sin\left(\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= -\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) \\ &= -\tfrac{1}{2}\sqrt{2}(\tfrac{1}{2}) + \tfrac{1}{2}\sqrt{3}(\tfrac{1}{2}\sqrt{2}) \\ &= \tfrac{1}{4}(\sqrt{6} - \sqrt{2}). \quad (1)\end{aligned}$$

Similarly,

$$\begin{aligned}\cos\left(\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\ &= \tfrac{1}{2}(\tfrac{1}{2}\sqrt{2}) + \tfrac{1}{2}\sqrt{3}(-\tfrac{1}{2}\sqrt{2}) \\ &= \tfrac{1}{4}(\sqrt{6} + \sqrt{2}). \quad (2)\end{aligned}$$

Summarizing,

$$\sin\left(\frac{\pi}{12}\right) = \tfrac{1}{4}(\sqrt{6} - \sqrt{2}) \approx 0.25881 \quad (3)$$

$$\cos\left(\frac{\pi}{12}\right) = \tfrac{1}{4}(\sqrt{6} + \sqrt{2}) \approx 0.96592 \quad (4)$$

$$\tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3} \approx 0.26794 \quad (5)$$

$$\csc\left(\frac{\pi}{12}\right) = \sqrt{6} + \sqrt{2} \approx 3.86370 \quad (6)$$

$$\sec\left(\frac{\pi}{12}\right) = \sqrt{6} - \sqrt{2} \approx 1.03527 \quad (7)$$

$$\cot\left(\frac{\pi}{12}\right) = 2 + \sqrt{3} \approx 3.73205. \quad (8)$$

Trigonometry Values—$\pi/15$

$$\begin{aligned}\sin\left(\frac{\pi}{15}\right) &= \sin\left(\frac{\pi}{6} - \frac{\pi}{10}\right) \\ &= \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{10}\right) - \sin\left(\frac{\pi}{10}\right)\cos\left(\frac{\pi}{6}\right) \\ &= \frac{1}{2}\sqrt{\frac{1}{8}(5+\sqrt{5})} - \frac{\sqrt{3}}{2}\frac{1}{4}(\sqrt{5}-1) \\ &= \tfrac{1}{16}(2\sqrt{3} - 2\sqrt{15} + \sqrt{40 + 8\sqrt{5}}) \quad (1)\end{aligned}$$

and

$$\begin{aligned}\cos\left(\frac{\pi}{15}\right) &= \cos\left(\frac{\pi}{6} - \frac{\pi}{10}\right) \\ &= \cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{10}\right) + \sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{10}\right) \\ &= \frac{\sqrt{3}}{2}\sqrt{\frac{1}{8}(5+\sqrt{5})} + \frac{1}{2}\frac{1}{4}(\sqrt{5}-1) \\ &= \tfrac{1}{8}(\sqrt{30 + 6\sqrt{5}} + \sqrt{5} - 1). \quad (2)\end{aligned}$$

Summarizing,

$$\sin\left(\frac{\pi}{15}\right) = \tfrac{1}{16}(2\sqrt{3} - 2\sqrt{15} + \sqrt{40 + 8\sqrt{5}}) \approx 0.20791 \quad (3)$$

$$\sin\left(\frac{2\pi}{15}\right) = \tfrac{1}{8}(\sqrt{3} + \sqrt{15} - \sqrt{10 - 2\sqrt{5}}) \approx 0.40673 \quad (4)$$

$$\cos\left(\frac{\pi}{15}\right) = \tfrac{1}{8}(\sqrt{30 + 6\sqrt{5}} + \sqrt{5} - 1) \approx 0.97814 \quad (5)$$

$$\cos\left(\frac{2\pi}{15}\right) = \tfrac{1}{8}(\sqrt{30 - 6\sqrt{5}} + 1) \approx 0.91354 \quad (6)$$

$$\tan\left(\frac{\pi}{15}\right) = \tfrac{1}{2}(3\sqrt{3} - \sqrt{15} - \sqrt{50 - 22\sqrt{5}}) \approx 0.21255. \quad (7)$$

Trigonometry Values—$\pi/16$

$$\begin{aligned}\sin\left(\frac{\pi}{16}\right) &= \sin\left(\frac{1}{2}\cdot\frac{\pi}{8}\right) \\ &= \sqrt{\tfrac{1}{2}\left(1 - \cos\frac{\pi}{8}\right)} = \sqrt{\tfrac{1}{2}\left(1 - \tfrac{1}{2}\sqrt{2+\sqrt{2}}\right)} \\ &= \sqrt{\tfrac{1}{2} - \tfrac{1}{4}\sqrt{2+\sqrt{2}}} = \tfrac{1}{2}\sqrt{2 - \sqrt{2+\sqrt{2}}} \quad (1)\end{aligned}$$

$$\begin{aligned}\cos\left(\frac{\pi}{16}\right) &= \cos\left(\frac{1}{2}\cdot\frac{\pi}{8}\right) \\ &= \sqrt{\frac{1}{2}\left(1 + \cos\frac{\pi}{8}\right)} = \sqrt{\tfrac{1}{2}\left(1 + \tfrac{1}{2}\sqrt{2+\sqrt{2}}\right)} \\ &= \sqrt{\tfrac{1}{2} + \tfrac{1}{4}\sqrt{2+\sqrt{2}}} = \tfrac{1}{2}\sqrt{2 + \sqrt{2+\sqrt{2}}} \quad (2)\end{aligned}$$

$$\begin{aligned}\tan\left(\frac{\pi}{16}\right) &= \sqrt{\frac{2 - \sqrt{2+\sqrt{2}}}{2 + \sqrt{2+\sqrt{2}}}} \\ &= \sqrt{4 + 2\sqrt{2}} - \sqrt{2} - 1. \quad (3)\end{aligned}$$

Summarizing,

$$\sin\left(\frac{\pi}{16}\right) = \tfrac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}} \approx 0.19509 \tag{4}$$

$$\sin\left(\frac{3\pi}{16}\right) = \tfrac{1}{2}\sqrt{2-\sqrt{2-\sqrt{2}}} \approx 0.55557 \tag{5}$$

$$\cos\left(\frac{\pi}{16}\right) = \tfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}} \approx 0.98079 \tag{6}$$

$$\cos\left(\frac{3\pi}{16}\right) = \tfrac{1}{2}\sqrt{2+\sqrt{2-\sqrt{2}}} \approx 0.83147 \tag{7}$$

$$\tan\left(\frac{\pi}{16}\right) = \sqrt{4+2\sqrt{2}}-\sqrt{2}-1 \approx 0.19891. \tag{8}$$

Trigonometry Values—$\pi/17$

Rather surprisingly, trigonometric functions of $n\pi/17$ for n an integer can be expressed in terms of sums, products, and finite root extractions because 17 is a FERMAT PRIME. This makes the HEPTADECAGON a CONSTRUCTIBLE, as first proved by Gauss. Although Gauss did not actually explicitly provide a construction, he did derive the trigonometric formulas below using a series of intermediate variables from which the final expressions were then built up.

Let

$$\begin{aligned}
\epsilon &\equiv \sqrt{17+\sqrt{17}}\\
\epsilon^* &\equiv \sqrt{17-\sqrt{17}}\\
\alpha &\equiv \sqrt{\sqrt{34}+6\sqrt{17}+(\sqrt{34}-\sqrt{2})\epsilon^*-8\sqrt{2}\,\epsilon}\\
\beta &\equiv 2\sqrt{17+3\sqrt{17}-2\sqrt{2}\,\epsilon-\sqrt{2}\,\epsilon^*}.
\end{aligned}$$

Then

$$\begin{aligned}
\sin\left(\frac{\pi}{17}\right) &= \tfrac{1}{8}[34-2\sqrt{17}-2\sqrt{2}\,\epsilon^*\\
&\quad -2\sqrt{68+12\sqrt{17}+2\sqrt{2}(\sqrt{17}-1)\epsilon^*-16\sqrt{2}\,\epsilon}\,]^{1/2}\\
&\approx 0.18375\\
\cos\left(\frac{\pi}{17}\right) &= \tfrac{1}{8}[30+2\sqrt{17}+2\sqrt{2}\,\epsilon^*\\
&\quad +2\sqrt{68+12\sqrt{17}+2\sqrt{2}(\sqrt{17}-1)\epsilon^*-16\sqrt{2}\,\epsilon}\,]^{1/2}\\
&\approx 0.98297\\
\sin\left(\frac{2\pi}{17}\right) &= \tfrac{1}{16}[136-8\sqrt{17}+4\sqrt{2}(1-\sqrt{17})\epsilon^*+16\sqrt{2}\,\epsilon\\
&\quad +2(\sqrt{2}-\sqrt{34}-2\epsilon^*)\sqrt{34+6\sqrt{17}+(\sqrt{34}-\sqrt{2})\epsilon^*-8\sqrt{2}\epsilon}\,]^{1/2}\\
&\approx 0.36124\\
\cos\left(\frac{2\pi}{17}\right) &= \tfrac{1}{16}[-1+\sqrt{17}+\sqrt{2}\epsilon^*\\
&\quad +\sqrt{68+12\sqrt{17}-2\sqrt{2}(1-\sqrt{17})\epsilon^*-16\sqrt{2}\,\epsilon}\,]\\
&\approx 0.0.93247\\
\sin\left(\frac{4\pi}{17}\right) &= \tfrac{1}{128}(-\sqrt{2}+\sqrt{34}+2\epsilon^*+2\alpha)\\
&\quad \times\sqrt{68-4\sqrt{17}-2(\sqrt{34}-\sqrt{2})\epsilon^*+8\sqrt{2}\,\epsilon+\alpha(\sqrt{2}-\sqrt{34}-2\epsilon^*)}\\
&\approx 0.0.67370\\
\sin\left(\frac{8\pi}{17}\right) &= \tfrac{1}{16}[136-8\sqrt{17}+8\sqrt{2}\epsilon-2(\sqrt{34}-3\sqrt{2})\epsilon^*\\
&\quad -2\beta(1-\sqrt{17}-\sqrt{2}\epsilon^*)\,]^{1/2}\\
&\approx 0.99573\\
\cos\left(\frac{8\pi}{17}\right) &= \tfrac{1}{16}(-1+\sqrt{17}+\sqrt{2}\,\epsilon^*\\
&\quad -2\sqrt{17+3\sqrt{17}-\sqrt{2}\,\epsilon^*-2\sqrt{2}\,\epsilon}\,).\\
&\approx 0.09227
\end{aligned}$$

There are some interesting analytic formulas involving the trigonometric functions of $n\pi/17$. Define

$$\begin{aligned}
P(x) &\equiv (x-1)(x-2)(x^2+1)\\
g_1(x) &\equiv \frac{2+\sqrt{P(x)}}{1-x}\\
g_4(x) &\equiv \frac{2-\sqrt{P(x)}}{1-x}\\
f_i(x) &\equiv \tfrac{1}{4}[g_i(x)-1]\\
a &\equiv \tfrac{1}{4}\tan^{-1}4,
\end{aligned}$$

where $i=1$ or 4. Then

$$\begin{aligned}
f_1(\tan a) &= \cos\left(\frac{2\pi}{17}\right)\\
f_4(\tan a) &= \cos\left(\frac{8\pi}{17}\right).
\end{aligned}$$

see also CONSTRUCTIBLE POLYGON, FERMAT PRIME, HEPTADECAGON

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Trigonometry Values—$\pi/18$

The exact values of $\cos(\pi/18)$ and $\sin(\pi/18)$ are given by infinite NESTED RADICALS.

$$\sin\left(\frac{\pi}{18}\right) = \tfrac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\ldots}}}} \approx 0.17365,$$

where the sequence of signs $+$, $+$, $-$ repeats with period 3, and

$$\cos\left(\frac{\pi}{18}\right) = \tfrac{1}{6}\sqrt{3}\left(\sqrt{8-\sqrt{8-\sqrt{8+\sqrt{8-\ldots}}}}+1\right) \approx 0.98481,$$

where the sequence of signs $-$, $-$, $+$ repeats with period 3.

Trigonometry Values—$\pi/20$

$$\sin\left(\frac{\pi}{20}\right) = \sin\left(\frac{1}{2}\frac{\pi}{10}\right) = \sqrt{\frac{1}{2}\left(1-\cos\frac{\pi}{10}\right)}$$
$$= \tfrac{1}{4}\sqrt{8-2\sqrt{10+2\sqrt{5}}}$$
$$\approx 0.15643\ldots \quad (1)$$
$$\cos\left(\frac{\pi}{20}\right) = \cos\left(\frac{1}{2}\frac{\pi}{10}\right) = \sqrt{\frac{1}{2}\left(1+\cos\frac{\pi}{10}\right)}$$
$$= \tfrac{1}{4}\sqrt{8+2\sqrt{10+2\sqrt{5}}}$$
$$\approx 0.98768\ldots \quad (2)$$
$$\tan\left(\frac{\pi}{20}\right) = 1+\sqrt{5}-\sqrt{5+2\sqrt{5}}$$
$$\approx 0.15838. \quad (3)$$

An interesting near-identity is given by

$$\frac{1}{4}\left[\cos(\tfrac{1}{10}) + \cosh(\tfrac{1}{10}) + 2\cos(\tfrac{1}{20}\sqrt{2})\cosh(\tfrac{1}{20}\sqrt{2})\right] \approx 1. \quad (4)$$

In fact, the left-hand side is approximately equal to $1 + 2.480 \times 10^{-13}$.

Trigonometry Values—0

By the definition of the trigonometric functions,

$$\sin 0 = 0$$
$$\cos 0 = 1$$
$$\tan 0 = 0$$
$$\csc 0 = \infty$$
$$\sec 0 = 1$$
$$\cot 0 = \infty.$$

Trigyrate Rhombicosidodecahedron

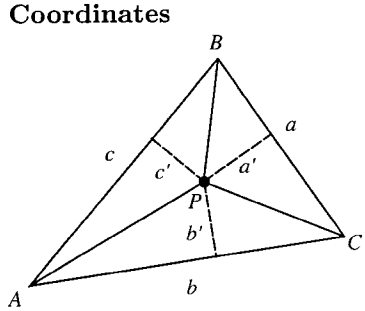
see JOHNSON SOLID

Trihedron

The TRIPLE of unit ORTHOGONAL VECTORS **T**, **N**, and **B** (TANGENT VECTOR, NORMAL VECTOR, and BINORMAL VECTOR).

see also BINORMAL VECTOR, NORMAL VECTOR, TANGENT VECTOR

Trilinear Coordinates



Given a TRIANGLE ΔABC, the trilinear coordinates of a point P with respect to ΔABC are an ordered TRIPLE of numbers, each of which is PROPORTIONAL to the directed distance from P to one of the side lines. Trilinear coordinates are denoted $\alpha : \beta : \gamma$ or (α, β, γ) and also are known as BARYCENTRIC COORDINATES, HOMOGENEOUS COORDINATES, or "trilinears."

In trilinear coordinates, the three VERTICES A, B, and C are given by $1:0:0$, $0:1:0$, and $0:0:1$. Let the point P in the above diagram have trilinear coordinates $\alpha : \beta : \gamma$ and lie at distances a', b', and c' from the sides BC, AC, and AB, respectively. Then the distances $a' = k\alpha$, $b' = k\beta$, and $c' = k\gamma$ can be found by writing Δ_a for the AREA of ΔBPC, and similarly for Δ_b and Δ_c. We then have

$$\Delta = \Delta_a + \Delta_b + \Delta_c = \tfrac{1}{2}aa' + \tfrac{1}{2}bb' + \tfrac{1}{2}cc'$$
$$= \tfrac{1}{2}(ak\alpha + bk\beta + ck\gamma) = \tfrac{1}{2}k(a\alpha + b\beta + c\gamma). \quad (1)$$

so

$$k \equiv \frac{2\Delta}{a\alpha + b\beta + c\gamma}, \quad (2)$$

where Δ is the AREA of ΔABC and a, b, and c are the lengths of its sides. When the values of the coordinates are taken as the actual lengths (i.e., the trilinears are chosen so that $k = 1$), the coordinates are known as EXACT TRILINEAR COORDINATES.

Trilinear coordinates are unchanged when each is multiplied by any constant μ, so

$$t_1 : t_2 : t_3 = \mu t_1 : \mu t_2 : \mu t_3. \quad (3)$$

When normalized so that

$$t_1 + t_2 + t_3 = 1, \quad (4)$$

trilinear coordinates are called AREAL COORDINATES. The trilinear coordinates of the line

$$ux + vy + wz = 0 \quad (5)$$

are

$$u : v : w = ad_A : bd_B : cd_C, \quad (6)$$

where d_i is the POINT-LINE DISTANCE from VERTEX A to the LINE.

Trilinear coordinates for some common POINTS are summarized in the following table, where A, B, and C are the angles at the corresponding vertices and a, b, and c are the opposite side lengths.

Point	Triangle Center Function
centroid M	$\csc A$, $1/a$
circumcenter O	$\cos A$
de Longchamps point	$\cos A - \cos B \cos C$
equal detour point	$\sec(\frac{1}{2}A)\cos(\frac{1}{2}B)\cos(\frac{1}{2}C) + 1$
Feuerbach point F	$1 - \cos(B - C)$
incenter I	1
isoperimetric point	$\sec(\frac{1}{2}A)\cos(\frac{1}{2}B)\cos(\frac{1}{2}C) - 1$
Lemoine point	a
nine-point center N	$\cos(B - C)$
orthocenter H	$\cos B \cos C$
vertex A	$1:0:0$
vertex B	$0:1:0$
vertex C	$0:0:1$

To convert trilinear coordinates to a vector position for a given triangle specified by the x- and y-coordinates of its axes, pick two UNIT VECTORS along the sides. For instance, pick

$$\hat{\mathbf{a}} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{7}$$

$$\hat{\mathbf{c}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{8}$$

where these are the UNIT VECTORS BC and AB. Assume the TRIANGLE has been labeled such that $A = \mathbf{x}_1$ is the lower rightmost VERTEX and $C = \mathbf{x}_2$. Then the VECTORS obtained by traveling l_a and l_c along the sides and then inward PERPENDICULAR to them must meet

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + l_c \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - k\gamma \begin{bmatrix} c_2 \\ -c_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + l_a \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - k\alpha \begin{bmatrix} a_2 \\ -a_1 \end{bmatrix}. \tag{9}$$

Solving the two equations

$$x_1 + l_c c_1 - k\gamma c_2 = x_2 l_a a_1 - k\alpha a_2 \tag{10}$$

$$y_1 + l_c c_2 + k\gamma c_1 = y_2 l_a a_2 + k\alpha a_1, \tag{11}$$

gives

$$l_c = \frac{k\alpha(a_1c_1 + a_2c_2) - \gamma k({c_1}^2 + {c_2}^2) + c_2(x_1 - x_2) + c_1(y_2 - y_1)}{a_1c_2 - a_2c_1} \tag{12}$$

$$l_a = \frac{k\alpha({a_1}^2 + {a_2}^2) - \gamma k(a_1c_1 + a_2c_2) + a_2(x_1 - x_2) + a_1(y_2 - y_1)}{a_1c_2 - a_2c_1}. \tag{13}$$

But $\hat{\mathbf{a}}$ and $\hat{\mathbf{c}}$ are UNIT VECTORS, so

$$l_c = \frac{k\alpha(a_1c_1 + a_2c_2) - \gamma k + c_2(x_1 - x_2) + c_1(y_2 - y_1)}{a_1c_2 - a_2c_1} \tag{14}$$

$$l_a = \frac{k\alpha - \gamma k(a_1c_1 + a_2c_2) + a_2(x_1 - x_2) + a_1(y_2 - y_1)}{a_1c_2 - a_2c_1}. \tag{15}$$

And the VECTOR coordinates of the point $\alpha : \beta : \gamma$ are then

$$\mathbf{x} = \mathbf{x}_1 + l_c \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - k\gamma \begin{bmatrix} c_2 \\ -c_1 \end{bmatrix}. \tag{16}$$

see also AREAL COORDINATES, EXACT TRILINEAR COORDINATES, ORTHOCENTRIC COORDINATES, POWER CURVE, QUADRIPLANAR COORDINATES, TRIANGLE, TRILINEAR POLAR

References

Boyer, C. B. *History of Analytic Geometry.* New York: Yeshiva University, 1956.

Casey, J. "The General Equation—Trilinear Co-Ordinates." Ch. 10 in *A Treatise on the Analytical Geometry of the Point, Line, Circle, and Conic Sections, Containing an Account of Its Most Recent Extensions, with Numerous Examples, 2nd ed., rev. enl.* Dublin: Hodges, Figgis, & Co., pp. 333–348, 1893.

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, pp. 67–71, 1959.

Coxeter, H. S. M. *Introduction to Geometry, 2nd ed.* New York: Wiley, 1969.

Coxeter, H. S. M. "Some Applications of Trilinear Coordinates." *Linear Algebra Appl.* **226–228**, 375–388, 1995.

Kimberling, C. "Triangle Centers and Central Triangles." *Congr. Numer.* **129**, 1–295, 1998.

Trilinear Line

A LINE is given in TRILINEAR COORDINATES by

$$l\alpha + m\beta + n\gamma = 0.$$

see also LINE, TRILINEAR COORDINATES

Trilinear Polar

Given a TRIANGLE CENTER $X = l : m : n$, the line

$$l\alpha + m\beta + n\gamma = 0$$

is called the trilinear polar of X^{-1} and is denoted L.

see also CHASLES'S POLARS THEOREM

Trillion

The word trillion denotes different numbers in American and British usage. In the American system, one trillion equals 10^{12}. In the British, French, and German systems, one trillion equals 10^{18}.

see also BILLION, LARGE NUMBER, MILLION

Trimagic Square

If replacing each number by its square or cube in a MAGIC SQUARE produces another MAGIC SQUARE, the square is said to be a trimagic square. Trimagic squares of order 32, 64, 81, and 128 are known. Tarry gave a method for constructing a trimagic square of order 128, Cazalas a method for trimagic squares of orders 64 and 81, and R. V. Heath a method for constructing an order 64 trimagic square which is different from Cazalas's (Kraitchik 1942).

Trimagic squares are also called TREBLY MAGIC SQUARES, and are 3-MULTIMAGIC SQUARES.

see also BIMAGIC SQUARE, MAGIC SQUARE, MULTIMAGIC SQUARE

References
Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 212–213, 1987.
Kraitchik, M. "Multimagic Squares." §7.10 in *Mathematical Recreations.* New York: W. W. Norton, pp. 176–178, 1942.

Trimean

The trimean is defined to be

$$TM \equiv \tfrac{1}{4}(H_1 + 2M + H_2),$$

where H_i are the HINGES and M is the MEDIAN. Press *et al.* (1992) call this TUKEY'S TRIMEAN. It is an L-ESTIMATE.

see also HINGE, L-ESTIMATE, MEAN, MEDIAN (STATISTICS)

References
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, p. 694, 1992.
Tukey, J. W. *Explanatory Data Analysis.* Reading, MA: Addison-Wesley, pp. 46–47, 1977.

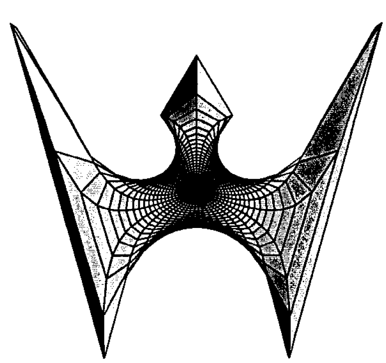
Trimorphic Number

A number n such that the last digits of n^3 are the same as n. 49 is trimorphic since $49^3 = 117649$ (Wells 1986, p. 124). The first few are 1, 4, 5, 6, 9, 24, 25, 49, 51, 75, 76, 99, 125, 249, 251, 375, 376, 499, ...

see also AUTOMORPHIC NUMBER, NARCISSISTIC NUMBER, SUPER-3 NUMBER

References
Wells, D. *The Penguin Dictionary of Curious and Interesting Numbers.* Middlesex, England: Penguin Books, 1986.

Trinoid



A MINIMAL SURFACE discovered by L. P. M. Jorge and W. Meeks III in 1983 with ENNEPER-WEIERSTRAß PARAMETERIZATION

$$f = \frac{1}{(\zeta^3 - 1)^2} \tag{1}$$

$$g = \zeta^2 \tag{2}$$

(Dickson 1990). Explicitly, it is given by

$$x = \Re\left[\frac{re^{i\theta}}{3(1 + re^{i\theta} + r^2e^{2i\theta})} - \frac{4\ln(re^{i\theta} - 1)}{9} + \frac{2\ln(1 + re^{i\theta} + r^2e^{2i\theta})}{9}\right] \tag{3}$$

$$y = -\tfrac{1}{9}\Im\left[\frac{-3re^{i\theta}(1 + re^{i\theta})}{r^3e^{3i\theta} - 1} + \frac{4\sqrt{3}\,(r^3e^{3i\theta} - 1)\tan^{-1}\left(\frac{1+2re^{i\theta}}{\sqrt{3}}\right)}{r^3e^{3i\theta} - 1}\right] \tag{4}$$

$$z = \Re\left[-\frac{2}{3} - \frac{2}{3(r^3e^{3i\theta} - 1)}\right], \tag{5}$$

for $\theta \in [0, 2\pi)$ and $r \in [0, 4]$.

see also MINIMAL SURFACE

References
Dickson, S. "Minimal Surfaces." *Mathematica J.* **1**, 38–40, 1990.
Wolfram Research "Mathematica Version 2.0 Graphics Gallery." `http://www.mathsource.com/cgi-bin/MathSource/Applications/Graphics/3D/0207-155`.

Trinomial

A POLYNOMIAL with three terms.

see also BINOMIAL, MONOMIAL, POLYNOMIAL

Trinomial Identity

$$(x^2 + axy + by^2)(t^2 + atu + bu^2) = r^2 + ars + bs^2, \tag{1}$$

where

$$r = xt - byu \tag{2}$$

$$s = yt + xu + ayu. \tag{3}$$

Trinomial Triangle

The NUMBER TRIANGLE obtained by starting with a row containing a single "1" and the next row containing three 1s and then letting subsequent row elements be computed by summing the elements above to the left, directly above, and above to the right:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & 1 & 1 & & & \\ & & 1 & 2 & 3 & 2 & 1 & & \\ & 1 & 3 & 6 & 7 & 6 & 3 & 1 & \\ 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \end{array}$$

(Sloane's A027907). The nth row can also be obtained by expanding $(1+x+x^2)^n$ and taking coefficients:

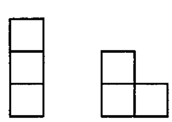
$$(1+x+x^2)^0 = 1$$
$$(1+x+x^2)^1 = 1+x+x^2$$
$$(1+x+x^2)^2 = 1+2x+3x^2+2x^3+x^4$$

and so on.

see also PASCAL'S TRIANGLE

References
Sloane, N. J. A. Sequence A027907 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Triomino



The two 3-POLYOMINOES are called triominoes, and are also known as the TROMINOES. The left triomino above is "STRAIGHT," while the right triomino is called "right" or L-.

see also L-POLYOMINO, POLYOMINO, STRAIGHT POLYOMINO

References
Gardner, M. "Polyominoes." Ch. 13 in *The Scientific American Book of Mathematical Puzzles & Diversions.* New York: Simon and Schuster, pp. 124–140, 1959.
Hunter, J. A. H. and Madachy, J. S. *Mathematical Diversions.* New York: Dover, pp. 80–81, 1975.
Lei, A. "Tromino." `http://www.cs.ust.hk/~philipl/omino/tromino.html`

Trip-Let

A 3-dimensional solid which is shaped in such a way that its projections along three mutually perpendicular axes are three different letters of the alphabet. Hofstadter (1989) has constructed such a solid for the letters G, E, and B.

see also CORK PLUG

References
Hofstadter, D. R. *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Vintage Books, cover and pp. xiv, 1, and 273, 1989.

Triple

A group of three elements, also called a TRIAD.

see also AMICABLE TRIPLE, MONAD, PAIR, PYTHAGOREAN TRIPLE, QUADRUPLET, QUINTUPLET, TETRAD, TRIAD, TWINS

Triple-Free Set

A SET of POSITIVE integers is called weakly triple-free if, for any integer x, the SET $\{x, 2x, 3x\} \not\subset S$. It is called strongly triple-free if $x \in S$ IMPLIES $2x \notin S$ and $3x \notin S$. Define

$$p(n) = \max\{|S| : S \subset \{1, 2, \ldots, n\} \text{ is weakly triple-free}\}$$
$$q(n) = \max\{|S| : S \subset \{1, 2, \ldots, n\} \text{ is strongly triple-free}\},$$

where $|S|$ denotes the CARDINALITY of S, then

$$\lim_{n\to\infty} \frac{p(n)}{n} \geq \tfrac{4}{5}$$

and

$$\lim_{n\to\infty} \frac{q(n)}{n} = 0.6134752692\ldots$$

(Finch).

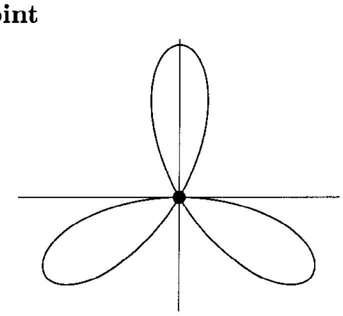
see also DOUBLE-FREE SET

References
Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/triple/triple.html`.

Triple Jacobi Product

see JACOBI TRIPLE PRODUCT

Triple Point



A point where a curve intersects itself along three arcs. The above plot shows the triple point at the ORIGIN of the TRIFOLIUM $(x^2+y^2)^2 + 3x^2y - y^3 = 0$.

see also DOUBLE POINT, QUADRUPLE POINT

References
Walker, R. J. *Algebraic Curves.* New York: Springer-Verlag, pp. 57–58, 1978.

Triple Scalar Product

see SCALAR TRIPLE PRODUCT

Triple Vector Product

see VECTOR TRIPLE PRODUCT

Triplet

see TRIPLE

Triplicate-Ratio Circle

see LEMOINE CIRCLE

Trisected Perimeter Point

A triangle center which has a TRIANGLE CENTER FUNCTION

$$\alpha = bc(v - c + a)(v - a + b),$$

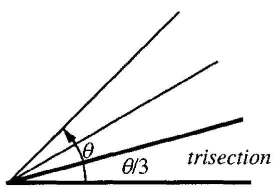
where v is the unique REAL ROOT of

$$2x^3 - 3(a+b+c)x^2 + (a^2 + b^2 + c^2 + 8bc + 8ca + 8ab)x - (b^2c + c^2a + a^2b + 5bc^2 + 5ca^2 + 5ab^2 + 9abc) = 0.$$

References

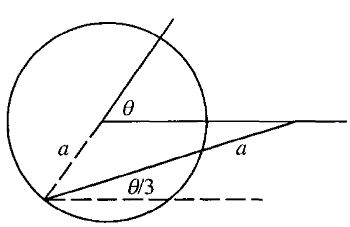
Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.

Trisection



Angle trisection is the division of an *arbitrary* ANGLE into three equal ANGLES. It was one of the three GEOMETRIC PROBLEMS OF ANTIQUITY for which solutions using only COMPASS and STRAIGHTEDGE were sought. The problem was algebraically proved impossible by Wantzel (1836).

Although trisection is not possible for a general ANGLE using a Greek construction, there are some specific angles, such as $\pi/2$ and π radians (90° and 180°, respectively), which *can* be trisected. Furthermore, some ANGLES are geometrically trisectable, but cannot be constructed in the first place, such as $3\pi/7$ (Honsberger 1991). In addition, trisection of an arbitrary angle *can* be accomplished using a *marked* RULER (a NEUSIS CONSTRUCTION) as illustrated below (Courant and Robbins 1996).



An ANGLE can also be divided into three (or any WHOLE NUMBER) of equal parts using the QUADRATRIX OF HIPPIAS or TRISECTRIX.

see also ANGLE BISECTOR, MACLAURIN TRISECTRIX, QUADRATRIX OF HIPPIAS, TRISECTRIX

References

Bogomolny, A. "Angle Trisection." http://www.cut-the-knot.com/pythagoras/archi.html.
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 190–191, 1996.
Courant, R. and Robbins, H. "Trisecting the Angle." §3.3.3 in *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, pp. 137–138, 1996.
Coxeter, H. S.M. "Angle Trisection." §2.2 in *Introduction to Geometry, 2nd ed.* New York: Wiley, p. 28, 1969.
Dixon, R. *Mathographics.* New York: Dover, pp. 50–51, 1991.
Dörrie, H. "Trisection of an Angle." §36 in *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, pp. 172–177, 1965.
Dudley, U. *The Trisectors.* Washington, DC: Math. Assoc. Amer., 1994.
Geometry Center. "Angle Trisection." http://www.geom.umn.edu:80/docs/forum/angtri/.
Honsberger, R. *More Mathematical Morsels.* Washington, DC: Math. Assoc. Amer., pp. 25–26, 1991.
Ogilvy, C. S. "Angle Trisection." *Excursions in Geometry.* New York: Dover, pp. 135–141, 1990.
Wantzel, P. L. "Recherches sur les moyens de reconnaître si un Problème de Géométrie peut se résoudre avec la règle et le compas." *J. Math. pures appliq.* **1**, 366–372, 1836.

Trisectrix

see CATALAN'S TRISECTRIX, MACLAURIN TRISECTRIX

Trisectrix of Catalan

see CATALAN'S TRISECTRIX

Trisectrix of Maclaurin

see MACLAURIN TRISECTRIX

Triskaidecagon

see TRIDECAGON

Triskaidekaphobia

The number 13 is traditionally associated with bad luck. This superstition leads some people to fear or avoid anything involving this number, a condition known as triskaidekaphobia. Triskaidekaphobia leads to interesting practices such as the numbering of floors as 1, 2, ..., 11, 12, 14, 15, ..., *omitting the number 13,* in many high-rise hotels.

see also 13, BAKER'S DOZEN, FRIDAY THE THIRTEENTH, TRISKAIDEKAPHOBIA

Tritangent

The tritangent of a CUBIC SURFACE is a PLANE which intersects the surface in three mutually intersecting lines. Each intersection of two lines is then a tangent point of the surface.

see also CUBIC SURFACE

References

Hunt, B. "Algebraic Surfaces." http://www.mathematik.uni-kl.de/~wwwagag/Galerie.html.

Tritangent Triangle

see EXCENTRAL TRIANGLE

Trivial

According to the Nobel Prize-winning physicist Richard Feynman (1985), mathematicians designate any THEOREM as "trivial" once a proof has been obtained—no matter how difficult the theorem was to prove in the first place. There are therefore exactly two types of true mathematical propositions: trivial ones, and those which have not yet been proven.

see also PROOF, THEOREM

References

Feynman, R. P. and Leighton, R. *Surely You're Joking, Mr. Feynman!* New York: Bantam Books, 1985.

Trivialization

In the definition of a FIBER BUNDLE $f : E \to B$, the homeomorphisms $g_U : f^{-1}(U) \to U \times F$ that commute with projection are called local trivializations for the FIBER BUNDLE f.

see also FIBER BUNDLE

Trochoid

The curve described by a point at a distance b from the center of a rolling CIRCLE of RADIUS a.

$$x = a\phi - b\sin\phi$$
$$y = a - b\cos\phi.$$

If $b < a$, the curve is a CURTATE CYCLOID. If $b = a$, the curve is a CYCLOID. If $b > a$, the curve is a PROLATE CYCLOID.

see also CURTATE CYCLOID, CYCLOID, PROLATE CYCLOID

References

Lee, X. "Trochoid." `http://www.best.com/~xah/SpecialPlaneCurves_dir/Trochoid_dir/trochoid.html`.

Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, pp. 46–50, 1991.

Yates, R. C. "Trochoids." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 233–236, 1952.

Tromino

see TRIOMINO

True

A statement which is rigorously known to be correct. A statement which is not true is called FALSE, although certain statements can be proved to be rigorously UNDECIDABLE within the confines of a given set of assumptions and definitions. Regular two-valued LOGIC allows statements to be only true or FALSE, but FUZZY LOGIC treats "truth" as a continuum which can have any value between 0 and 1.

see also ALETHIC, FALSE, FUZZY LOGIC, LOGIC, TRUTH TABLE, UNDECIDABLE

Truncate

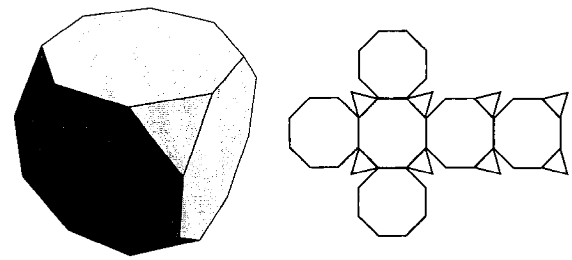
To truncate a REAL NUMBER is to remove its nonintegral part. Truncation of a number x therefore corresponds to taking the FLOOR FUNCTION $\lfloor x \rfloor$.

see also CEILING FUNCTION, FLOOR FUNCTION, NINT, ROUND

Truncated Cone

see CONICAL FRUSTUM

Truncated Cube



An ARCHIMEDEAN SOLID whose DUAL POLYHEDRON is the TRIAKIS OCTAHEDRON. It has SCHLÄFLI SYMBOL t{4, 3}. It is also UNIFORM POLYHEDRON U_9 and has WYTHOFF SYMBOL $2\,3\,|\,4$. Its faces are $8\{3\}+6\{8\}$. The INRADIUS, MIDRADIUS, and CIRCUMRADIUS for $a = 1$ are

$$r = \tfrac{1}{17}(5+2\sqrt{2})\sqrt{7+4\sqrt{2}} \approx 1.63828$$
$$\rho = \tfrac{1}{2}(2+\sqrt{2}) \approx 1.70711$$
$$R = \tfrac{1}{2}\sqrt{7+4\sqrt{2}} \approx 1.77882.$$

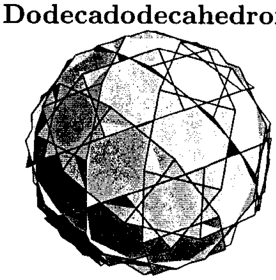
References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 138, 1987.

Truncated Cuboctahedron

see GREAT RHOMBICUBOCTAHEDRON (ARCHIMEDEAN)

Truncated Dodecadodecahedron



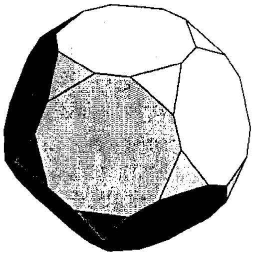
The UNIFORM POLYHEDRON U_{59}, also called the QUASITRUNCATED DODECAHEDRON, whose DUAL POLYHEDRON is the MEDIAL DISDYAKIS TRIACONTAHEDRON. It has SCHLÄFLI SYMBOL t'$\left\{\begin{matrix}\frac{5}{2}\\5\end{matrix}\right\}$ and WYTHOFF SYMBOL $2\,\frac{5}{3}\,|\,5$. Its faces are $12\{10\}+30\{4\}+12\{\frac{10}{3}\}$. Its CIRCUMRADIUS for $a = 1$ is

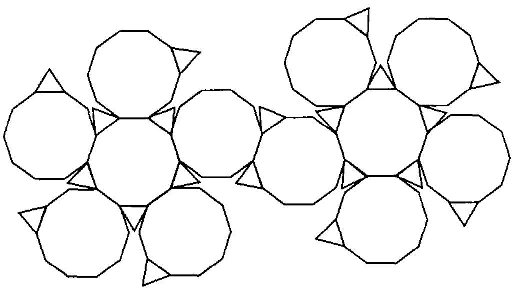
$$R = \tfrac{1}{2}\sqrt{11}.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 152–153, 1989.

Truncated Dodecahedron

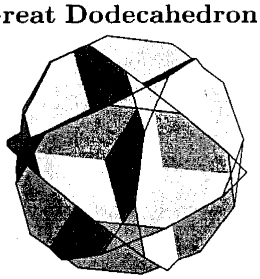




An ARCHIMEDEAN SOLID whose DUAL POLYHEDRON is the TRIAKIS ICOSAHEDRON. It has SCHLÄFLI SYMBOL t{5,3}. It is also UNIFORM POLYHEDRON U_{26} and has WYTHOFF SYMBOL $2\,3\,|\,5$. Its faces are $20\{3\}+12\{10\}$. The INRADIUS, MIDRADIUS, and CIRCUMRADIUS for $a=1$ are

$$r = \tfrac{5}{488}(17\sqrt{2}+3\sqrt{10})\sqrt{37+15\sqrt{5}} \approx 2.88526$$
$$\rho = \tfrac{1}{4}(5+3\sqrt{5}) \approx 2.92705$$
$$R = \tfrac{1}{4}\sqrt{74+30\sqrt{5}} \approx 2.96945.$$

Truncated Great Dodecahedron



The UNIFORM POLYHEDRON U_{37} whose DUAL POLYHEDRON is the SMALL STELLAPENTAKIS DODECAHEDRON. It has SCHLÄFLI SYMBOL $t\{5,\tfrac{5}{2}\}$. It has WYTHOFF SYMBOL $2\,\tfrac{5}{2}\,5$. Its faces are $12\{\tfrac{5}{2}\}+12\{10\}$. Its CIRCUMRADIUS for $a=1$ is

$$R = \tfrac{1}{4}\sqrt{34+10\sqrt{5}}.$$

see also GREAT ICOSAHEDRON

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, p. 115, 1971.

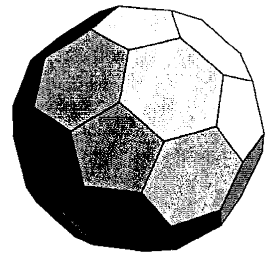
Truncated Great Icosahedron

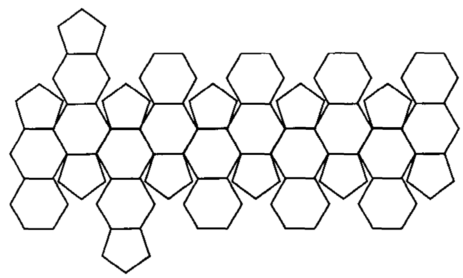
see GREAT TRUNCATED ICOSAHEDRON

Truncated Hexahedron

see TRUNCATED CUBE

Truncated Icosahedron





An ARCHIMEDEAN SOLID used in the construction of SOCCER BALLS. Its DUAL POLYHEDRON is the PENTAKIS DODECAHEDRON. It has SCHLÄFLI SYMBOL t{3,5}. It is also UNIFORM POLYHEDRON U_{25} and has WYTHOFF SYMBOL $2\,5\,|\,3$. Its faces are $20\{6\}+12\{5\}$. The INRADIUS, MIDRADIUS, and CIRCUMRADIUS for $a=1$ are

$$r = \tfrac{9}{872}(21+\sqrt{5})\sqrt{58+18\sqrt{5}} \approx 2.37713$$
$$\rho = \tfrac{3}{4}(1+\sqrt{5}) \approx 2.42705$$
$$R = \tfrac{1}{4}\sqrt{58+18\sqrt{5}} \approx 2.47802.$$

Truncated Icosidodecahedron

see GREAT RHOMBICOSIDODECAHEDRON (ARCHIMEDEAN)

Truncated Octahedral Number

A FIGURATE NUMBER which is constructed as an OCTAHEDRAL NUMBER with a SQUARE PYRAMID removed from each of the six VERTICES,

$$TO_n = O_{3n-2} - 6P_{n-1} = \tfrac{1}{3}(3n-2)[2(3n-2)^2+1],$$

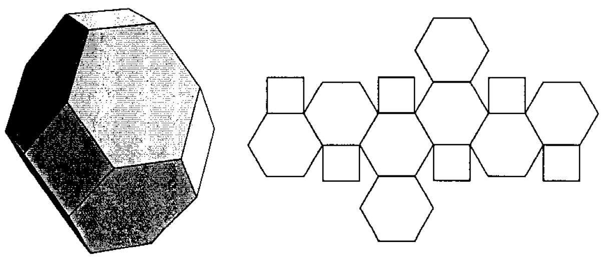
where O_n is an OCTAHEDRAL NUMBER and P_n is a PYRAMIDAL NUMBER. The first few are 1, 38, 201, 586, ... (Sloane's A005910). The GENERATING FUNCTION for the truncated octahedral numbers is

$$\frac{x(6x^3+55x^2+34x+1)}{(x-1)^4} = x + 38x^2 + 201x^3 + \ldots.$$

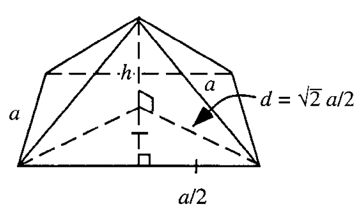
see also OCTAHEDRAL NUMBER

References
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, p. 52, 1996.
Sloane, N. J. A. Sequence A005910/M5266 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Truncated Octahedron



An ARCHIMEDEAN SOLID, also known as the MECON, whose DUAL POLYHEDRON is the TETRAKIS HEXAHEDRON. It is also UNIFORM POLYHEDRON U_8 and has SCHLÄFLI SYMBOL t{3,4} and WYTHOFF SYMBOL 2 4 | 3. The faces of the truncated octahedron are 8{6}+6{4}. The truncated octahedron has the O_h OCTAHEDRAL GROUP of symmetries.



The solid can be formed from an OCTAHEDRON via TRUNCATION by removing six SQUARE PYRAMIDS, each with edge slant height $a = s/3$ and height h, where s is the side length of the original OCTAHEDRON. From the above diagram, the height and base area of the SQUARE PYRAMID are

$$h = \sqrt{a^2 - d^2} = \tfrac{1}{2}\sqrt{2}\,a \qquad (1)$$

$$A_b = a^2. \qquad (2)$$

The VOLUME of the truncated octahedron is then given by the VOLUME of the OCTAHEDRON

$$V_{\text{octahedron}} = \tfrac{1}{3}\sqrt{2}\,s^3 = 9\sqrt{2}\,a^3 \qquad (3)$$

minus six times the volume of the SQUARE PYRAMID,

$$V = V_{\text{octahedron}} - 6(\tfrac{1}{3}A_b h) = (9\sqrt{2} - \sqrt{2})a^3 = 8\sqrt{2}\,a^3. \qquad (4)$$

The truncated octahedron is a SPACE-FILLING POLYHEDRON. The INRADIUS, MIDRADIUS, and CIRCUMRADIUS for $a = 1$ are

$$r = \tfrac{9}{20}\sqrt{10} \approx 1.42302 \qquad (5)$$

$$\rho = \tfrac{3}{2} = 1.5 \qquad (6)$$

$$R = \tfrac{1}{2}\sqrt{10} \approx 1.58114. \qquad (7)$$

see also OCTAHEDRON, SQUARE PYRAMID, TRUNCATION

References
Coxeter, H. S. M. *Regular Polytopes, 3rd ed.* New York: Dover, pp. 29–30 and 257, 1973.

Truncated Polyhedron

A polyhedron with truncated faces, given by the SCHLÄFLI SYMBOL $\mathrm{t}\left\{\begin{smallmatrix} p \\ q \end{smallmatrix}\right\}$.

see also RHOMBIC POLYHEDRON, SNUB POLYHEDRON

Truncated Pyramid

see PYRAMIDAL FRUSTUM

Truncated Square Pyramid

The truncated square pyramid is a special case of a PYRAMIDAL FRUSTUM for a SQUARE PYRAMID. Let the base and top side lengths of the truncated pyramid be a and b, and let the height be h. Then the VOLUME of the solid is

$$V = \tfrac{1}{3}(a^2 + ab + b^2)h.$$

This FORMULA was known to the Egyptians ca. 1850 BC. The Egyptians cannot have proved it without calculus, however, since Dehn showed in 1900 that no proof of this equation exists which does not rely on the concept of continuity (and therefore some form of INTEGRATION).

see also FRUSTUM, PYRAMID, PYRAMIDAL FRUSTUM, SQUARE PYRAMID

Truncated Tetrahedral Number

A FIGURATE NUMBER constructed by taking the $(3n-2)$th TETRAHEDRAL NUMBER and removing the $(n-1)$th TETRAHEDRAL NUMBER from each of the four corners,

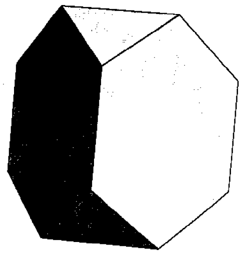
$$\mathrm{Ttet}_n \equiv \mathrm{Te}_{3n-3} - 4\mathrm{Te}_{n-1} = \tfrac{1}{6}n(23n^2 - 27n + 10).$$

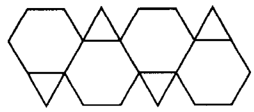
The first few are 1, 16, 68, 180, 375, ... (Sloane's A005906). The GENERATING FUNCTION for the truncated tetrahedral numbers is

$$\frac{x(10x^2 + 12x + 1)}{(x-1)^4} = x + 16x^2 + 89x^3 + 180x^4 + \ldots.$$

References
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 46–47, 1996.
Sloane, N. J. A. Sequence A005906/M5002 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Truncated Tetrahedron



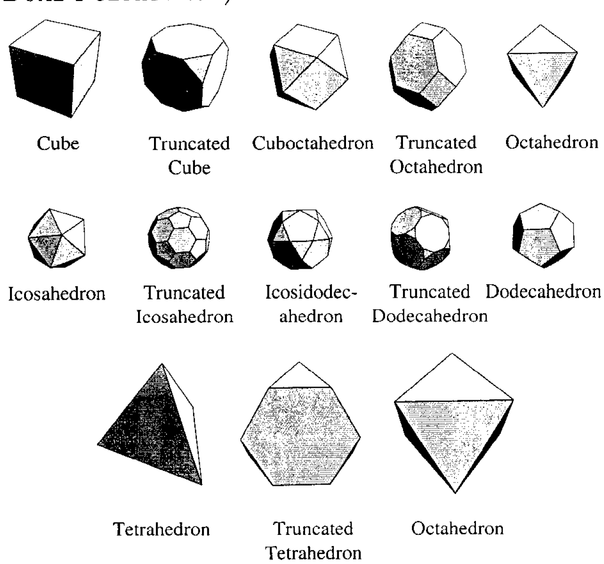


An ARCHIMEDEAN SOLID whose dual is the TRIAKIS TETRAHEDRON. It has SCHLÄFLI SYMBOL t{3,3}. It is also UNIFORM POLYHEDRON U_2 and has WYTHOFF SYMBOL $2\,3\,|\,3$. Its faces are $4\{3\}+4\{6\}$. The INRADIUS, MIDRADIUS, and CIRCUMRADIUS for a truncated tetrahedron with $a=1$ are

$$\begin{aligned} r &= \tfrac{9}{44}\sqrt{22} \approx 0.95940 \\ \rho &= \tfrac{3}{4}\sqrt{2} \approx 1.06066 \\ R &= \tfrac{1}{4}\sqrt{22} \approx 1.17260. \end{aligned}$$

Truncation

The removal of portions of SOLIDS falling outside a set of symmetrically placed planes. The five PLATONIC SOLIDS belong to one of the following three truncation series (which, in the first two cases, carry the solid to its DUAL POLYHEDRON).



see also STELLATION, TRUNCATED CUBE, TRUNCATED DODECAHEDRON, TRUNCATED ICOSAHEDRON, TRUNCATED OCTAHEDRON, TRUNCATED TETRAHEDRON, VERTEX FIGURE

Truth Table

A truth table is a 2-D array with $n+1$ columns. The first n columns correspond to the possible values of n inputs, and the last column to the operation being performed. The rows list all possible combinations of inputs together with the corresponding outputs. For example, the following truth table shows the result of the binary AND operator acting on two inputs A and B, each of which may be true or false.

A	B	$A \wedge B$
F	F	F
F	T	F
T	F	F
T	T	T

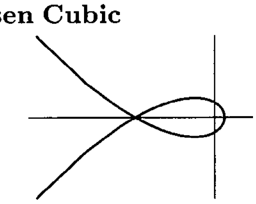
see also AND, MULTIPLICATION TABLE, OR, XOR

Tschebyshev

An alternative spelling of the name "Chebyshev."

see also CHEBYSHEV APPROXIMATION FORMULA, CHEBYSHEV CONSTANTS, CHEBYSHEV DEVIATION, CHEBYSHEV DIFFERENTIAL EQUATION, CHEBYSHEV FUNCTION, CHEBYSHEV-GAUSS QUADRATURE, CHEBYSHEV INEQUALITY, CHEBYSHEV INTEGRAL, CHEBYSHEV PHENOMENON, CHEBYSHEV POLYNOMIAL OF THE FIRST KIND, CHEBYSHEV POLYNOMIAL OF THE SECOND KIND, CHEBYSHEV QUADRATURE, CHEBYSHEV-RADAU QUADRATURE, CHEBYSHEV-SYLVESTER CONSTANT

Tschirnhausen Cubic



The Tschirnhausen cubic is a plane curve given by

$$a = r\cos^3\left(\tfrac{1}{3}\theta\right),$$

and is also known as CATALAN'S TRISECTRIX and L'HOSPITAL'S CUBIC. The name Tschirnhaus's cubic is given in R. C. Archibald's 1900 paper attempting to classify curves (MacTutor Archive). Tschirnhaus's cubic is the NEGATIVE PEDAL CURVE of a PARABOLA with respect to the FOCUS.

References

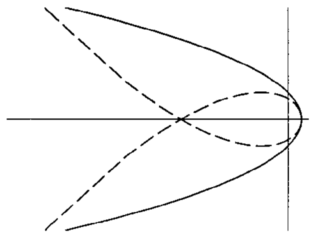
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 87–90, 1972.

MacTutor History of Mathematics Archive. "Tschirnhaus's Cubic." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Tschirnhaus.html`.

Tschirnhausen Cubic Caustic

The CAUSTIC of the TSCHIRNHAUSEN CUBIC taking the RADIANT POINT as the pole is NEILE'S PARABOLA.

Tschirnhausen Cubic Pedal Curve



The PEDAL CURVE to the TSCHIRNHAUSEN CUBIC for PEDAL POINT at the origin is the PARABOLA

$$x = 1 - t^2$$
$$y = 2t.$$

see also PARABOLA, PEDAL CURVE, PEDAL POINT, TSCHIRNHAUSEN CUBIC

Tschirnhausen Transformation

A transformation of a POLYNOMIAL equation $f(x) = 0$ which is of the form $y = g(x)/h(x)$ where g and h are POLYNOMIALS and $h(x)$ does not vanish at a root of $f(x) = 0$. The CUBIC EQUATION is a special case of such a transformation. Tschirnhaus (1683) showed that a POLYNOMIAL of degree $n > 2$ can be reduced to a form in which the x^{n-1} and x^{n-2} terms have 0 COEFFICIENTS. In 1786, E. S. Bring showed that a general QUINTIC EQUATION can be reduced to the form

$$x^5 + px + q = 0.$$

In 1834, G. B. Jerrard showed that a Tschirnhaus transformation can be used to eliminate the x^{n-1}, x^{n-2}, *and* x^{n-3} terms for a general POLYNOMIAL equation of degree $n > 3$.

see also BRING QUINTIC FORM, CUBIC EQUATION

References
Boyer, C. B. *A History of Mathematics.* New York: Wiley, pp. 472–473, 1968.
Tschirnhaus. *Acta Eruditorum.* 1683.

Tubular Neighborhood

The tubular embedding of a SUBMANIFOLD $M^m \subset N^n$ of another MANIFOLD N^n is an EMBEDDING $t : M \times \mathbb{B}^{n-m} \to N$ such that $t(x, 0) = x$ whenever $x \in M$, where $\mathbb{B}^{n-m}$ is the unit BALL in $\mathbb{R}^{n-m}$ centered at 0. The tubular neighborhood is also called the PRODUCT NEIGHBORHOOD.

see also BALL, EMBEDDING, PRODUCT NEIGHBORHOOD

References
Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, pp. 34–35, 1976.

Tucker Circles

Let three equal lines P_1Q_1, P_2Q_2, and P_3Q_3 be drawn ANTIPARALLEL to the sides of a triangle so that two (say P_2Q_2 and P_3Q_3) are on the same side of the third line as $A_2P_2Q_3A_3$. Then $P_2Q_3P_3Q_2$ is an isosceles TRAPEZOID, i.e., P_3Q_2, P_1Q_3, and P_2Q_1 are parallel to the respective sides. The MIDPOINTS C_1, C_2, and C_3 of the antiparallels are on the respective symmedians and divide them proportionally.

If T divides KO in the same ratio, TC_1, TC_2, TC_3 are parallel to the radii OA_1, OA_2, and OA_3 and equal. Since the antiparallels are perpendicular to the symmedians, they are equal chords of a circle with center T which passes through the six given points. This circle is called the Tucker circle.

If

$$c \equiv \frac{\overline{KC_1}}{\overline{KA_1}} = \frac{\overline{KC_2}}{\overline{KA_2}} = \frac{\overline{KC_3}}{\overline{KA_3}} = \frac{\overline{KT}}{\overline{KO}},$$

then the radius of the Tucker circle is

$$R\sqrt{c^2 + (1-c)^2 \tan\omega},$$

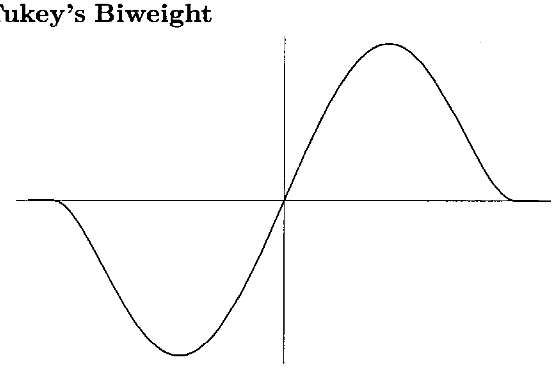
where ω is the BROCARD ANGLE.

The COSINE CIRCLE, LEMOINE CIRCLE, and TAYLOR CIRCLE are Tucker circles.

see also ANTIPARALLEL, BROCARD ANGLE, COSINE CIRCLE, LEMOINE CIRCLE, TAYLOR CIRCLE

References
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 271–277 and 300–301, 1929.

Tukey's Biweight



The function

$$\psi(z) = \begin{cases} z\left(1 - \frac{z^2}{c^2}\right)^2 & \text{for } |z| < c \\ 0 & \text{for } |z| > c \end{cases}$$

sometimes used in ROBUST ESTIMATION. It has a minimum at $z = -c/\sqrt{3}$ and a maximum at $z = c/\sqrt{3}$, where

$$\psi'(z) = 1 - \frac{3x^2}{c^2} = 0,$$

and an inflection point at $z = 0$, where

$$\psi''(z) = -\frac{6z}{c^2} = 0.$$

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, p. 697, 1992.

Tukey's Trimean

see TRIMEAN

Tunnel Number

Let a KNOT K be n-EMBEDDABLE. Then its tunnel number is a KNOT invariant which is related to n.

see also EMBEDDABLE KNOT

References

Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, p. 114, 1994.

Turán Graph

The (n, k)-Turán graph is the EXTREMAL GRAPH on n VERTICES which contains no k-CLIQUE. In other words, the Turán graph has the maximum possible number of EDGES of any n-vertex graph not containing a COMPLETE GRAPH K_k. TURÁN'S THEOREM gives the maximum number of edges $t(n, k)$ for the (n, k)-Turán graph. For $k = 3$,

$$t(n,3) = \tfrac{1}{4}n^4,$$

so the Turán graph is given by the COMPLETE BIPARTITE GRAPHS

$$\begin{cases} K_{n/2,n/2} & n \text{ even} \\ K_{(n-1)/2,(n+1)/2} & n \text{ odd.} \end{cases}$$

see also CLIQUE, COMPLETE BIPARTITE GRAPH, TURÁN'S THEOREM

References

Aigner, M. "Turán's Graph Theorem." *Amer. Math. Monthly* **102**, 808–816, 1995.

Turán's Inequalities

For a set of POSITIVE γ_k, $k = 0, 1, 2\ldots$, Turán's inequalities are given by

$$\gamma_k{}^2 - \gamma_{k-1}\gamma_{k+1} \geq 0$$

for $k = 1, 2, \ldots$.

see also JENSEN POLYNOMIAL

References

Csordas, G.; Varga, R. S.; and Vincze, I. "Jensen Polynomials with Applications to the Riemann ζ-Function." *J. Math. Anal. Appl.* **153**, 112–135, 1990.

Szegő, G. *Orthogonal Polynomials, 4th ed.* Providence, RI: Amer. Math. Soc., p. 388, 1975.

Turán's Theorem

Let $G(V, E)$ be a GRAPH with VERTICES V and EDGES E on n VERTICES without a k-CLIQUE. Then

$$t(n,k) \leq \frac{(k-2)n^2}{2(k-1)},$$

where $t(n, k) = |E|$ is the EDGE NUMBER. More precisely, the K-GRAPH $K_{n_1,\ldots,n_{k-1}}$ with $|n_i - n_j| \leq 1$ for $i \neq j$ is the unique GRAPH without a k-CLIQUE with the maximal number of EDGES $t(n, k)$.

see also CLIQUE, K-GRAPH, TURÁN GRAPH

References

Aigner, M. "Turán's Graph Theorem." *Amer. Math. Monthly* **102**, 808–816, 1995.

Turbine

A VECTOR FIELD on a CIRCLE in which the directions of the VECTORS are all at the same ANGLE to the CIRCLE.

see also CIRCLE, VECTOR FIELD

Turing Machine

A theoretical computing machine which consists of an infinitely long magnetic tape on which instructions can be written and erased, a single-bit register of memory, and a processor capable of carrying out the following instructions: move the tape right, move the tape left, change the state of the register based on its current value and a value on the tape, and write or erase a value on the tape. The machine keeps processing instructions until it reaches a particular state, causing it to halt. Determining whether a Turing machine will halt for a given input and set of rules is called the HALTING PROBLEM.

see also BUSY BEAVER, CELLULAR AUTOMATON, CHAITIN'S OMEGA, CHURCH-TURING THESIS, COMPUTABLE NUMBER, HALTING PROBLEM, UNIVERSAL TURING MACHINE

References

Penrose, R. "Algorithms and Turning Machines." Ch. 2 in *The Emperor's New Mind: Concerning Computers, Minds, and the Laws of Physics.* Oxford, England: Oxford University Press, pp. 30–73, 1989.

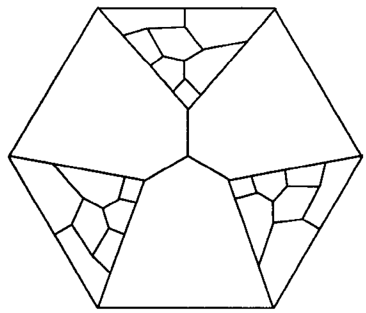
Turing, A. M. "On Computable Numbers, with an Application to the Entscheidungsproblem." *Proc. London Math. Soc. Ser. 2* **42**, 230–265, 1937.

Turing, A. M. "Correction to: On Computable Numbers, with an Application to the Entscheidungsproblem." *Proc. London Math. Soc. Ser. 2* **43**, 544–546, 1938.

Turning Angle

see TANGENTIAL ANGLE

Tutte's Graph



A counterexample to TAIT'S HAMILTONIAN GRAPH CONJECTURE given by Tutte (1946). A simpler counterexample was later given by Kozyrev and Grinberg.

see also HAMILTONIAN CIRCUIT, TAIT'S HAMILTONIAN GRAPH CONJECTURE

References

Honsberger, R. *Mathematical Gems I.* Washington, DC: Math. Assoc. Amer., pp. 82–89, 1973.

Saaty, T. L. and Kainen, P. C. *The Four-Color Problem: Assaults and Conquest.* New York: Dover, p. 112, 1986.

Tutte, W. T. "On Hamiltonian Circuits." *J. London Math. Soc.* **21**, 98–101, 1946.

Tutte Polynomial

Let G be a GRAPH, and let $\mathrm{ea}(T)$ denote the cardinality of the set of externally active edges of a spanning tree T of G and $\mathrm{ia}(T)$ denote the cardinality of the set of internally active edges of T. Then

$$t_G(x,y)=\sum_{T\subseteq G} x^{\mathrm{ia}(T)}y^{\mathrm{ea}(T)}.$$

References

Gessel, I. M. and Sagan, B. E. "The Tutte Polynomial of a Graph, Depth-First Search, and Simplicial Complex Partitions." *Electronic J. Combinatorics* **3**, No. 2, R9, 1–36, 1996. `http://www.combinatorics.org/Volume_3/volume3_2.html#R9.`

Tutte, W. T. "A Contribution to the Theory of Chromatic Polynomials." *Canad. J. Math.* **6**, 80–91, 1953.

Tutte's Theorem

Let G be a GRAPH and S a SUBGRAPH of G. Let the number of ODD components in $G-S$ be denoted S', and $|S|$ the number of VERTICES of S. The condition $|S|\geq S'$ for every SUBSET of VERTICES is NECESSARY and SUFFICIENT for G to have a 1-FACTOR.

see also FACTOR (GRAPH)

References

Honsberger, R. "Lovász' Proof of a Theorem of Tutte." Ch. 14 in *Mathematical Gems II.* Washington, DC: Math. Assoc. Amer., pp. 147–157, 1976.

Tutte, W. T. "The Factorization of Linear Graphs." *J. London Math. Soc.* **22**, 107–111, 1947.

Twin Peaks

For an INTEGER $n\geq 2$, let $\mathrm{lpf}(x)$ denote the LEAST PRIME FACTOR of n. A PAIR of INTEGERS (x,y) is called a twin peak if

1. $x<y$,
2. $\mathrm{lpf}(x)=\mathrm{lpf}(y)$,
3. For all z, $x<z<y$ IMPLIES $\mathrm{lpf}(z)<\mathrm{lpf}(x)$.

A broken-line graph of the least prime factor function resembles a jagged terrain of mountains. In terms of this terrain, a twin peak consists of two mountains of equal height with no mountain of equal or greater height between them. Denote the height of twin peak (x,y) by $p=\mathrm{lpf}(x)=\mathrm{lpf}(y)$. By definition of the LEAST PRIME FACTOR function, p must be PRIME.

Call the distance between two twin peaks (x,y)

$$s\equiv y-x.$$

Then s must be an EVEN multiple of p; that is, $s=kp$ where k is EVEN. A twin peak with $s=kp$ is called a kp-twin peak. Thus we can speak of $2p$-twin peaks, $4p$-twin peaks, etc. A kp-twin peak is fully specified by k, p, and x, from which we can easily compute $y\equiv x+kp$.

The set of kp-twin peaks is periodic with period $q=p\#$, where $p\#$ is the PRIMORIAL of p. That is, if (x,y) is a kp-twin peak, then so is $(x+q,y+q)$. A fundamental kp-twin peak is a twin peak having x in the fundamental period $[0,q)$. The set of fundamental kp-twin peaks is symmetric with respect to the fundamental period; that is, if (x,y) is a twin peak on $[0,q)$, then so is $(q-y,q-x)$.

The question of the EXISTENCE of twin peaks was first raised by David Wilson in the math-fun mailing list on Feb. 10, 1997. Wilson already had privately showed the EXISTENCE of twin peaks of height $p\leq 13$ to be unlikely, but was unable to rule them out altogether. Later that same day, John H. Conway, Johan de Jong, Derek Smith, and Manjul Bhargava collaborated to discover the first twin peak. Two hours at the blackboard revealed that $p=113$ admits the $2p$-twin peak

$$x=126972592296404970720882679404584182254788131,$$

which settled the EXISTENCE question. Immediately thereafter, Fred Helenius found the smaller $2p$-twin peak with $p=89$ and

$$x=9503844926749390990454854843625839.$$

The effort now shifted to finding the least PRIME p admitting a $2p$-twin peak. On Feb. 12, 1997, Fred Helenius found $p=71$, which admits 240 fundamental $2p$-twin peaks, the least being

$$x=7310131732015251470110369.$$

Helenius's results were confirmed by Dan Hoey, who also computed the least $2p$-twin peak $L(2p)$ and number of fundamental $2p$-twin peaks $N(2p)$ for $p = 73$, 79, and 83. His results are summarized in the following table.

p	$L(2p)$	$N(2p)$
71	7310131732015251470110369	240
73	2061519317176132799110061	40296
79	3756800873017263196139951	164440
83	6316254452384500173544921	6625240

The $2p$-twin peak of height $p = 73$ is the smallest known twin peak. Wilson found the smallest known $4p$-twin peak with $p = 1327$, as well as another very large $4p$-twin peak with $p = 3203$. Richard Schroeppel noted that the latter twin peak is at the high end of its fundamental period and that its reflection within the fundamental period $[0, p\#)$ is smaller.

Many open questions remain concerning twin peaks, e.g.,

1. What is the smallest twin peak (smallest n)?
2. What is the least PRIME p admitting a $4p$-twin peak?
3. Do $6p$-twin peaks exist?
4. Is there, as Conway has argued, an upper bound on the span of twin peaks?
5. Let $p < q < r$ be PRIME. If p and r each admit kp-twin peaks, does q then necessarily admit a kp-twin peak?

see also ANDRICA'S CONJECTURE, DIVISOR FUNCTION, LEAST COMMON MULTIPLE, LEAST PRIME FACTOR

Twin Prime Conjecture

Adding a correction proportional to $1/\ln p$ to a computation of BRUN'S CONSTANT ending with $\ldots + 1/p + 1/(p+2)$ will give an estimate with error less than $c(\sqrt{p}\,\ln p)^{-1}$. An extended form of the conjecture states that

$$P_x(p, p+2) \sim 2\Pi_2 \int_2^x \frac{dx}{(\ln x)^2},$$

where Π_2 is the TWIN PRIMES CONSTANT. The twin prime conjecture is a special case of the more general PRIME PATTERNS CONJECTURE corresponding to the set $S = \{0, 2\}$.

see also BRUN'S CONSTANT, PRIME ARITHMETIC PROGRESSION, PRIME CONSTELLATION, PRIME PATTERNS CONJECTURE, TWIN PRIMES

Twin Primes

Twin primes are PRIMES (p, q) such that $p - q = 2$. The first few twin primes are $n \pm 1$ for $n = 4$, 6, 12, 18, 30, 42, 60, 72, 102, 108, 138, 150, 180, 192, 198, 228, 240, 270, 282, ... (Sloane's A014574). Explicitly, these are (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), ... (Sloane's A001359 and A006512).

Let $\pi_2(n)$ be the number of twin primes p and $p+2$ such that $p \leq n$. It is not known if there are an infinite number of such PRIMES (Shanks 1993), but all twin primes except (3, 5) are of the form $6n \pm 1$. J. R. Chen has shown there exists an INFINITE number of PRIMES p such that $p + 2$ has at most two factors (Le Lionnais 1983, p. 49). Bruns proved that there exists a computable INTEGER x_0 such that if $x \geq x_0$, then

$$\pi_2(x) < \frac{100x}{(\ln x)^2} \tag{1}$$

(Ribenboim 1989, p. 201). It has been shown that

$$\pi_2(x) \leq c \prod_{p>2} \left[1 - \frac{1}{(p-1)^2}\right] \frac{x}{(\ln x)^2} \left[1 + \mathcal{O}\left(\frac{\ln \ln x}{\ln x}\right)\right], \tag{2}$$

where c has been reduced to $68/9 \approx 7.5556$ (Fouvry and Iwaniec 1983), $128/17 \approx 7.5294$ (Fouvry 1984), 7 (Bombieri *et al.* 1986), 6.9075 (Fouvry and Grupp 1986), and 6.8354 (Wu 1990). The bound on c is further reduced to 6.8324107886 in a forthcoming thesis by Haugland (1998). This calculation involved evaluation of 7-fold integrals and fitting of three different parameters. Hardy and Littlewood conjectured that $c = 2$ (Ribenboim 1989, p. 202).

Define

$$E \equiv \liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\ln p_n}. \tag{3}$$

If there are an infinite number of twin primes, then $E = 0$. The best upper limit to date is $E \leq \frac{1}{4} + \pi/16 = 0.44634\ldots$ (Huxley 1973, 1977). The best previous values were 15/16 (Ricci), $(2 + \sqrt{3})/8 = 0.46650\ldots$ (Bombieri and Davenport 1966), and $(2\sqrt{2} - 1)/4 = 0.45706\ldots$ (Pil'Tai 1972), as quoted in Le Lionnais (1983, p. 26).

Some large twin primes are $10{,}006{,}428 \pm 1$, $1{,}706{,}595 \times 2^{11235} \pm 1$, and $571{,}305 \times 2^{7701} \pm 1$. An up-to-date table of known twin primes with 2000 or more digits follows. An extensive list is maintained by **Caldwell**.

$(p, p+1)$	dig.	Reference
$260{,}497{,}545 \times 2^{6625} \pm 1$	2003	Atkin & Rickert 1984
$43{,}690{,}485{,}351{,}513 \times 10^{1995} \pm 1$	2009	Dubner, Atkin 1985
$2{,}846!!!! \pm 1$	2151	Dubner 1992
$10{,}757{,}0463 \times 10^{2250} \pm 1$	2259	Dubner, Atkin 1985
$663{,}777 \times 2^{7650} \pm 1$	2309	Brown *et al.* 1989
$75{,}188{,}117{,}004 \times 10^{2298} \pm 1$	2309	Dubner 1989
$571305 \times 2^{7701} \pm 1$	2324	Brown *et al.* 1989
$1{,}171{,}452{,}282 \times 10^{2490} \pm 1$	2500	Dubner 1991
$459 \cdot 2^{8529} \pm 1$	2571	Dubner 1993
$1{,}706{,}595 \cdot 2^{11235} \pm 1$	3389	Noll *et al.* 1989
$4{,}655{,}478{,}828 \cdot 10^{3429} \pm 1$	3439	Dubner 1993
$1{,}692{,}923{,}232 \cdot 10^{4020} \pm 1$	4030	Dubner 1993
$6{,}797{,}727 \cdot 2^{15328} \pm 1$	4622	Forbes 1995
$697{,}053{,}813 \cdot 2^{16352} \pm 1$	4932	Indlekofer & Ja'rai 1994
$570{,}918{,}348 \cdot 10^{5120} \pm 1$	5129	Dubner 1995
$242{,}206{,}083 \cdot 2^{38880} \pm 1$	11713	Indlekofer & Ja'rai 1995

The last of these is the largest known twin prime pair. In 1995, Nicely discovered a flaw in the Intel® Pentium™ microprocessor by computing the reciprocals of 824,633,702,441 and 824,633,702,443, which should have been accurate to 19 decimal places but were incorrect from the tenth decimal place on (Cipra 1995, 1996; Nicely 1996).

If $n \geq 2$, the INTEGERS n and $n+2$ form a pair of twin primes IFF

$$4[(n-1)!+1]+n \equiv 0 \pmod{n(n+2)}. \tag{4}$$

$n = pp'$ where (p, p') is a pair of twin primes IFF

$$\phi(n)\sigma(n) = (n-3)(n+1) \tag{5}$$

(Ribenboim 1989). The values of $\pi_2(n)$ were found by Brent (1976) up to $n = 10^{11}$. T. Nicely calculated them up to 10^{14} in his calculation of BRUN'S CONSTANT. The following table gives the number less than increasing powers of 10 (Sloane's A007508).

n	$\pi_2(n)$
10^3	35
10^4	205
10^5	1224
10^6	8,169
10^7	58,980
10^8	440,312
10^9	3,424,506
10^{10}	27,412,679
10^{11}	224,376,048
10^{12}	1,870,585,220
10^{13}	15,834,664,872
10^{14}	135,780,321,665

see also BRUN'S CONSTANT, DE POLIGNAC'S CONJECTURE PRIME CONSTELLATION, SEXY PRIMES, TWIN PRIME CONJECTURE, TWIN PRIMES CONSTANT

References

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Twin Primes Constant

The twin primes constant Π_2 is defined by

$$\Pi_2 \equiv \prod_{\substack{p>2\\ p \text{ prime}}} \left[1 - \frac{1}{(p-1)^2}\right] \tag{1}$$

$$\begin{aligned}\ln(\tfrac{1}{2}\Pi_2) &= \sum_{\substack{p\geq 3\\ p \text{ prime}}} \ln\left[\frac{p(p-2)}{(p-1)^2}\right]\\ &= \sum_{\substack{p\geq 3\\ p \text{ prime}}} \left[\ln\left(1-\frac{2}{p}\right) - 2\ln\left(1-\frac{1}{p}\right)\right]\\ &= -\sum_{j=2}^{\infty} \frac{2^j-2}{j} \sum_{\substack{p\geq 3\\ p \text{ prime}}} p^{-j},\end{aligned} \tag{2}$$

where the ps in sums and products are taken over PRIMES only. Flajolet and Vardi (1996) give series with accelerated convergence

$$\Pi_2 = \prod_{n=2}^{\infty} [\zeta(n)(1-2^{-n})]^{-I_n} \tag{3}$$

$$= \tfrac{3}{4}\tfrac{15}{16}\tfrac{35}{36} \prod_{n=2}^{\infty} [\zeta(n)(1-2^{-n})(1-3^{-n})(1-5^{-n}) \times(1-7^{-n})]^{-I_n}, \tag{4}$$

with

$$I_n \equiv \frac{1}{n} \sum_{d|n} \mu(d) 2^{n/d}, \tag{5}$$

where $\mu(x)$ is the MÖBIUS FUNCTION. (4) has convergence like $\sim (11/2)^{-n}$.

The most accurately known value of Π_2 is

$$\Pi_2 = 0.6601618158 \ldots. \tag{6}$$

Le Lionnais (1983, p. 30) calls C_2 the SHAH-WILSON CONSTANT, and $2C_2$ the twin prime constant (Le Lionnais 1983, p. 37).

see also BRUN'S CONSTANT, GOLDBACH CONJECTURE, MERTENS CONSTANT

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Twins

see BROTHERS, PAIR

Twirl

A ROTATION combined with an EXPANSION or DILATION.

see also SCREW, SHIFT

Twist

The twist of a ribbon measures how much it twists around its axis and is defined as the integral of the incremental twist around the ribbon. Letting Lk be the linking number of the two components of a ribbon, Tw be the twist, and Wr be the WRITHE, then

$$\mathrm{Lk}(R) = \mathrm{Tw}(R) + \mathrm{Wr}(R)$$

(Adams 1994, p. 187).

see also SCREW, WRITHE

References

Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, 1994.

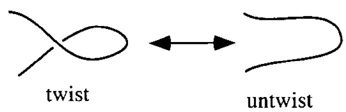
Twist Map

A class of AREA-PRESERVING MAPS of the form

$$\theta_{i+1} = \theta_i + 2\pi\alpha(r_i)$$
$$r_{i+1} = r_i,$$

which maps CIRCLES into CIRCLES but with a twist resulting from the $\alpha = \alpha(r_i)$ term.

Twist Move



The REIDEMEISTER MOVE of type II.

see also REIDEMEISTER MOVES

Twist Number

see WRITHE

Twist-Spun Knot

A generalization of SPUN KNOTS due to Zeeman. This method produces 4-D KNOT types that cannot be produced by ordinary spinning.

see also SPUN KNOT

Twisted Chevalley Groups

FINITE SIMPLE GROUPS of LIE-TYPE of ORDERS 14, 52, 78, 133, and 248. They are denoted ${}^3D_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$, ${}^2F_4(2^n)'$, $G_2(q)$,${}^2G_2(3^n)$, ${}^2B(2^n)$.

see also CHEVALLEY GROUPS, FINITE GROUP, SIMPLE GROUP, TITS GROUP

References

Wilson, R. A. "ATLAS of Finite Group Representation." `http://for.mat.bham.ac.uk/atlas#twi`.

Twisted Conic

see SKEW CONIC

Twisted Sphere

see CORKSCREW SURFACE

Two

see 2

Two-Ears Theorem

Except for TRIANGLES, every SIMPLE POLYGON has at least two nonoverlapping EARS.

see also EAR, ONE-MOUTH THEOREM, PRINCIPAL VERTEX

References

Meisters, G. H. "Principal Vertices, Exposed Points, and Ears." *Amer. Math. Monthly* **87**, 284–285, 1980.

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Two-Point Distance

see POINT-POINT DISTANCE—1-D, POINT-POINT DISTANCE—2-D, POINT-POINT DISTANCE—3-D

Two Triangle Theorem

see DESARGUES' THEOREM

Tychonof Compactness Theorem

The topological product of any number of COMPACT SPACES is COMPACT.

Type

Whitehead and Russell (1927) devised a hierarchy of "types" in order to eliminate self-referential statements from *Principia Mathematica*, which purported to derive all of mathematics from logic. A set of the lowest type contained only objects (not sets), a set of the next higher type could contain only objects or sets of the lower type, and so on. Unfortunately, GÖDEL'S INCOMPLETENESS THEOREM showed that both *Principia Mathematica* and all consistent formal systems must be incomplete.

see also GÖDEL'S INCOMPLETENESS THEOREM

References

Hofstadter, D. R. *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Vintage Books, pp. 21–22, 1989.

Whitehead, A. N. and Russell, B. *Principia Mathematica.* New York: Cambridge University Press, 1927.

Type I Error

An error in a STATISTICAL TEST which occurs when a true hypothesis is rejected (a false negative in terms of the NULL HYPOTHESIS).

see also NULL HYPOTHESIS, SENSITIVITY, SPECIFICITY, STATISTICAL TEST, TYPE II ERROR

Type II Error

An error in a STATISTICAL TEST which occurs when a false hypothesis is accepted (a false positive in terms of the NULL HYPOTHESIS).

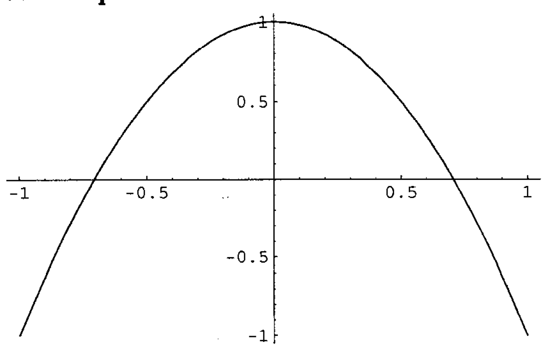
see also NULL HYPOTHESIS, SENSITIVITY, SPECIFICITY, STATISTICAL TEST, TYPE I ERROR

U

U-Number

see ULAM SEQUENCE

Ulam Map



$$f(x) = 1 - 2x^2$$

for $x \in [-1, 1]$. Fixed points occur at $x = -1$, $1/2$, and order 2 fixed points at $x = (1 \pm \sqrt{5})/4$. The INVARIANT DENSITY of the map is

$$\rho(y) = \frac{1}{\pi\sqrt{1-y^2}}.$$

References

Beck, C. and Schlögl, F. *Thermodynamics of Chaotic Systems: An Introduction.* Cambridge, England: Cambridge University Press, p. 194, 1995.

Ulam Number

see ULAM SEQUENCE

Ulam's Problem

see COLLATZ PROBLEM

Ulam Sequence

The Ulam sequence $\{a_i\} = (u, v)$ is defined by $a_1 = u$, $a_2 = v$, with the general term a_n for $n > 2$ given by the least INTEGER expressible uniquely as the SUM of two distinct earlier terms. The numbers so produced are sometimes called U-NUMBERS or ULAM NUMBERS.

The first few numbers in the (1, 2) Ulam sequence are 1, 2, 3, 4, 6, 8, 11, 13, 16, ... (Sloane's A002858). Here, the first term after the initial 1, 2 is obviously 3 since $3 = 1 + 2$. The next term is $4 = 1 + 3$. (We don't have to worry about $4 = 2 + 2$ since it is a sum of a single term instead of *unique* terms.) 5 is not a member of the sequence since it is representable in *two* ways, $5 = 1 + 4 = 2 + 3$, but $6 = 2 + 4$ is a member.

Proceeding in the manner, we can generate Ulam sequences for any (u, v), examples of which are given below.

$$\begin{aligned}
(1,2) &= \{1,2,3,4,6,8,11,13,16,18,\ldots\}\\
(1,3) &= \{1,3,4,5,6,8,10,12,17,21,\ldots\}\\
(1,4) &= \{1,4,5,6,7,8,10,16,18,19,\ldots\}\\
(1,5) &= \{1,5,6,7,8,9,10,12,20,22,\ldots\}\\
(2,3) &= \{2,3,5,7,8,9,13,14,18,19,\ldots\}\\
(2,4) &= \{2,4,6,8,12,16,22,26,32,36,\ldots\}\\
(2,5) &= \{2,5,7,9,11,12,13,15,19,23,\ldots\}.
\end{aligned}$$

Schmerl and Spiegel (1994) proved that Ulam sequences $(2, v)$ for ODD $v \geq 5$ have exactly two EVEN terms. Ulam sequences with only finitely many EVEN terms eventually must have periodic successive differences (Finch 1991, 1992abc). Cassaigne and Finch (1995) proved that the Ulam sequences $(4, v)$ for $5 \leq v \equiv 1 \pmod 4$ have exactly three EVEN terms.

The Ulam sequence can be generalized by the s-ADDITIVE SEQUENCE.

see also GREEDY ALGORITHM, s-ADDITIVE SEQUENCE, STÖHR SEQUENCE

References

Cassaigne, J. and Finch, S. "A Class of 1-Additive Sequences and Quadratic Recurrences." *Exper. Math* **4**, 49–60, 1995.

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Ultrametric

An ultrametric is a METRIC which satisfies the following strengthened version of the TRIANGLE INEQUALITY,

$$d(x, z) \leq \max(d(x, y), d(y, z))$$

for all x, y, z. At least two of $d(x,y)$, $d(y,z)$, and $d(x,z)$ are the same.

Let X be a SET, and let $X^{\mathbb{N}}$ (where $\mathbb{N}$ is the SET of NATURAL NUMBERS) denote the collection of sequences of elements of X (i.e., all the possible sequences x_1, x_2, x_3, ...). For sequences $a = (a_1, a_2, \ldots)$, $b = (b_1, b_2, \ldots)$, let n be the number of initial places where the sequences agree, i.e., $a_1 = b_1$, $a_2 = b_2$, ..., $a_n = b_n$, but $a_{n+1} \neq b_{n+1}$. Take $n = 0$ if $a_1 \neq b_1$. Then defining $d(a,b) = 2^{-n}$ gives an ultrametric.

The p-ADIC NUMBER metric is another example of an ultrametric.

see also METRIC, p-ADIC NUMBER

Ultraradical

A symbol which can be used to express solutions not obtainable by finite ROOT extraction. The solution to the irreducible QUINTIC EQUATION

$$x^5 + x = a$$

is written $\sqrt[5]{a}$.

see also RADICAL

Ultraspherical Differential Equation

$$(1 - x^2)y'' - (2\alpha + 1)xy' + n(n + 2\alpha)y = 0. \quad (1)$$

Alternate forms are

$$(1-x^2)Y'' + (2\lambda - 3)xY' + (n+1)(n+2\lambda-1)Y = 0, \quad (2)$$

where

$$Y = (1 - x^2)^{\lambda - 1/2} P_n^{(\lambda)}(x), \quad (3)$$

$$\frac{d^2u}{dx^2} + \left[\frac{(n+\lambda)^2}{1-x^2} + \frac{\frac{1}{2} + \lambda - \lambda^2 + \frac{1}{4}x^2}{(1-x^2)^2}\right] u = 0, \quad (4)$$

where

$$u = (1 - x^2)^{\lambda/2 + 1/4} P_n^{(\lambda)}(x), \quad (5)$$

and

$$\frac{d^2u}{d\theta^2} + \left[(n+\lambda)^2 + \frac{\lambda(1-\lambda)}{\sin^2\theta}\right] u = 0, \quad (6)$$

where

$$u = \sin^\lambda\theta \, P_n^{(\lambda)}(\cos\theta). \quad (7)$$

The solutions are the ULTRASPHERICAL FUNCTIONS $P_n^{(\lambda)}(x)$. For integral n with $\alpha < 1/2$, the function converges to the ULTRASPHERICAL POLYNOMIALS $C_n^{(\alpha)}(x)$.

References

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 547–549, 1953.

Ultraspherical Function

A function defined by a POWER SERIES whose coefficients satisfy the RECURRENCE RELATION

$$a_{j+2} = a_j \frac{(k+j)(k+j+2\alpha) - n(n+2\alpha)}{(k+j+1)(k+j+2)}.$$

For $x \neq -1$, the function converges for $\alpha < 1/2$ and diverges for $\alpha > 1/2$.

Ultraspherical Polynomial

The ultraspherical polynomials are solutions $P_n^{(\lambda)}(x)$ to the ULTRASPHERICAL DIFFERENTIAL EQUATION for INTEGER n and $\alpha < 1/2$. They are generalizations of LEGENDRE POLYNOMIALS to $(n+2)$-D space and are proportional to (or, depending on the normalization, equal to) the GEGENBAUER POLYNOMIALS $C_n^{(\lambda)}(x)$, denoted in *Mathematica*® (Wolfram Research, Champaign, IL) `GegenbauerC[n,lambda,x]`. The ultraspherical polynomials are also JACOBI POLYNOMIALS with $\alpha = \beta$. They are given by the GENERATING FUNCTION

$$\frac{1}{(1 - 2xt + t^2)^\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(x) t^n, \quad (1)$$

and can be given explicitly by

$$P_n^{(\lambda)}(x) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(2\lambda)} \frac{\Gamma(n + 2\lambda)}{\Gamma(n + \lambda + \frac{1}{2})} P_n^{(\lambda-1/2, \lambda-1/2)}(x), \quad (2)$$

where $P_n^{(\lambda-1/2,\lambda-1/2)}$ is a JACOBI POLYNOMIAL (Szegő 1975, p. 80). The first few ultraspherical polynomials are

$$P_0^{(\lambda)}(x) = 1 \quad (3)$$

$$P_1^{(\lambda)}(x) = 2\lambda x \quad (4)$$

$$P_2^{(\lambda)}(x) = -\lambda + 2\lambda(1+\lambda)x^2 \quad (5)$$

$$P_3^{(\lambda)}(x) = -2\lambda(1+\lambda)x + \tfrac{4}{3}\lambda(1+\lambda)(2+\lambda)x^3. \quad (6)$$

In terms of the HYPERGEOMETRIC FUNCTIONS,

$$P_n^{(\lambda)}(x) = \binom{n + 2\lambda - 1}{n} \times {}_2F_1(-n, n + 2\lambda; \lambda + \tfrac{1}{2}; \tfrac{1}{2}(1-x)) \quad (7)$$

$$= 2^n \binom{n + \lambda - 1}{n} (x-1)^n \times {}_2F_1\left(-n, -n - \lambda + \tfrac{1}{2}; -2n - 2\lambda + 1; \frac{2}{1-x}\right) \quad (8)$$

$$= \binom{n + 2\lambda + 1}{n} \left(\frac{x+1}{2}\right)^n \times {}_2F_1\left(-n, -n - \lambda + \tfrac{1}{2}; \lambda + \tfrac{1}{2}; \frac{x-1}{x+1}\right). \quad (9)$$

They are normalized by

$$\int_{-1}^{1}(1-x^2)^{\lambda-1/2}[P_n^{(\lambda)}]^2\,dx = 2^{1-2\lambda}\pi\frac{\Gamma(n+2\lambda)}{(n+\lambda)\Gamma^2(\lambda)\Gamma(n+1)}. \quad (10)$$

Derivative identities include

$$\frac{d}{dx}P_n^{(\lambda)}(x) = 2\lambda P_{n-1}^{(\lambda+1)}(x) \quad (11)$$

$$(1-x^2)\frac{d}{dx}[P_n^{(\lambda)}] = [2(n+\lambda)]^{-1}[(n+2\lambda-1) \times(n+2\lambda)P_{n-1}^{(\lambda)}(x) - n(n+1)P_{n+1}^{(\lambda)}(x)] \quad (12)$$

$$= -nxP_n^{(\lambda)}(x) + (n+2\lambda-1)P_{n-1}^{(\lambda)}(x) \quad (13)$$

$$= (n+2\lambda)xP_n^{(\lambda)}(x) - (n+1)P_{n+1}^{(\lambda)}(x) \quad (14)$$

$$nP_n^{(\lambda)}(x) = x\frac{d}{dx}[P_n^{(\lambda)}(x)] - \frac{d}{dx}[P_{n-1}^{(\lambda)}(x)] \quad (15)$$

$$(n+2\lambda)P_n^{(\lambda)}(x) = \frac{d}{dx}[P_{n+1}^{(\lambda)}(x)] - x\frac{d}{dx}[P_n^{(\lambda)}(x)] \quad (16)$$

$$\frac{d}{dx}[P_{n+1}^{(\lambda)}(x) - P_{n-1}^{(\lambda)}(x)] = 2(n+\lambda)P_n^{(\lambda)}P_n^{(\lambda)}(x) \quad (17)$$

$$= 2\lambda[P_n^{(\lambda+1)}(x) - P_{n-2}^{(\lambda+1)}(x)] \quad (18)$$

(Szegő 1975, pp. 80–83).

A RECURRENCE RELATION is

$$nP_n^{(\lambda)}(x) = 2(n+\lambda-1)xP_{n-1}^{(\lambda)}(x) - (n+2\lambda-2)P_{n-2}^{(\lambda)}(x) \quad (19)$$

for $n = 2, 3, \ldots$.

Special double-ν FORMULAS also exist

$$P_{2\nu}^{(\lambda)}(x) = \binom{2\nu+2\lambda-1}{2\nu}{}_2F_1(-\nu,\nu+\lambda;\lambda+\tfrac{1}{2};1-x^2) \quad (20)$$

$$= (-1)^\nu\binom{\nu+\lambda-1}{\nu}{}_2F_1(-\nu,\nu+\lambda;\tfrac{1}{2};x^2) \quad (21)$$

$$P_{2\nu+1}^{(\lambda)}(x) = \binom{2\nu+2\lambda}{2\nu+1}x\,{}_2F_1(-\nu,\nu+\lambda+1;\lambda+\tfrac{1}{2};1-x^2) \quad (22)$$

$$= (-1)^\nu 2\lambda\binom{\nu+\lambda}{\nu}x\,{}_2F_1(-\nu,\nu+\lambda+1;\tfrac{3}{2};x^2). \quad (23)$$

Special values are given in the following table.

λ	Special Polynomial
$\frac{1}{2}$	Legendre
1	Chebyshev polynomial of the second kind

Koschmieder (1920) gives representations in terms of ELLIPTIC FUNCTIONS for $\alpha = -3/4$ and $\alpha = -2/3$.

see also BIRTHDAY PROBLEM, CHEBYSHEV POLYNOMIAL OF THE SECOND KIND, ELLIPTIC FUNCTION, HYPERGEOMETRIC FUNCTION, JACOBI POLYNOMIAL

References

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Umbilic Point

A point on a surface at which the CURVATURE is the same in any direction.

Umbral Calculus

The study of certain properties of FINITE DIFFERENCES. The term was coined by Sylvester from the word "umbra" (meaning "shadow" in Latin), and reflects the fact that for many types of identities involving sequences of polynomials with POWERS a^n, "shadow" identities are obtained when the polynomials are changed to discrete values and the exponent in a^n is changed to the POCHHAMMER SYMBOL $(a)_n \equiv a(a-1)\cdots(a-n+1)$.

For example, NEWTON'S FORWARD DIFFERENCE FORMULA written in the form

$$f(x+a) = \sum_{n=0}^{\infty}\frac{(a)_n\Delta^n f(x)}{n!} \quad (1)$$

with $f(x+a) \equiv f_{x+a}$ looks suspiciously like a finite analog of the TAYLOR SERIES expansion

$$f(x+a) = \sum_{n=0}^{\infty}\frac{a^n\tilde{D}^n f(x)}{n!}, \quad (2)$$

where $\tilde{D}$ is the DIFFERENTIAL OPERATOR. Similarly, the CHU-VANDERMONDE IDENTITY

$$(x+a)_n = \sum_{k=0}^{\infty}\binom{n}{k}(a)_k(x)_{n-k} \quad (3)$$

with $\binom{n}{k}$ a BINOMIAL COEFFICIENT, looks suspiciously like an analog of the BINOMIAL THEOREM

$$(x+a)^n = \sum_{k=0}^{\infty}\binom{n}{k}a^k x^{n-k} \quad (4)$$

(Di Bucchianico and Loeb).

see also BINOMIAL THEOREM, CHU-VANDERMONDE IDENTITY, FINITE DIFFERENCE

References

Roman, S. and Rota, G.-C. "The Umbral Calculus." *Adv. Math.* **27**, 95–188, 1978.

Roman, S. *The Umbral Calculus.* New York: Academic Press, 1984.

Umbrella

see WHITNEY UMBRELLA

Unambiguous

see WELL-DEFINED

Unbiased

A quantity which does not exhibit BIAS. An ESTIMATOR $\hat{\theta}$ is an UNBIASED ESTIMATOR of θ if

$$\langle\hat{\theta}\rangle = \theta.$$

see also BIAS (ESTIMATOR), ESTIMATOR

Uncia

$$1 \text{ uncia} \equiv \tfrac{1}{12}.$$

The word *uncia* was Latin for a unit equal to 1/12 of another unit called the *as*. The words "inch" (1/12 of a foot) and "ounce" (originally 1/12 of a pound and still 1/12 of a "Troy pound," now used primarily to weigh precious metals) are derived from the word *uncia.*

see also CALCUS, HALF, QUARTER, SCRUPLE, UNIT FRACTION

References

Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, p. 4, 1996.

Uncorrelated

Variables x_i and x_j are said to be uncorrelated if their COVARIANCE is zero:

$$\operatorname{cov}(x_i, x_j) = 0.$$

INDEPENDENT STATISTICS are always uncorrelated, but the converse is not necessarily true.

see also COVARIANCE, INDEPENDENT STATISTICS

Uncountable Set

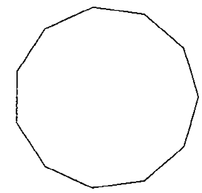
see UNCOUNTABLY INFINITE SET

Uncountably Infinite Set

An INFINITE SET which is not a COUNTABLY INFINITE SET.

see also ALEPH-0, ALEPH-1, COUNTABLE SET, COUNTABLY INFINITE SET, FINITE, INFINITE

Undecagon



The unconstructible 11-sided POLYGON with SCHLÄFLI SYMBOL $\{11\}$.

see also DECAGON, DODECAGON, TRIGONOMETRY VALUES—$\pi/11$

Undecidable

Not DECIDABLE as a result of being neither formally provable nor unprovable.

see also GÖDEL'S INCOMPLETENESS THEOREM, RICHARDSON'S THEOREM

Undecillion

In the American system, 10^{36}.

see also LARGE NUMBER

Undetermined Coefficients Method

Given a nonhomogeneous ORDINARY DIFFERENTIAL EQUATION, select a differential operator which will annihilate the right side, and apply it to both sides. Find the solution to the homogeneous equation, plug it into the left side of the original equation, and solve for constants by setting it equal to the right side. The solution is then obtained by plugging the determined constants into the homogeneous equation.

see also ORDINARY DIFFERENTIAL EQUATION

Undulating Number

A number of the form $aba\cdots$, $abab\cdots$, etc. The first few nontrivial undulants (with the stipulation that $a \neq b$) are 101, 121, 131, 141, 151, 161, 171, 181, 191, 202, 212, ... (Sloane's A046075). Including the trivial 1- and 2-digit undulants and dropping the requirement that $a \neq b$ gives Sloane's A033619.

The first few undulating SQUARES are 121, 484, 676, 69696, ... (Sloane's A016073), with no larger such numbers of fewer than a million digits (Pickover 1995). Several tricks can be used to speed the search for square undulating numbers, especially by examining the possible patterns of ending digits. For example, the only possible sets of four trailing digits for undulating SQUARES are 0404, 1616, 2121, 2929, 3636, 6161, 6464, 6969, 8484, and 9696.

The only undulating POWER $n^p = aba\cdots$ for $3 \leq p \leq 31$ and up to 100 digits is $7^3 = 343$ (Pickover 1995). A large undulating prime is given by $7+720(100^{49}-1)/99$ (Pickover 1995).

A binary undulant is a Power of 2 whose base-10 representation contains one or both of the sequences $010\cdots$ and $101\cdots$. The first few are 2^n for $n = 103$, 107, 138, 159, 179, 187, 192, 199, 205, ... (Sloane's A046076). The smallest n for which an undulating sequence of *exactly* d-digit occurs for $d = 3$, 4, ... are $n = 103$, 138, 875, 949, 6617, 1802, 14545, ... (Sloane's A046077). An undulating binary sequence of length 10 occurs for $n = 1,748,219$ (Pickover 1995).

References

Pickover, C. A. "Is There a Double Smoothly Undulating Integer?" In *Computers, Pattern, Chaos and Beauty.* New York: St. Martin's Press, 1990.

Pickover, C. A. "The Undulation of the Monks." Ch. 20 in *Keys to Infinity.* New York: W. H. Freeman, pp. 159–161 1995.

Sloane, N. J. A. Sequences A016073, A033619, A046075, A046076, and A046077 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Unduloid

A Surface of Revolution with constant Nonzero Mean Curvature also called an Onduloid. It is a Roulette obtained from the path described by the Foci of a Conic Section when rolled on a Line. This curve then generates an unduloid when revolved about the Line. These curves are special cases of the shapes assumed by soap film spanning the gap between prescribed boundaries. The unduloid of a Parabola gives a Catenoid.

see also Calculus of Variations, Catenoid, Roulette

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Unexpected Hanging Paradox

A Paradox also known as the Surprise Examination Paradox or Prediction Paradox.

A prisoner is told that he will be hanged on some day between Monday and Friday, but that he will not know on which day the hanging will occur before it happens. He cannot be hanged on Friday, because if he were still alive on Thursday, he would know that the hanging will occur on Friday, but he has been told he will not know the day of his hanging in advance. He cannot be hanged Thursday for the same reason, and the same argument shows that he cannot be hanged on any other day. Nevertheless, the executioner unexpectedly arrives on some day other than Friday, surprising the prisoner.

This Paradox is similar to that in Robert Louis Stevenson's "The Imp in the Bottle," in which you are offered the opportunity to buy, for whatever price you wish, a bottle containing a genie who will fulfill your every desire. The only catch is that the bottle must thereafter be resold for a price smaller than what you paid for it, or you will be condemned to live out the rest of your days in excrutiating torment. Obviously, no one would buy the bottle for 1¢ since he would have to give the bottle away, but no one would accept the bottle knowing he would be unable to get rid of it. Similarly, no one would buy it for 2¢, and so on. However, for some reasonably large amount, it will always be possible to find a next buyer, so the bottle will be bought (Paulos 1995).

see also Sorites Paradox

References

Chow, T. Y. "The Surprise Examination or Unexpected Hanging Paradox." *Amer. Math. Monthly* **105**, 41–51, 1998.

Clark, D. "How Expected is the Unexpected Hanging?" *Math. Mag.* **67**, 55–58, 1994.

Gardner, M. "The Paradox of the Unexpected Hanging." Ch. 1 in *The Unexpected Hanging and Other Mathematical Diversions.* Chicago, IL: Chicago University Press, 1991.

Margalit, A. and Bar-Hillel, M. "Expecting the Unexpected." *Philosophia* **13**, 263–288, 1983.

Pappas, T. "The Paradox of the Unexpected Exam." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 147, 1989.

Paulos, J. A. *A Mathematician Reads the Newspaper.* New York: BasicBooks, p. 97, 1995.

Quine, W. V. O. "On a So-Called Paradox." *Mind* **62**, 65–67, 1953.

Unfinished Game

see Sharing Problem

Unhappy Number

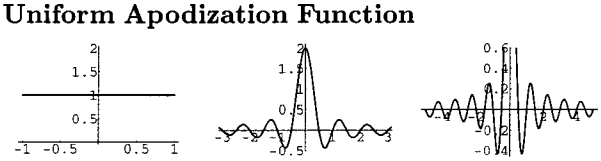
A number which is not Happy is said to be unhappy.

see also Happy Number

Unicursal Circuit

A Circuit in which an entire Graph is traversed in one route. An example of a curve which can be traced unicursally is the Mohammed Sign.

Uniform Apodization Function



An Apodization Function

$$f(x) = 1, \tag{1}$$

having INSTRUMENT FUNCTION

$$I(x) = \int_{-a}^{a} e^{-2\pi ikx}\,dx = -\frac{1}{2\pi ik}(e^{-2\pi ika} - e^{2\pi ikx})$$
$$= \frac{\sin(2\pi ka)}{\pi k} = 2a\,\mathrm{sinc}(2\pi ka). \qquad (2)$$

The peak (in units of a) is 2. The extrema are given by letting $\beta \equiv 2\pi ka$ and solving

$$\frac{d}{d\beta}(\beta\sin\beta) = \frac{\sin\beta - \beta\cos\beta}{\beta^2} = 0 \qquad (3)$$

$$\sin\beta - \beta\cos\beta = 0 \qquad (4)$$

$$\tan\beta = \beta. \qquad (5)$$

Solving this numerically gives $\beta_0 = 0$, $\beta_1 = 4.49341$, $\beta_2 = 7.72525$, ... for the first few solutions. The second of these is the peak POSITIVE sidelobe, and the third is the peak NEGATIVE sidelobe. As a fraction of the peak, they are 0.128375 and -0.217234. The FULL WIDTH AT HALF MAXIMUM is found by setting $I(x) = 1$

$$\mathrm{sinc}(x) = \tfrac{1}{2}, \qquad (6)$$

and solving for $x_{1/2}$, yielding

$$x_{1/2} = 2\pi k_{1/2}a = 1.89549. \qquad (7)$$

Therefore, with $L \equiv 2a$,

$$\mathrm{FWHM} = 2k_{1/2} = \frac{0.603353}{a} = \frac{1.20671}{L}. \qquad (8)$$

see also APODIZATION FUNCTION

Uniform Boundedness Principle

If a "pointwise-bounded" family of continuous linear OPERATORS from a BANACH SPACE to a NORMED SPACE is "uniformly bounded." Symbolically, if $\sup ||T_i(x)||$ is FINITE for each x in the unit BALL, then $\sup ||T_i||$ is FINITE. The theorem is also called the BANACH-STEINHAUS THEOREM.

References

Zeidler, E. *Applied Functional Analysis: Applications to Mathematical Physics.* New York: Springer-Verlag, 1995.

Uniform Convergence

A SERIES $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent to $S(x)$ for a set E of values of x if, for each $\epsilon > 0$, an INTEGER N can be found such that

$$|S_n(x) - S(x)| < \epsilon \qquad (1)$$

for $n \geq N$ and all $x \in E$. To test for uniform convergence, use ABEL'S UNIFORM CONVERGENCE TEST or the WEIERSTRAß M-TEST. If individual terms $u_n(x)$ of a uniformly converging series are continuous, then

1. The series sum
$$f(x) = \sum_{n=1}^{\infty} u_n(x) \qquad (2)$$
is continuous,
2. The series may be integrated term by term
$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty}\int_a^b u_n(x)\,dx, \qquad (3)$$
and
3. The series may be differentiated term by term
$$\frac{d}{dx}f(x) = \sum_{n=1}^{\infty} n\frac{d}{dx}u_n(x). \qquad (4)$$

see also ABEL'S THEOREM, ABEL'S UNIFORM CONVERGENCE TEST, WEIERSTRAß M-TEST

References

Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 299–301, 1985.

Uniform Distribution

A distribution which has constant probability is called a uniform distribution, sometimes also called a RECTANGULAR DISTRIBUTION. The probability density function and cumulative distribution function for a *continuous* uniform distribution are

$$P(x) = \begin{cases} \frac{1}{b-a} & \text{for } a < x < b \\ 0 & \text{for } x < a,\ x > b \end{cases} \qquad (1)$$

$$D(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \leq x < b \\ 1 & \text{for } x \geq b. \end{cases} \qquad (2)$$

With $a = 0$ and $b = 1$, these can be written

$$P(x) = \tfrac{1}{2}\,\mathrm{sgn}(x) - \mathrm{sgn}(x-1) \qquad (3)$$
$$D(x) = \tfrac{1}{2}[1 - (1-x)^2\,\mathrm{sgn}(1-x) + x\,\mathrm{sgn}(x)]. \qquad (4)$$

The CHARACTERISTIC FUNCTION is

$$\phi(t) = \frac{2}{ht}\sin(\tfrac{1}{2}ht)e^{imt}, \qquad (5)$$

where

$$a = m - \tfrac{1}{2}h \qquad (6)$$
$$b = m + \tfrac{1}{2}h. \qquad (7)$$

The MOMENT-GENERATING FUNCTION is

$$M(t) = \langle e^{xt}\rangle = \int_a^b \frac{e^{xt}}{b-a}dx = \left[\frac{e^{xt}}{t(b-a)}\right]_a^b, \qquad (8)$$

so

$$M(t) = \begin{cases} \frac{e^{tb}-e^{ta}}{t(b-a)} & \text{for } t \neq 0 \\ 0 & \text{for } t = 0, \end{cases} \tag{9}$$

and

$$\begin{aligned} M'(t) &= \frac{1}{b-a}\left[\frac{1}{t}(be^{bt} - ae^{at}) - \frac{1}{t^2}(e^{bt} - e^{at})\right] \\ &= \frac{e^{bt}(bt-1) - e^{at}(at-1)}{(b-a)t^2}. \end{aligned} \tag{10}$$

The function is not differentiable at zero, so the MOMENTS cannot be found using the standard technique. They can, however, be found by direct integration. The MOMENTS about 0 are

$$\mu_1' = \tfrac{1}{2}(a+b) \tag{11}$$
$$\mu_2' = \tfrac{1}{3}(a^2 + ab + b^2) \tag{12}$$
$$\mu_3' = \tfrac{1}{4}(a+b)(a^2+b^2) \tag{13}$$
$$\mu_4' = \tfrac{1}{5}(a^4 + a^3b + a^2b^2 + ab^3 + b^4). \tag{14}$$

The MOMENTS about the MEAN are

$$\mu_1 = 0 \tag{15}$$
$$\mu_2 = \tfrac{1}{12}(b-a)^2 \tag{16}$$
$$\mu_3 = 0 \tag{17}$$
$$\mu_4 = \tfrac{1}{80}(b-a)^4, \tag{18}$$

so the MEAN, VARIANCE, SKEWNESS, and KURTOSIS are

$$\mu = \tfrac{1}{2}(a+b) \tag{19}$$
$$\sigma^2 = \mu_2 = \tfrac{1}{12}(b-a)^2 \tag{20}$$
$$\gamma_1 = \frac{\mu_3}{\sigma^{3/2}} = 0 \tag{21}$$
$$\gamma_2 = -\tfrac{6}{5}. \tag{22}$$

The probability distribution function and cumulative distributions function for a *discrete* uniform distribution are

$$P(n) = \frac{1}{N} \tag{23}$$
$$D(n) = \frac{n}{N} \tag{24}$$

for $n = 1, \ldots, N$. The MOMENT-GENERATING FUNCTION is

$$\begin{aligned} M(t) = \langle e^{nt} \rangle &= \sum_{n=1}^{N} \frac{1}{N} e^{nt} = \frac{1}{N} \frac{e^t - e^{t(N+1)}}{1 - e^t} \\ &= \frac{e^t(1 - e^{Nt})}{N(1 - e^t)}. \end{aligned} \tag{25}$$

The MOMENTS about 0 are

$$\mu_m' = \frac{1}{N} \sum_{n=1}^{N} n^m, \tag{26}$$

so

$$\mu_1' = \tfrac{1}{2}(N+1) \tag{27}$$
$$\mu_2' = \tfrac{1}{6}(N+1)(2N+1) \tag{28}$$
$$\mu_3' = \tfrac{1}{4}N(N+1)^2 \tag{29}$$
$$\mu_4' = \tfrac{1}{30}(N+1)(2N+1)(3N^2+3N-1), \tag{30}$$

and the MOMENTS about the MEAN are

$$\mu_2 = \tfrac{1}{12}(N-1)(N+1) \tag{31}$$
$$\mu_3 = 0 \tag{32}$$
$$\mu_4 = \tfrac{1}{240}(N-1)(N+1)(3N^2-7). \tag{33}$$

The MEAN, VARIANCE, SKEWNESS, and KURTOSIS are

$$\mu = \tfrac{1}{2}(N+1) \tag{34}$$
$$\sigma^2 = \mu_2 = \tfrac{1}{12}(N-1)(N+1) \tag{35}$$
$$\gamma_1 = \frac{\mu_3}{\sigma^{3/2}} = 0 \tag{36}$$
$$\gamma_2 = \frac{6(N^2+1)}{5(N-1)(N+1)}. \tag{37}$$

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 531 and 533, 1987.

Uniform Polyhedron

The uniform polyhedra are POLYHEDRA with identical VERTICES. Coxeter *et al.* (1954) conjectured that there are 75 such polyhedra in which only two faces are allowed to meet at an EDGE, and this was subsequently proven. (However, when any EVEN number of faces may meet, there are 76 polyhedra.) If the five pentagonal PRISMS are included, the number rises to 80.

The VERTICES of a uniform polyhedron all lie on a SPHERE whose center is their CENTROID. The VERTICES joined to another VERTEX lie on a CIRCLE.

Source code and binary programs for generating and viewing the uniform polyhedra are also available at `http://www.math.technion.ac.il/~rl/kaleido/`. The following depictions of the polyhedra were produced by R. Maeder's `UniformPolyhedra.m` package for *Mathematica*® (Wolfram Research, Champaign, IL). Due to a limitation in *Mathematica*'s renderer, uniform polyhedra 69, 72, 74, and 75 cannot be displayed using this package.

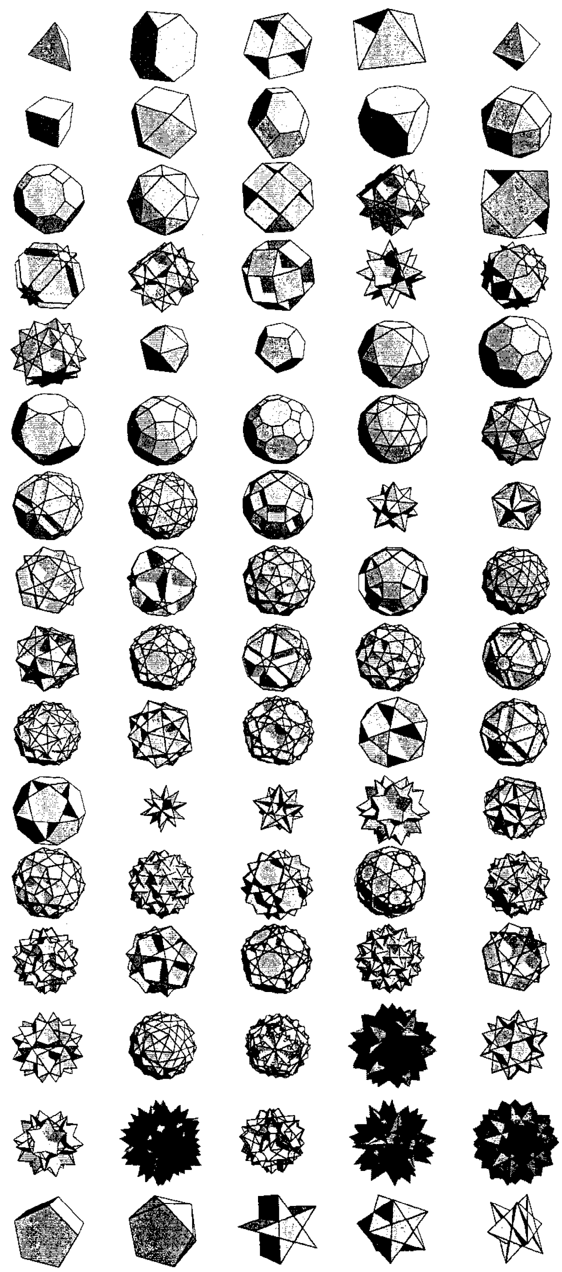
Uniform Polyhedron

n	Name/Dual
1	tetrahedron
	tetrahedron
2	truncated tetrahedron
	triakis tetrahedron
3	octahemioctahedron
	octahemioctacron
4	tetrahemihexahedron
	tetrahemihexacron
5	octahedron
	cube
6	cube
	octahedron
7	cuboctahedron
	rhombic dodecahedron
8	truncated octahedron
	tetrakis hexahedron
9	truncated cube
	triakis octahedron
10	small rhombicuboctahedron
	deltoidal icositetrahedron
11	truncated cuboctahedron
	disdyakis dodecahedron
12	snub cube
	pentagonal icositetrahedron
13	small cubicuboctahedron
	small hexacronic icositetrahedron
14	great cubicuboctahedron
	great hexacronic icositetrahedron
15	cubohemioctahedron
	hexahemioctahedron
16	cubitruncated cuboctahedron
	tetradyakis hexahedron
17	great rhombicuboctahedron
	great deltoidal icositetrahedron
18	small rhombihexahedron
	small rhombihexacron
19	stellated truncated hexahedron
	great triakis octahedron
20	great truncated cuboctahedron
	great disdyakis dodecahedron
21	great rhombihexahedron
	great rhombihexacron
22	icosahedron
	dodecahedron
23	dodecahedron
	icosahedron
24	icosidodecahedron
	rhombic triacontahedron
25	truncated icosahedron
	pentakis dodecahedron

n	Name/Dual
26	truncated dodecahedron
	triakis icosahedron
27	small rhombicosidodecahedron
	deltoidal hexecontahedron
28	truncated icosidodecahedron
	disdyakis triacontahedron
29	snub dodecahedron
	pentagonal hexecontahedron
30	small ditrigonal icosidodecahedron
	small triambic icosahedron
31	small icosicosidodecahedron
	small icosacronic hexecontahedron
32	small snub icosicosidodecahedron
	small hexagonal hexecontahedron
33	small dodecicosidodecahedron
	small dodecacronic hexecontahedron
34	small stellated dodecahedron
	great dodecahedron
35	great dodecahedron
	small stellated dodecahedron
36	dodecadodecahedron
	medial rhombic triacontahedron
37	truncated great dodecahedron
	small stellapentakis dodecahedron
38	rhombidodecadodecahedron
	medial deltoidal hexecontahedron
39	small rhombidodecahedron
	small rhombidodecacron
40	snub dodecadodecahedron
	medial pentagonal hexecontahedron
41	ditrigonal dodecadodecahedron
	medial triambic icosahedron
42	great ditrigonal dodecicosidodecahedron
	great ditrigonal dodecacronic hexecontahedron
43	small ditrigonal dodecicosidodecahedron
	small ditrigonal dodecacronic hexecontahedron
44	icosidodecadodecahedron
	medial icosacronic hexecontahedron
45	icositruncated dodecadodecahedron
	tridyakis icosahedron
46	snub icosidodecadodecahedron
	medial hexagonal hexecontahedron
47	great ditrigonal icosidodecahedron
	great triambic icosahedron
48	great icosicosidodecahedron
	great icosacronic hexecontahedron
49	small icosihemidodecahedron
	small icosihemidodecacron
50	small dodecicosahedron
	small dodecicosacron

n	Name/Dual
51	small dodecahemidodecahedron
	small dodecahemidodecacron
52	great stellated dodecahedron
	great icosahedron
53	great icosahedron
	great stellated dodecahedron
54	great icosidodecahedron
	great rhombic triacontahedron
55	great truncated icosahedron
	great stellapentakis dodecahedron
56	rhombicosahedron
	rhombicosacron
57	great snub icosidodecahedron
	great pentagonal hexecontahedron
58	small stellated truncated dodecahedron
	great pentakis dodecahedron
59	truncated dodecadodecahedron
	medial disdyakis triacontahedron
60	inverted snub dodecadodecahedron
	medial inverted pentagonal hexecontahedron
61	great dodecicosidodecahedron
	great dodecacronic hexecontahedron
62	small dodecahemicosahedron
	small dodecahemicosacron
63	great dodecicosahedron
	great dodecicosacron
64	great snub dodecicosidodecahedron
	great hexagonal hexecontahedron
65	great dodecahemicosahedron
	great dodecahemicosacron
66	great stellated truncated dodecahedron
	great triakis icosahedron
67	great rhombicosidodecahedron
	great deltoidal hexecontahedron
68	great truncated icosidodecahedron
	great disdyakis triacontahedron
69	great inverted snub icosidodecahedron
	great inverted pentagonal hexecontahedron
70	great dodecahemidodecahedron
	great dodecahemidodecacron
71	great icosihemidodecahedron
	great icosihemidodecacron
72	small retrosnub icosicosidodecahedron
	small hexagrammic hexecontahedron
73	great rhombidodecahedron
	great rhombidodecacron
74	great retrosnub icosidodecahedron
	great pentagrammic hexecontahedron
75	great dirhombicosidodecahedron
	great dirhombicosidodecacron

n	Name/Dual
76	pentagonal prism
	pentagonal dipyramid
77	pentagonal antiprism
	pentagonal deltahedron
78	pentagrammic prism
	pentagrammic dipyramid
79	pentagrammic antiprism
	pentagrammic deltahedron
80	pentagrammic crossed antiprism
	pentagrammic concave deltahedron



see also ARCHIMEDEAN SOLID, AUGMENTED POLYHEDRON, JOHNSON SOLID, KEPLER-POINSOT SOLID, PLATONIC SOLID, POLYHEDRON, VERTEX FIGURE, WYTHOFF SYMBOL

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Uniform Variate

A RANDOM NUMBER which lies within a specified range (which can, without loss of generality, be taken as [0, 1]), with a UNIFORM DISTRIBUTION.

References
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Uniform Deviates." §7.1 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 267–277, 1992.

Unimodal Distribution

A DISTRIBUTION such as the GAUSSIAN DISTRIBUTION which has a single "peak."

see also BIMODAL DISTRIBUTION

Unimodal Sequence

A finite SEQUENCE which first increases and then decreases. A SEQUENCE $\{s_1, s_2, \ldots, s_n\}$ is unimodal if there exists a t such that

$$s_1 \leq s_2 \leq \ldots \leq s_t$$

and

$$s_t \geq s_{t+1} \geq \ldots \geq s_n.$$

Unimodular Group

A group whose left HAAR MEASURE equals its right HAAR MEASURE.

see also HAAR MEASURE

References
Knapp, A. W. "Group Representations and Harmonic Analysis, Part II." *Not. Amer. Math. Soc.* **43**, 537–549, 1996.

Unimodular Matrix

A MATRIX A with INTEGER elements and DETERMINANT $\det(\mathsf{A}) = \pm 1$, also called a UNIT MATRIX.

The inverse of a unimodular matrix is another unimodular matrix. A POSITIVE unimodular matrix has $\det(\mathsf{A}) = +1$. The nth POWER of a POSITIVE UNIMODULAR MATRIX

$$\mathsf{M} \equiv \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \tag{1}$$

is

$$\mathsf{M}^n = \begin{bmatrix} m_{11}U_{n-1}(a) - U_{n-2}(a) & m_{12}U_{n-1}(a) \\ m_{21}U_{n-1}(a) & m_{22}U_{n-1}(a) - U_{n-2}(a) \end{bmatrix}, \tag{2}$$

where

$$a \equiv \tfrac{1}{2}(m_{11} + m_{22}) \tag{3}$$

and the U_n are CHEBYSHEV POLYNOMIALS OF THE SECOND KIND,

$$U_m(x) = \frac{\sin[(m+1)\cos^{-1} x]}{\sqrt{1-x^2}}. \tag{4}$$

see also CHEBYSHEV POLYNOMIAL OF THE SECOND KIND

References
Born, M. and Wolf, E. *Principles of Optics: Electromagnetic Theory of Propagation, Interference, and Diffraction of Light, 6th ed.* New York: Pergamon Press, p. 67, 1980.
Goldstein, H. *Classical Mechanics, 2nd ed.* Reading, MA: Addison-Wesley, p. 149, 1980.

Unimodular Transformation

A transformation $\mathbf{x}' = \mathsf{A}\mathbf{x}$ is unimodular if the DETERMINANT of the MATRIX A satisfies

$$\det(\mathsf{A}) = \pm 1.$$

A NECESSARY and SUFFICIENT condition that a linear transformation transform a lattice to itself is that the transformation be unimodular.

Union

The union of two sets A and B is the set obtained by combining the members of each. This is written $A \cup B$, and is pronounced "A union B" or "A cup B." The union of sets A_1 through A_n is written $\bigcup_{i=1}^n A_i$.

Let A, B, C, ... be sets, and let $P(S)$ denote the probability of S. Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \quad (1)$$

Similarly,

$$\begin{aligned} P(A \cup B \cup C) &= P[A \cup (B \cup C)] \\ &= P(A) + P(B \cup C) - P[A \cap (B \cup C)] \\ &= P(A) + [P(B) + P(C) - P(B \cap C)] \\ &\quad - P[(A \cap B) \cup (A \cap C)] \\ &= P(A) + P(B) + P(C) - P(B \cap C) \\ &\quad - \{P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]\} \\ &= P(A) + P(B) + P(C) - P(A \cap B) \\ &\quad - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C). \quad (2) \end{aligned}$$

If A and B are DISJOINT, by definition $P(A \cap B) = 0$, so

$$P(A \cup B) = P(A) + P(B). \quad (3)$$

Continuing, for a set of n disjoint elements E_1, E_2, ..., E_n

$$P\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n P(E_i), \quad (4)$$

which is the COUNTABLE ADDITIVITY PROBABILITY AXIOM. Now let

$$E_i \equiv A \cap B_i, \quad (5)$$

then

$$P\left(\bigcup_{i=1}^n E \cap B_i\right) = \sum_{i=1}^n P(E \cap B_i). \quad (6)$$

see also INTERSECTION, OR

Uniplanar Double Point

see ISOLATED SINGULARITY

Unipotent

A p-ELEMENT x of a GROUP G is unipotent if $F^*(C_G(x))$ is a p-GROUP, where F^* is the generalized FITTING SUBGROUP.

see also FITTING SUBGROUP, p-ELEMENT, p-GROUP

Unique

The property of being the only possible solution (perhaps modulo a constant, class of transformation, etc.).

see also ALEKSANDROV'S UNIQUENESS THEOREM, EXISTENCE, MAY-THOMASON UNIQUENESS THEOREM

Unique Factorization Theorem

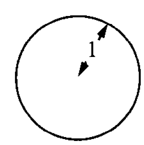
see FUNDAMENTAL THEOREM OF ARITHMETIC

Unit

A unit is an element in a RING that has a multiplicative inverse. If n is an ALGEBRAIC INTEGER which divides every ALGEBRAIC INTEGER in the FIELD, n is called a unit in that FIELD. A given FIELD may contain an infinity of units. The units of $\mathbb{Z}_n$ are the elements RELATIVELY PRIME to n. The units in $\mathbb{Z}_n$ which are SQUARES are called QUADRATIC RESIDUES.

see also EISENSTEIN UNIT, FUNDAMENTAL UNIT, PRIME UNIT, QUADRATIC RESIDUE

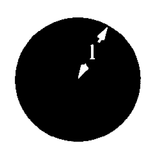
Unit Circle



A CIRCLE of RADIUS 1, such as the one used to defined the functions of TRIGONOMETRY.

see also UNIT DISK, UNIT SQUARE

Unit Disk



A DISK with RADIUS 1.

see also FIVE DISKS PROBLEM, UNIT CIRCLE, UNIT SQUARE

Unit Fraction

A unit fraction is a FRACTION with NUMERATOR 1, also known as an EGYPTIAN FRACTION. Any RATIONAL NUMBER has infinitely many unit fraction representations, although only finitely many have a given fixed number of terms. Each FRACTION x/y with y ODD has a unit fraction representation in which each DENOMINATOR is ODD (Breusch 1954; Guy 1994, p. 160). Every x/y has a t-term representation where $t = \mathcal{O}(\sqrt{\log y})$ (Vose 1985).

There are a number of ALGORITHMS (including the BINARY REMAINDER METHOD, CONTINUED FRACTION UNIT FRACTION ALGORITHM, GENERALIZED REMAINDER METHOD, GREEDY ALGORITHM, REVERSE GREEDY ALGORITHM, SMALL MULTIPLE METHOD, and SPLITTING ALGORITHM) for decomposing an arbitrary FRACTION into unit fractions.

see also CALCUS, HALF, QUARTER, SCRUPLE, UNCIA

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Beeckmans, L. "The Splitting Algorithm for Egyptian Fractions." *J. Number Th.* **43**, 173–185, 1993.

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Vose, M. "Egyptian Fractions." *Bull. London Math. Soc.* **17**, 21, 1985.

Wagon, S. "Egyptian Fractions." §8.6 in *Mathematica in Action.* New York: W. H. Freeman, pp. 271–277, 1991.

Unit Matrix

see UNIMODULAR MATRIX

Unit Point

The point in the PLANE with Cartesian coordinates (1, 1).

References

Woods, F. S. *Higher Geometry: An Introduction to Advanced Methods in Analytic Geometry.* New York: Dover, p. 9, 1961.

Unit Ring

A unit ring is a set together with two BINARY OPERATORS $S(+,*)$ satisfying the following conditions:

1. Additive associativity: For all $a, b, c \in S$, $(a+b)+c = a+(b+c)$,
2. Additive commutativity: For all $a, b \in S$, $a+b = b+a$,
3. Additive identity: There exists an element $0 \in S$ such that for all $a \in S : 0+a = a+0 = a$,
4. Additive inverse: For every $a \in S$, there exists a $-a \in S$ such that $a+(-a) = (-a)+a = 0$,
5. Multiplicative associativity: For all $a, b, c \in S$, $(a*b)*c = a*(b*c)$,
6. Multiplicative identity: There exists an element $1 \in S$ such that for all $a \in S$, $1*a = a*1 = a$,
7. Left and right distributivity: For all $a, b, c \in S$, $a*(b+c) = (a*b)+(a*c)$ and $(b+c)*a = (b*a)+(c*a)$.

Thus, a unit ring is a RING with a multiplicative identity.

see also BINARY OPERATOR, RING

References

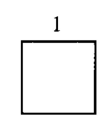
Rosenfeld, A. *An Introduction to Algebraic Structures.* New York: Holden-Day, 1968.

Unit Sphere

A SPHERE of RADIUS 1.

see also SPHERE, UNIT CIRCLE

Unit Square



A SQUARE with side lengths 1. *The* unit square usually means the one with coordinates (0, 0), (1, 0), (1, 1), (0, 1) in the real plane, or 0, 1, $1+i$, and i in the COMPLEX PLANE.

see also HEILBRONN TRIANGLE PROBLEM, UNIT CIRCLE, UNIT DISK

Unit Step

see HEAVISIDE STEP FUNCTION

Unit Vector

A VECTOR of unit length. The unit vector $\hat{\mathbf{v}}$ having the same direction as a given (nonzero) vector $\mathbf{v}$ is defined by

$$\hat{\mathbf{v}} \equiv \frac{\mathbf{v}}{|\mathbf{v}|},$$

where $|\mathbf{v}|$ denotes the NORM of $\mathbf{v}$, is the unit vector in the same direction as the (finite) VECTOR $\mathbf{v}$. A unit VECTOR in the $\mathbf{x}_n$ direction is given by

$$\hat{\mathbf{x}}_n \equiv \frac{\frac{\partial \mathbf{r}}{\partial x_n}}{\left|\frac{\partial \mathbf{r}}{\partial x_n}\right|},$$

where $\mathbf{r}$ is the RADIUS VECTOR.

see also NORM, RADIUS VECTOR, VECTOR

Unital

A BLOCK DESIGN of the form $(q^3 + 1, q + 1, 1)$.

References

Dinitz, J. H. and Stinson, D. R. "A Brief Introduction to Design Theory." Ch. 1 in *Contemporary Design Theory: A Collection of Surveys* (Ed. J. H. Dinitz and D. R. Stinson). New York: Wiley, pp. 1–12, 1992.

Unitary Aliquot Sequence

An ALIQUOT SEQUENCE computed using the analog of the RESTRICTED DIVISOR FUNCTION $s^*(n)$ in which only UNITARY DIVISORS are included.

see also ALIQUOT SEQUENCE, UNITARY SOCIABLE NUMBERS

References

Guy, R. K. "Unitary Aliquot Sequences." §B8 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 63–65, 1994.

Unitary Amicable Pair

A PAIR of numbers m and n such that

$$\sigma^*(m) = \sigma^*(n) = m + n,$$

where $\sigma^*(n)$ is the sum of UNITARY DIVISORS. Hagis (1971) and García (1987) give 82 such pairs. The first few are (114, 126), (1140, 1260), (18018, 22302), (32130, 40446), ... (Sloane's A002952 and A002953).

References

García, M. "New Unitary Amicable Couples." *J. Recr. Math.* **19**, 12–14, 1987.

Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 57, 1994.

Hagis, P. "Relatively Prime Amicable Numbers of Opposite Parity." *Math. Comput.* **25**, 915–918, 1971.

Sloane, N. J. A. Sequences A002952/M5372 and A002953/M5389 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Unitary Divisor

A DIVISOR d of c for which

$$\mathrm{GCD}(d, c/d) = 1,$$

where GCD is the GREATEST COMMON DIVISOR.

see also DIVISOR, GREATEST COMMON DIVISOR, UNITARY PERFECT NUMBER

References

Guy, R. K. "Unitary Perfect Numbers." §B3 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 53–59, 1994.

Unitary Group

The unitary group $U_n(q)$ is the set of $n \times n$ UNITARY MATRICES.

see also LIE-TYPE GROUP, UNITARY MATRIX

References

Wilson, R. A. "ATLAS of Finite Group Representation." `http://for.mat.bham.ac.uk/atlas#unit`.

Unitary Matrix

A unitary matrix is a MATRIX U for which

$$\mathsf{U}^\dagger = \mathsf{U}^{-1}, \tag{1}$$

where $\dagger$ denotes the ADJOINT OPERATOR. This guarantees that

$$\mathsf{U}^\dagger \mathsf{U} = 1. \tag{2}$$

Unitary matrices leave the length of a COMPLEX vector unchanged. The product of two unitary matrices is itself unitary. If U is unitary, then so is U^{-1}. A SIMILARITY TRANSFORMATION of a HERMITIAN MATRIX with a unitary matrix gives

$$\begin{aligned}(uau^{-1})^\dagger &= [(ua)(u^{-1})]^\dagger = (u^{-1})^\dagger(ua)^\dagger = (u^\dagger)^\dagger(a^\dagger u^\dagger)\\ &= uau^\dagger = uau^{-1}.\end{aligned} \tag{3}$$

For REAL MATRICES, HERMITIAN is the same as ORTHOGONAL. Unitary matrices are NORMAL MATRICES.

If M is a unitary matrix, then the PERMANENT

$$|\operatorname{perm}(\mathsf{M})| \leq 1 \tag{4}$$

(Minc 1978, p. 25, Vardi 1991).

see also ADJOINT OPERATOR, HERMITIAN MATRIX, NORMAL MATRIX, ORTHOGONAL MATRIX, PERMANENT

References

Arfken, G. "Hermitian Matrices, Unitary Matrices." §4.5 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 209–217, 1985.

Minc, H. *Permanents.* Reading, MA: Addison-Wesley, 1978.

Vardi, I. "Permanents." §6.1 in *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, pp. 108 and 110–112, 1991.

Unitary Multiperfect Number

A number n which is an INTEGER multiple k of the SUM of its UNITARY DIVISORS $\sigma^*(n)$ is called a unitary k-multiperfect number. There are no ODD unitary multiperfect numbers.

References

Guy, R. K. "Unitary Perfect Numbers." §B3 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 53–59, 1994.

Unitary Multiplicative Character

A MULTIPLICATIVE CHARACTER is called unitary if it has ABSOLUTE VALUE 1 everywhere.

see also CHARACTER (MULTIPLICATIVE)

Unitary Perfect Number

A number n which is the sum of its UNITARY DIVISORS with the exception of n itself. There are no ODD unitary perfect numbers, and it has been conjectured that there are only a FINITE number of EVEN ones. The first few are 6, 60, 90, 87360, 146361946186458562560000, ... (Sloane's A002827).

References

Guy, R. K. "Unitary Perfect Numbers." §B3 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 53–59, 1994.

Sloane, N. J. A. Sequence A002827/M4268 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Wall, C. R. "On the Largest Odd Component of a Unitary Perfect Number." *Fib. Quart.* **25**, 312–316, 1987.

Unitary Sociable Numbers

SOCIABLE NUMBERS computed using the analog of the RESTRICTED DIVISOR FUNCTION $s^*(n)$ in which only UNITARY DIVISORS are included.

see also SOCIABLE NUMBERS

References

Guy, R. K. "Unitary Aliquot Sequences." §B8 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 63–65, 1994.

Unitary Transformation

A transformation of the form

$$\mathsf{A}' = \mathsf{U}\mathsf{A}\mathsf{U}^\dagger,$$

where † denotes the ADJOINT OPERATOR.

see also ADJOINT OPERATOR, TRANSFORMATION

Unitary Unimodular Group

see SPECIAL UNITARY GROUP

Unity

The number 1. There are n nth ROOTS OF UNITY, known as the DE MOIVRE NUMBERS.

see also 1, PRIMITIVE ROOT OF UNITY

Univalent Function

A function or transformation f in which $f(z)$ does not overlap z.

Univariate Function

A FUNCTION of a single variable (e.g., $f(x)$, $g(z)$, $\theta(\xi)$, etc.).

see also MULTIVARIATE FUNCTION

Univariate Polynomial

A POLYNOMIAL in a single variable. In common usage, univariate POLYNOMIALS are sometimes simply called "POLYNOMIALS."

see also POLYNOMIAL

Universal Graph

see COMPLETE GRAPH

Universal Statement

A universal statement S is a FORMULA whose FREE variables are all in the scope of universal quantifiers.

Universal Turing Machine

A TURING MACHINE which, by appropriate programming using a finite length of input tape, can act as *any* TURING MACHINE whatsoever.

see CHAITIN'S CONSTANT, HALTING PROBLEM, TURING MACHINE

References

Penrose, R. *The Emperor's New Mind: Concerning Computers, Minds, and the Laws of Physics.* Oxford: Oxford University Press, pp. 51–57, 1989.

Unknot

A closed loop which is not KNOTTED. In the 1930s, by making use of REIDEMEISTER MOVES, Reidemeister first proved that KNOTS exist which are distinct from the unknot. He proved this by COLORING each part of a knot diagram with one of three colors.

The KNOT SUM of two unknots is another unknot.

The JONES POLYNOMIAL of the unknot is defined to give the normalization

$$V(t) = 1.$$

Haken (1961) devised an ALGORITHM to tell if a knot projection is the unknot. The ALGORITHM is so complicated, however, that it has never been implemented. Although it is not immediately obvious, the unknot is a PRIME KNOT.

see also COLORABLE, KNOT, KNOT THEORY, LINK, REIDEMEISTER MOVES, UNKNOTTING NUMBER

References

Haken, W. "Theorie der Normalflachen." *Acta Math.* **105**, 245–375, 1961.

Unknotting Number

The smallest number of times a KNOT must be passed through itself to untie it. Lower bounds can be computed using relatively straightforward techniques, but it is in general difficult to determine exact values. Many unknotting numbers can be determined from a knot's SIGNATURE. A KNOT with unknotting number 1 is a PRIME KNOT (Scharlemann 1985). It is not always true that the unknotting number is achieved in a projection with the minimal number of crossings.

The following table is from Kirby (1997, pp. 88–89), with the values for 10_{139} and 10_{152} taken from Kawamura. The unknotting numbers for 10_{154} and 10_{161} can be found using MENASCO'S THEOREM (Stoimenow 1998).

3_1	1	8_9	1	9_{10}	2 or 3	9_{32}	1 or 2
4_1	1	8_{10}	1 or 2	9_{11}	2	9_{33}	1
5_1	2	8_{11}	1	9_{12}	1	9_{34}	1
5_2	1	8_{12}	2	9_{13}	2 or 3	9_{35}	2 or 3
6_1	1	8_{13}	1	9_{14}	1	9_{36}	2
6_2	1	8_{14}	1	9_{15}	2	9_{37}	2
6_3	1	8_{15}	2	9_{16}	3	9_{38}	2 or 3
7_1	3	8_{16}	2	9_{17}	2	9_{39}	1
7_2	1	8_{17}	1	9_{18}	2	9_{40}	2
7_3	2	8_{18}	2	9_{19}	1	9_{41}	2
7_4	2	8_{19}	3	9_{20}	2	9_{42}	1
7_5	2	8_{20}	1	9_{21}	1	9_{43}	2
7_6	1	8_{21}	1	9_{22}	1	9_{44}	1
7_7	1	9_1	4	9_{23}	2	9_{45}	1
8_1	1	9_2	1	9_{24}	1	9_{46}	2
8_2	2	9_3	3	9_{25}	2	9_{47}	2
8_3	2	9_4	2	9_{26}	1	9_{48}	2
8_4	2	9_5	2	9_{27}	1	9_{49}	2 or 3
8_5	2	9_6	3	9_{28}	1	10_{139}	4
8_6	2	9_7	2	9_{29}	1	10_{152}	4
8_7	1	9_8	2	9_{30}	1	10_{154}	3
8_8	2	9_9	3	9_{31}	2	10_{161}	3

see also BENNEQUIN'S CONJECTURE, MENASCO'S THEOREM, MILNOR'S CONJECTURE, SIGNATURE (KNOT)

References

Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, pp. 57–64, 1994.

Cipra, B. "From Knot to Unknot." *What's Happening in the Mathematical Sciences, Vol. 2.* Providence, RI: Amer. Math. Soc., pp. 8–13, 1994.

Kawamura, T. "The Unknotting Numbers of 10_{139} and 10_{152} are 4." To appear in *Osaka J. Math.* `http://ms421sun.ms.u-tokyo.ac.jp/~kawamura/worke.html`.

Kirby, R. (Ed.) "Problems in Low-Dimensional Topology." *AMS/IP Stud. Adv. Math., 2.2, Geometric Topology (Athens, GA, 1993).* Providence, RI: Amer. Math. Soc., pp. 35–473, 1997.

Scharlemann, M. "Unknotting Number One Knots are Prime." *Invent. Math.* **82**, 37–55, 1985.

Stoimenow, A. "Positive Knots, Closed Braids and the Jones Polynomial." Rev. May, 1997. `http://www.informatik.hu-berlin.de/~stoimeno/pos.ps.gz`.

Weisstein, E. W. "Knots and Links." `http://www.astro.virginia.edu/~eww6n/math/notebooks/Knots.m`.

Unless

If A is true unless B, then not-B implies A, but B does not necessarily imply not-A.

see also PRECISELY UNLESS

Unlesss

see PRECISELY UNLESS

Unmixed

A homogeneous IDEAL defining a projective ALGEBRAIC VARIETY is unmixed if it has no embedded PRIME divisors.

Unpoke Move

see POKE MOVE

Unsafe

A position in a GAME is unsafe if the person who plays next can win. Every unsafe position can be made SAFE by at least one move.

see also GAME, SAFE

Unsolved Problem

see PROBLEM

Unstable Improper Node

A FIXED POINT for which the STABILITY MATRIX has equal POSITIVE EIGENVALUES.

see also ELLIPTIC FIXED POINT (DIFFERENTIAL EQUATIONS), FIXED POINT, HYPERBOLIC FIXED POINT (DIFFERENTIAL EQUATIONS), STABLE IMPROPER NODE, STABLE NODE, STABLE SPIRAL POINT, UNSTABLE NODE, UNSTABLE SPIRAL POINT, UNSTABLE STAR

References

Tabor, M. "Classification of Fixed Points." §1.4.b in *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, pp. 22–25, 1989.

Unstable Node

A FIXED POINT for which the STABILITY MATRIX has both EIGENVALUES POSITIVE, so $\lambda_1 > \lambda_2 > 0$.

see also ELLIPTIC FIXED POINT (DIFFERENTIAL EQUATIONS), FIXED POINT, HYPERBOLIC FIXED POINT (DIFFERENTIAL EQUATIONS), STABLE IMPROPER NODE, STABLE NODE, STABLE SPIRAL POINT, STABLE STAR, UNSTABLE IMPROPER NODE, UNSTABLE SPIRAL POINT, UNSTABLE STAR

References

Tabor, M. "Classification of Fixed Points." §1.4.b in *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, pp. 22–25, 1989.

Unstable Spiral Point

A FIXED POINT for which the STABILITY MATRIX has EIGENVALUES of the form $\lambda_\pm = \alpha \pm i\beta$ (with $\alpha, \beta > 0$).

see also ELLIPTIC FIXED POINT (DIFFERENTIAL EQUATIONS), FIXED POINT, HYPERBOLIC FIXED POINT (DIFFERENTIAL EQUATIONS), STABLE IMPROPER NODE, STABLE NODE, STABLE SPIRAL POINT, STABLE STAR, UNSTABLE IMPROPER NODE, UNSTABLE NODE, UNSTABLE STAR

References

Tabor, M. "Classification of Fixed Points." §1.4.b in *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, pp. 22–25, 1989.

Unstable Star

A FIXED POINT for which the STABILITY MATRIX has one zero EIGENVECTOR with POSITIVE EIGENVALUE $\lambda > 0$.

see also ELLIPTIC FIXED POINT (DIFFERENTIAL EQUATIONS), FIXED POINT, HYPERBOLIC FIXED POINT (DIFFERENTIAL EQUATIONS), STABLE IMPROPER NODE, STABLE NODE, STABLE SPIRAL POINT, STABLE STAR, UNSTABLE IMPROPER NODE, UNSTABLE NODE, UNSTABLE SPIRAL POINT

References

Tabor, M. "Classification of Fixed Points." §1.4.b in *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, pp. 22–25, 1989.

Untouchable Number

An untouchable number is an INTEGER which is not the sum of the PROPER DIVISORS of any other number. The first few are 2, 5, 52, 88, 96, 120, 124, 146, ... (Sloane's A005114). Erdős has proven that there are infinitely many. It is thought that 5 is the only ODD untouchable number.

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 840, 1972.

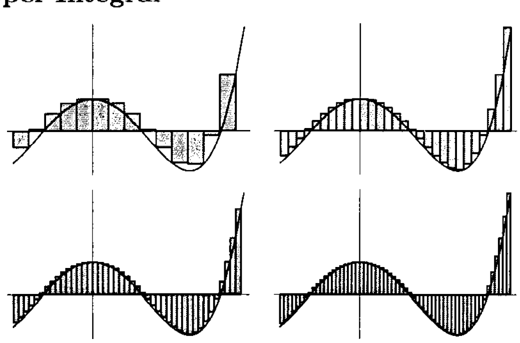
Guy, R. K. "Untouchable Numbers." §B10 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 66–67, 1994.

Sloane, N. J. A. Sequence A005114/M1552 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Upper Bound

see LEAST UPPER BOUND

Upper Integral



The limit of an UPPER SUM, when it exists, as the MESH SIZE approaches 0.

see also LOWER INTEGRAL, RIEMANN INTEGRAL, UPPER SUM

Upper Limit

Let the greatest term H of a SEQUENCE be a term which is greater than all but a finite number of the terms which are equal to H. Then H is called the upper limit of the SEQUENCE.

An upper limit of a SERIES

$$\text{upper} \lim_{n\to\infty} S_n = \overline{\lim_{n\to\infty}} S_n = k$$

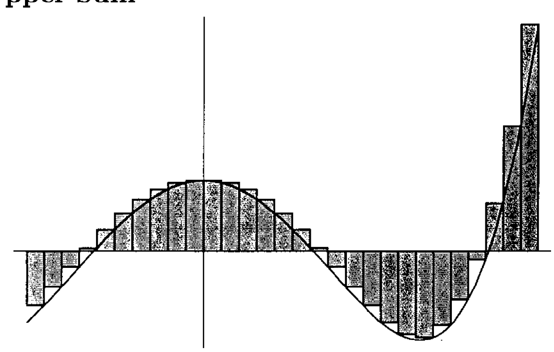
is said to exist if, for every $\epsilon > 0$, $|S_n - k| < \epsilon$ for infinitely many values of n and if no number larger than k has this property.

see also LIMIT, LOWER LIMIT

References

Bromwich, T. J. I'a and MacRobert, T. M. "Upper and Lower Limits of a Sequence." §5.1 in *An Introduction to the Theory of Infinite Series, 3rd ed.* New York: Chelsea, p. 40, 1991.

Upper Sum



For a given function $f(x)$ over a partition of a given interval, the upper sum is the sum of box areas $f(x_k^*)\Delta x_k$ using the greatest value of the function $f(x_k^*)$ in each subinterval Δx_k.

see also LOWER SUM, RIEMANN INTEGRAL, UPPER INTEGRAL

Upper-Trimmed Subsequence

The upper-trimmed subsequence of $x = \{x_n\}$ is the sequence $\lambda(x)$ obtained by dropping the first occurrence of n for each n. If x is a FRACTAL SEQUENCE, then $\lambda(x) = x$.

see also LOWER-TRIMMED SUBSEQUENCE

References

Kimberling, C. "Fractal Sequences and Interspersions." *Ars Combin.* **45**, 157–168, 1997.

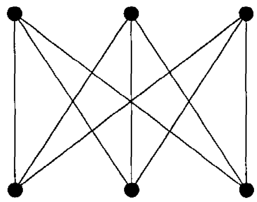
Upward Drawing

see HASSE DIAGRAM

Urchin

Kepler's original name for the SMALL STELLATED DODECAHEDRON.

Utility Graph



The utility problem asks, "Can a PLANAR GRAPH be constructed from each of three nodes ('house owners') to each of three other nodes ('wells')?" The answer is no, and the proof can be effected using the JORDAN CURVE THEOREM, while a more general result encompassing this one is the KURATOWSKI REDUCTION THEOREM. The utility graph UG is the graph showing the relationships described above. It is identical to the THOMSEN GRAPH and, in the more formal parlance of GRAPH THEORY, is known as the COMPLETE BIPARTITE GRAPH $K_{3,3}$.

see also COMPLETE BIPARTITE GRAPH, KURATOWSKI REDUCTION THEOREM, PLANAR GRAPH, THOMSEN GRAPH

References

Chartrand, G. "The Three Houses and Three Utilities Problem: An Introduction to Planar Graphs." §9.1 in *Introductory Graph Theory.* New York: Dover, pp. 191–202, 1985.

Ore, Ø. *Graphs and Their Uses.* New York: Random House, pp. 14–17, 1963.

Pappas, T. "Wood, Water, Grain Problem." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, pp. 175 and 233, 1989.

Utility Problem

see UTILITY GRAPH

V

Valence

see VALENCY

Valency

The number of EDGES at a GRAPH VERTEX.

Valuation

A generalization of the p-ADIC NUMBERS first proposed by Kürschák in 1913. A valuation $|\cdot|$ on a FIELD K is a FUNCTION from K to the REAL NUMBERS $\mathbb{R}$ such that the following properties hold for all $x, y \in K$:

1. $|x| \geq 0$,
2. $|x| = 0$ IFF $x = 0$,
3. $|xy| = |x|\,|y|$,
4. $|x| \leq 1$ IMPLIES $|1+x| \leq C$ for some constant $C \geq 1$ (independent of x).

If (4) is satisfied for $C = 2$, then $|\cdot|$ satisfies the TRIANGLE INEQUALITY,

4a. $|x+y| \leq |x| + |y|$ for all $x, y \in K$.

If (4) is satisfied for $C = 1$ then $|\cdot|$ satisfies the stronger TRIANGLE INEQUALITY

4b. $|x+y| \leq \max(|x|, |y|)$.

The simplest valuation is the ABSOLUTE VALUE for REAL NUMBERS. A valuation satisfying (4b) is called non-ARCHIMEDEAN VALUATION; otherwise, it is called ARCHIMEDEAN.

If $|\cdot|_1$ is a valuation on K and $\lambda \geq 1$, then we can define a new valuation $|\cdot|_2$ by

$$|x|_2 = |x|_1^\lambda. \tag{1}$$

This does indeed give a valuation, but possibly with a different constant C in AXIOM 4. If two valuations are related in this way, they are said to be equivalent, and this gives an equivalence relation on the collection of all valuations on K. Any valuation is equivalent to one which satisfies the triangle inequality (4a). In view of this, we need only to study valuations satisfying (4a), and we often view axioms (4) and (4a) as interchangeable (although this is not strictly true).

If two valuations are equivalent, then they are both non-ARCHIMEDEAN or both ARCHIMEDEAN. $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ with the usual Euclidean norms are Archimedean valuated fields. For any PRIME p, the p-ADIC NUMBERS $\mathbb{Q}_p$ with the p-adic valuation $|\cdot|_p$ is a non-Archimedean valuated field.

If K is any FIELD, we can define the trivial valuation on K by $|x| = 1$ for all $x \neq 0$ and $|0| = 0$, which is a non-Archimedean valuation. If K is a FINITE FIELD, then the only possible valuation over K is the trivial one. It can be shown that any valuation on $\mathbb{Q}$ is equivalent to one of the following: the trivial valuation, Euclidean absolute norm $|\cdot|$, or p-adic valuation $|\cdot|_p$.

The equivalence of any nontrivial valuation of $\mathbb{Q}$ to either the usual ABSOLUTE VALUE or to a p-ADIC NUMBER absolute value was proved by Ostrowski (1935). Equivalent valuations give rise to the same topology. Conversely, if two valuations have the same topology, then they are equivalent. A stronger result is the following: Let $|\cdot|_1, |\cdot|_2, \ldots, |\cdot|_k$ be valuations over K which are pairwise inequivalent and let $a_1, a_2, \ldots, a_k$ be elements of K. Then there exists an infinite sequence $(x_1, x_2, \ldots)$ of elements of K such that

$$\lim_{n\to\infty \text{ w.r.t. } |\cdot|_1} x_n = a_1 \tag{2}$$

$$\lim_{n\to\infty \text{ w.r.t. } |\cdot|_2} x_n = a_2, \tag{3}$$

etc. This says that inequivalent valuations are, in some sense, completely independent of each other. For example, consider the rationals $\mathbb{Q}$ with the 3-adic and 5-adic valuations $|\cdot|_3$ and $|\cdot|_5$, and consider the sequence of numbers given by

$$x_n = \frac{43 \cdot 5^n + 92 \cdot 3^n}{3^n + 5^n}. \tag{4}$$

Then $x_n \to 43$ as $n \to \infty$ with respect to $|\cdot|_3$, but $x_n \to 92$ as $n \to \infty$ with respect to $|\cdot|_5$, illustrating that a sequence of numbers can tend to two different limits under two different valuations.

A discrete valuation is a valuation for which the VALUATION GROUP is a discrete subset of the REAL NUMBERS $\mathbb{R}$. Equivalently, a valuation (on a FIELD K) is discrete if there exists a REAL NUMBER $\epsilon > 0$ such that

$$|x| \in (1-\epsilon, 1+\epsilon) \Rightarrow |x| = 1 \text{ for all } x \in K. \tag{5}$$

The p-adic valuation on $\mathbb{Q}$ is discrete, but the ordinary absolute valuation is not.

If $|\cdot|$ is a valuation on K, then it induces a metric

$$d(x,y) = |x-y| \tag{6}$$

on K, which in turn induces a TOPOLOGY on K. If $|\cdot|$ satisfies (4b) then the metric is an ULTRAMETRIC. We say that $(K, |\cdot|)$ is a complete valuated field if the METRIC SPACE is complete.

see also ABSOLUTE VALUE, LOCAL FIELD, METRIC SPACE, p-ADIC NUMBER, STRASSMAN'S THEOREM, ULTRAMETRIC, VALUATION GROUP

References

Cassels, J. W. S. *Local Fields.* Cambridge, England: Cambridge University Press, 1986.

Ostrowski, A. "Untersuchungen zur aritmetischen Theorie der Körper." *Math. Zeit.* **39**, 269–404, 1935.

Valuation Group

Let $(K, |\cdot|)$ be a valuated field. The valuation group G is defined to be the set

$$G = \{|x| : x \in K, x \neq 0\},$$

with the group operation being multiplication. It is a SUBGROUP of the POSITIVE REAL NUMBERS, under multiplication.

Valuation Ring

Let $(K, |\cdot|)$ be a non-Archimedean valuated field. Its valuation ring R is defined to be

$$R = \{x \in K : |x| \leq 1\}.$$

The valuation ring has maximal IDEAL

$$M = \{x \in K : |x| < 1\},$$

and the field R/M is called the residue field, class field, or field of digits. For example, if $K = \mathbb{Q}_p$ (p-adic numbers), then $R = Z_p$ (p-adic integers), $M = pZ_p$ (p-adic integers congruent to 0 mod p), and $R/M = \mathrm{GF}(p)$, the FINITE FIELD of order p.

Valuation Theory

The study of VALUATIONS which simplifies class field theory and the theory of algebraic function fields.

see also VALUATION

References
Iyanaga, S. and Kawada, Y. (Eds.). "Valuations." §425 in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, pp. 1350–1353, 1980.

Value

The quantity which a FUNCTION f takes upon application to a given quantity.

see also VALUE (GAME)

Value (Game)

The solution to a GAME in GAME THEORY. When a SADDLE POINT is present

$$\min_{i \leq m} \min_{j \leq n} a_{ij} = \min_{j \leq n} \max_{i \leq m} a_{ij} \equiv v,$$

and v is the value for pure strategies.

see also ABSOLUTE VALUE, GAME THEORY, MINIMAX THEOREM, VALUATION

Vampire Number

A number $v = xy$ with an EVEN number n of DIGITS formed by multiplying a pair of $n/2$-DIGIT numbers (where the DIGITS are taken from the original number in any order) x and y together. Pairs of trailing zeros are not allowed. If v is a vampire number, then x and y are called its "fangs." Examples of vampire numbers include

$$\begin{aligned}
1260 &= 21 \times 60 \\
1395 &= 15 \times 93 \\
1435 &= 35 \times 41 \\
1530 &= 30 \times 51 \\
1827 &= 21 \times 87 \\
2187 &= 27 \times 81 \\
6880 &= 80 \times 86
\end{aligned}$$

(Sloane's A014575). There are seven 4-digit vampires, 155 6-digit vampires, and 3382 8-digit vampires. General formulas can be constructed for special classes of vampires, such as the fangs

$$\begin{aligned}
x &= 25 \cdot 10^k + 1 \\
y &= 100(10^{k+1} + 52)/25,
\end{aligned}$$

giving the vampire

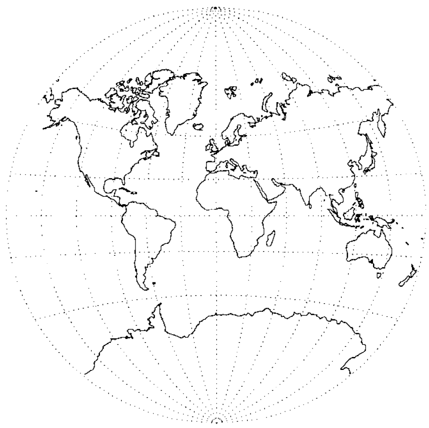
$$\begin{aligned}
v = xy &= (10^{k+1} + 52)10^{k+2} + 100(10^{k+1} + 52)/25 \\
&= x^* \cdot 10^{k+2} + t \\
&= 8(26 + 5 \cdot 10^k)(1 + 25 \cdot 10^k),
\end{aligned}$$

where x^* denotes x with the DIGITS reversed (Roushe and Rogers).

Pickover (1995) also defines pseudovampire numbers, in which the multiplicands have different number of digits.

References
Pickover, C. A. "Vampire Numbers." Ch. 30 in *Keys to Infinity.* New York: W. H. Freeman, pp. 227–231, 1995.
Pickover, C. A. "Vampire Numbers." *Theta* **9**, 11–13, Spring 1995.
Pickover, C. A. "Interview with a Number." *Discover* **16**, 136, June 1995.
Roushe, F. W. and Rogers, D. G. "Tame Vampires." Undated manuscript.
Sloane, N. J. A. Sequence A014575 in "An On-Line Version of the Encyclopedia of Integer Sequences."

van der Grinten Projection



A MAP PROJECTION given by the transformation

$$x = \operatorname{sgn}(\lambda - \lambda_0) \times \frac{\pi|A(G - P^2) - \sqrt{A^2(G - P^2)^2 - (P^2 + A^2)(G^2 - P^2)}|}{P^2 + A^2} \quad (1)$$

$$y = \operatorname{sgn}(\phi) \frac{\pi|PQ - A\sqrt{(A^2+1)(P^2+A^2) - Q^2}}{P^2 + A^2}, \quad (2)$$

where

$$A = \frac{1}{2}\left|\frac{\pi}{\lambda - \lambda_0} - \frac{\lambda - \lambda_0}{\pi}\right| \quad (3)$$

$$G = \frac{\cos\theta}{\sin\theta + \cos\theta - 1} \quad (4)$$

$$P = G\left(\frac{2}{\sin\theta} - 1\right) \quad (5)$$

$$\theta = \sin^{-1}\left|\frac{2\phi}{\pi}\right| \quad (6)$$

$$Q = A^2 + G. \quad (7)$$

The inverse FORMULAS are

$$\phi = \operatorname{sgn}(y)\pi\left[-m_1\cos(\theta_1 + \tfrac{1}{3}\pi) - \frac{c_2}{3c_3}\right] \quad (8)$$

$$\lambda = \frac{\pi|X^2 + Y^2 - 1 + \sqrt{1 + 2(X^2 - Y^2) + (X^2 + Y^2)^2}|}{2X} + \lambda_0, \quad (9)$$

where

$$X = \frac{x}{\pi} \quad (10)$$

$$Y = \frac{y}{\pi} \quad (11)$$

$$c_1 = -|Y|(1 + X^2 + Y^2) \quad (12)$$

$$c_2 = c_1 - 2Y^2 + X^2 \quad (13)$$

$$c_3 = -2c_1 + 1 + 2Y^2 + (X^2 + Y^2)^2 \quad (14)$$

$$d = \frac{Y^2}{c_3} + \frac{1}{27}\left(\frac{2c_2{}^3}{c_3{}^3} - \frac{9c_1c_2}{c_3{}^2}\right) \quad (15)$$

$$a_1 = \frac{1}{c_3}\left(c_1 - \frac{c_2{}^2}{3c_3}\right) \quad (16)$$

$$m_1 = 2\sqrt{-\tfrac{1}{3}a_1} \quad (17)$$

$$\theta_1 = \tfrac{1}{3}\cos^{-1}\left(\frac{3d}{a_1 m_1}\right). \quad (18)$$

References

Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, pp. 239–242, 1987.

van der Pol Equation

An ORDINARY DIFFERENTIAL EQUATION which can be derived from the RAYLEIGH DIFFERENTIAL EQUATION by differentiating and setting $y = y'$. It is an equation describing self-sustaining oscillations in which energy is fed into small oscillations and removed from large oscillations. This equation arises in the study of circuits containing vacuum tubes and is given by

$$y'' - \mu(1 - y^2)y' + y = 0.$$

see also RAYLEIGH DIFFERENTIAL EQUATION

References

Kreyszig, E. *Advanced Engineering Mathematics, 6th ed.* New York: Wiley, pp. 165–166, 1988.

van der Waerden Number

The threshold numbers proven to exist by VAN DER WAERDEN'S THEOREM. The first few are 1, 3, 9, 35, 178, ... (Sloane's A005346).

References

Goodman, J. E. and O'Rourke, J. (Eds.). *Handbook of Discrete & Computational Geometry.* Boca Raton, FL: CRC Press, p. 159, 1997.

Honsberger, R. *More Mathematical Morsels.* Washington, DC: Math. Assoc. Amer., p. 29, 1991.

Sloane, N. J. A. Sequence A005346/M2819 in "An On-Line Version of the Encyclopedia of Integer Sequences."

van der Waerden's Theorem

For any given POSITIVE INTEGERS k and r, there exists a threshold number $n(k, r)$ (known as a VAN DER WAERDEN NUMBER) such that no matter how the numbers 1, 2, ..., n are partitioned into k classes, at least one of the classes contains an ARITHMETIC PROGRESSION of length at least r. However, no FORMULA for $n(k, r)$ is known.

see also ARITHMETIC PROGRESSION

References

Honsberger, R. *More Mathematical Morsels.* Washington, DC: Math. Assoc. Amer., p. 29, 1991.

Khinchin, A. Y. "Van der Waerden's Theorem on Arithmetic Progressions." Ch. 1 in *Three Pearls of Number Theory.* New York: Dover, pp. 11–17, 1998.

van der Waerden, B. L. "Beweis einer Baudetschen Vermutung." *Nieuw Arch. Wiskunde* **15**, 212–216, 1927.

van Kampen's Theorem

In the usual diagram of inclusion homeomorphisms, if the upper two maps are injective, then so are the other two.

References

Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, pp. 74–75 and 369–373, 1976.

van Wijngaarden-Deker-Brent Method

see BRENT'S METHOD

Vandermonde Determinant

$$\Delta(x_1,\ldots,x_n) \equiv \begin{vmatrix} 1 & x_1 & {x_1}^2 & \cdots & {x_1}^{n-1} \\ 1 & x_2 & {x_2}^2 & \cdots & {x_2}^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & {x_n}^2 & \cdots & {x_n}^{n-1} \end{vmatrix} = \prod_{\substack{i,j \\ i>j}} (x_i - x_j)$$

(Sharpe 1987). For INTEGERS $a_1, \ldots, a_n$, $\Delta(a_1,\ldots,a_n)$ is divisible by $\prod_{i=1}^n (i-1)!$ (Chapman 1996).

see also VANDERMONDE MATRIX

References

Chapman, R. "A Polynomial Taking Integer Values." *Math. Mag.* **69**, 121, 1996.

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1111, 1979.

Sharpe, D. §2.9 in *Rings and Factorization.* Cambridge, England: Cambridge University Press, 1987.

Vandermonde Identity

see CHU-VANDERMONDE IDENTITY

Vandermonde Matrix

A type of matrix which arises in the LEAST SQUARES FITTING of POLYNOMIALS and the reconstruction of a DISTRIBUTION from the distribution's MOMENTS. The solution of an $n \times n$ Vandermonde matrix equation requires $\mathcal{O}(n^2)$ operations. A Vandermonde matrix of order n is of the form

$$\begin{bmatrix} 1 & x_1 & {x_1}^2 & \cdots & {x_1}^{n-1} \\ 1 & x_2 & {x_2}^2 & \cdots & {x_2}^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & {x_n}^2 & \cdots & {x_n}^{n-1} \end{bmatrix}.$$

see also TOEPLITZ MATRIX, TRIDIAGONAL MATRIX, VANDERMONDE DETERMINANT

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Vandermonde Matrices and Toeplitz Matrices." §2.8 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 82–89, 1992.

Vandermonde's Sum

see CHU-VANDERMONDE IDENTITY

Vandermonde Theorem

A special case of GAUSS'S THEOREM with a a NEGATIVE INTEGER $-n$:

$${}_2F_1(-n,b;c;1) = \frac{(c-b)_n}{(c)_n},$$

where ${}_2F_1(a,b;c;z)$ is a HYPERGEOMETRIC FUNCTION and $(a)_n$ is a POCHHAMMER SYMBOL (Bailey 1935, p. 3).

see also GAUSS'S THEOREM

References

Bailey, W. N. *Generalised Hypergeometric Series.* Cambridge, England: Cambridge University Press, 1935.

Vandiver's Criteria

Let p be a IRREGULAR PRIME, and let $P = rp+1$ be a PRIME with $P < p^2 - p$. Also let t be an INTEGER such that $t^3 \not\equiv 1 \pmod{P}$. For an IRREGULAR PAIR $(p, 2k)$, form the product

$$Q_{2k} = t^{-rd/2} \prod_{b=1}^m (t^{rb} - 1)^{b^{p-1-2k}},$$

where

$$m = \tfrac{1}{2}(p1 - 1)$$
$$d = \sum_{n=1}^m n^{p-2k}.$$

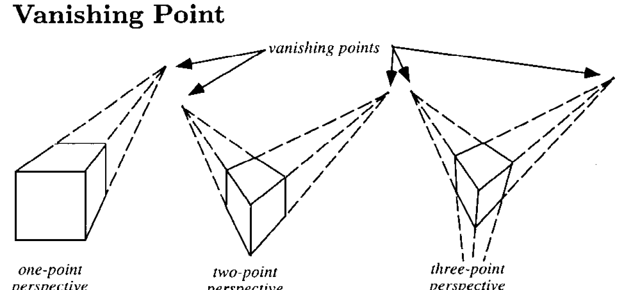
If ${Q_{2k}}^r \not\equiv 1 \pmod{P}$ for all such IRREGULAR PAIRS, then FERMAT'S LAST THEOREM holds for exponent p.

see also FERMAT'S LAST THEOREM, IRREGULAR PAIR, IRREGULAR PRIME

References

Johnson, W. "Irregular Primes and Cyclotomic Invariants." *Math. Comput.* **29**, 113–120, 1975.

Vanishing Point



The point or points to which the extensions of PARALLEL lines appear to converge in a PERSPECTIVE drawing.

see also PERSPECTIVE, PROJECTIVE GEOMETRY

References

Dixon, R. "Perspective Drawings." Ch. 3 in *Mathographics.* New York: Dover, pp. 79–88, 1991.

Varga's Constant

$$V \equiv \frac{1}{\Lambda} = 9.2890254919\ldots,$$

where Λ is the ONE-NINTH CONSTANT.

see also ONE-NINTH CONSTANT

Variance

For N samples of a variate having a distribution with *known* MEAN μ, the "population variance" (usually called "variance" for short, although the word "population" should be added when needed to distinguish it from the SAMPLE VARIANCE) is defined by

$$\begin{aligned}\operatorname{var}(x) &\equiv \frac{1}{N}\sum(x-\mu)^2 = \left\langle x^2 - 2\mu x + \mu^2\right\rangle \\ &= \left\langle x^2\right\rangle - \langle 2\mu x\rangle + \left\langle \mu^2\right\rangle \\ &= \left\langle x^2\right\rangle - 2\mu\langle x\rangle + \mu^2, \end{aligned} \tag{1}$$

where

$$\langle x\rangle \equiv \frac{1}{N}\sum_{i=1}^N x_i. \tag{2}$$

But since $\langle x\rangle$ is an UNBIASED ESTIMATOR for the MEAN

$$\mu \equiv \langle x\rangle, \tag{3}$$

it follows that the variance

$$\sigma^2 \equiv \operatorname{var}(x) = \left\langle x^2\right\rangle - \mu^2. \tag{4}$$

The population STANDARD DEVIATION is then defined as

$$\sigma \equiv \sqrt{\operatorname{var}(x)} = \sqrt{\langle x^2\rangle - \mu^2}. \tag{5}$$

A useful identity involving the variance is

$$\operatorname{var}(f(x)+g(x)) = \operatorname{var}(f(x)) + \operatorname{var}(g(x)). \tag{6}$$

Therefore,

$$\begin{aligned}\operatorname{var}(ax+b) &= \left\langle [(ax+b) - \langle ax+b\rangle]^2\right\rangle \\ &= \left\langle (ax+b-a\langle x\rangle - b)^2\right\rangle \\ &= \left\langle (ax-a\mu)^2\right\rangle = \left\langle a^2(x-\mu)^2\right\rangle \\ &= a^2\left\langle (x-\mu)^2\right\rangle = a^2\operatorname{var}(x) \qquad (7)\\ \operatorname{var}(b) &= 0. \qquad (8)\end{aligned}$$

If the population MEAN is not known, using the sample mean $\bar{x}$ instead of the population mean μ to compute

$$s^2 \equiv \hat{\sigma}_N^2 \equiv \frac{1}{N}\sum_{i=1}^N (x_i - \bar{x})^2 \tag{9}$$

gives a BIASED ESTIMATOR of the population variance. In such cases, it is appropriate to use a STUDENT'S t-DISTRIBUTION instead of a GAUSSIAN DISTRIBUTION.

However, it turns out (as discussed below) that an UNBIASED ESTIMATOR for the population variance is given by

$$s'^2 \equiv \hat{\sigma}'^2_N \equiv \frac{1}{N-1}\sum_{i=1}^N (x_i - \bar{x})^2. \tag{10}$$

The MEAN and VARIANCE of the sample standard deviation for a distribution with population mean μ and VARIANCE are

$$\mu_{s_N{}^2} = \frac{N-1}{N}s^2 \tag{11}$$

$$\sigma_{s_N{}^2}{}^2 = \frac{N-1}{N^3}[(N-1)\mu_4 - (N-3)\mu_2{}^2]. \tag{12}$$

The quantity $Ns_N{}^2/\sigma^2$ has a CHI-SQUARED DISTRIBUTION.

For multiple variables, the variance is given using the definition of COVARIANCE,

$$\begin{aligned}\operatorname{var}\left(\sum_{i=1}^n x_i\right) &= \operatorname{cov}\left(\sum_{i=1}^n x_i, \sum_{j=1}^m x_j\right) \\ &= \sum_{i=1}^n\sum_{j=1}^m \operatorname{cov}(x_i,x_j) \\ &= \sum_{i=1}^n\sum_{\substack{j=1\\ j=i}}^m \operatorname{cov}(x_i,x_j) + \sum_{i=1}^n\sum_{\substack{j=1\\ j\neq i}}^m \operatorname{cov}(x_i,x_j) \\ &= \sum_{i=1}^n \operatorname{cov}(x_i,x_j) + \sum_{i=1}^n\sum_{\substack{j=1\\ j\neq i}}^m \operatorname{cov}(x_i,x_j) \\ &= \sum_{i=1}^n \operatorname{var}(x_i) + 2\sum_{i=1}^n\sum_{j=i+1}^m \operatorname{cov}(x_i,x_j).\end{aligned} \tag{13}$$

A linear sum has a similar form:

$$\begin{aligned}\operatorname{var}\left(\sum_{i=1}^n a_i x_i\right) &= \operatorname{cov}\left(\sum_{i=1}^n a_i x_i, \sum_{j=1}^m a_j x_j\right) \\ &= \sum_{i=1}^n\sum_{j=1}^m a_i a_j \operatorname{cov}(x_i,x_j) \\ &= \sum_{i=1}^n a_i{}^2\operatorname{var}(x_i) + 2\sum_{i=1}^n\sum_{j=i+1}^m a_i a_j \operatorname{cov}(x_i,x_j).\end{aligned} \tag{14}$$

These equations can be expressed using the COVARIANCE MATRIX.

To estimate the population VARIANCE from a sample of N elements with a priori *unknown* MEAN (i.e., the MEAN is estimated from the sample itself), we need an

UNBIASED ESTIMATOR for σ. This is given by the k-STATISTIC k_2, where

$$k_2 = \frac{N}{N-1} m_2 \tag{15}$$

and $m_2 \equiv s^2$ is the SAMPLE VARIANCE

$$s^2 \equiv \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2. \tag{16}$$

Note that some authors prefer the definition

$$s'^2 \equiv \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2, \tag{17}$$

since this makes the sample variance an UNBIASED ESTIMATOR for the population variance.

When computing numerically, the MEAN must be computed before s^2 can be determined. This requires storing the set of sample values. It is possible to calculate s'^2 using a recursion relationship involving only the last sample as follows. Here, use μ_j to denote μ calculated from the first j samples (*not* the jth MOMENT)

$$\mu_j \equiv \frac{\sum_{i=1}^{j} x_i}{j}, \tag{18}$$

and $s_j{}^2$ denotes the value for the sample variance s'^2 calculated from the first j samples. The first few values calculated for the MEAN are

$$\mu_1 = x_1 \tag{19}$$

$$\mu_2 = \frac{1 \cdot \mu_1 + x_2}{2} \tag{20}$$

$$\mu_3 = \frac{2\mu_2 + x_3}{3}. \tag{21}$$

Therefore, for $j = 2, 3$ it is true that

$$\mu_j = \frac{(j-1)\mu_{j-1} + x_j}{j}. \tag{22}$$

Therefore, by induction,

$$\begin{aligned} \mu_{j+1} &= \frac{[(j+1)-1]\mu_{(j+1)-1} + x_{j+1}}{j+1} \\ &= \frac{j\mu_j + x_{j+1}}{j+1} \end{aligned} \tag{23}$$

$$\mu_{j+1}(j+1) = (j+1)\mu_j + (x_{j+1} - \mu_j) \tag{24}$$

$$\mu_{j+1} = \mu_j + \frac{x_{j+1} - \mu_j}{j+1}, \tag{25}$$

and

$$s_j{}^2 = \frac{\sum_{i=1}^{j} (x_i - \mu_j)^2}{j-1} \tag{26}$$

for $j \geq 2$, so

$$\begin{aligned} j s_{j+1}{}^2 &= j \frac{\sum_{i=1}^{j+1} (x_i - \mu_{j+1})^2}{j} = \sum_{i=1}^{j+1} (x_i - \mu_{j+1})^2 \\ &= \sum_{i=1}^{j+1} [(x_i - \mu_j)(\mu_j - \mu_{j+1})]^2 \\ &= \sum_{i=1}^{j+1} (x_i - \mu_j)^2 + \sum_{i=1}^{j+1} (\mu_j - \mu_{j+1})^2 \\ &\quad + 2\sum_{i=1}^{j+1} (x_i - \mu_j)(\mu_j - \mu_{j+1}). \end{aligned} \tag{27}$$

Working on the first term,

$$\begin{aligned} \sum_{i=1}^{j+1} (x_i - \mu_j)^2 &= \sum_{i=1}^{j} (x_i - \mu_j)^2 + (x_{j+1} - \mu_j)^2 \\ &= (j-1)s_j{}^2 + (x_{j+1} - \mu_j)^2. \end{aligned} \tag{28}$$

Use (24) to write

$$x_{j+1} - \mu_j = (j+1)(\mu_{j+1} - \mu_j), \tag{29}$$

so

$$\sum_{i=1}^{j+1} (x_i - \mu_j)^2 = (j-1)s_j{}^2 + (j+1)^2(\mu_{j+1} - \mu_j)^2. \tag{30}$$

Now work on the second term in (27),

$$\sum_{i=1}^{j+1} (\mu_j - \mu_{j+1})^2 = (j+1)(\mu_j - \mu_{j+1})^2. \tag{31}$$

Considering the third term in (27),

$$\begin{aligned} \sum_{i=1}^{j+1} (x_i - \mu_j)(\mu_j - \mu_{j+1}) &= (\mu_j - \mu_{j+1}) \sum_{i=1}^{j+1} (x_i - \mu_j) \\ &= (\mu_j - \mu_{j+1}) \left[\sum_{i=1}^{j} (x_i - \mu_j) + (x_{j+1} - \mu_j) \right] \\ &= (\mu_j - \mu_{j+1}) \left(x_{j+1} - \mu_j - j\mu_j + \sum_{i=1}^{j} x_i \right). \end{aligned} \tag{32}$$

But

$$\sum_{i=1}^{j} x_i = j\mu_j, \tag{33}$$

so

$$\begin{aligned} &\sum_{i=1}^{j+1} (\mu_j - \mu_{j+1})(x_{j+1} - \mu_j) \\ &\quad = \sum_{i=1}^{j+1} (\mu_j - \mu_{j+1})(j+1)(\mu_{j+1} - \mu_j) \\ &\quad = -(j+1)(\mu_j - \mu_{j+1})^2. \end{aligned} \tag{34}$$

Plugging (30), (31), and (34) into (27),

$$\begin{aligned} j s_{j+1}{}^2 &= [(j-1)s_j{}^2 + (j+1)^2(\mu_{j+1}-\mu_j)^2] \\ &\quad + [(j+1)(\mu_j-\mu_{j+1}) + 2[-(j+1)(\mu_j-\mu_{j+1})] \\ &= (j-1)s_j{}^2 + (j+1)^2(\mu_{j+1}-\mu_j)^2 \\ &\quad - (j+1)(\mu_j-\mu_{j+1})^2 \\ &= (j-1)s_j{}^2 + (j+1)[(j+1)-1](\mu_{j+1}-\mu_j)^2 \\ &= (j-1)s_j{}^2 + j(j+1)(\mu_{j+1}-\mu_j)^2, \end{aligned} \tag{35}$$

so

$$s_{j+1}{}^2 = \left(1-\frac{1}{j}\right) s_j{}^2 + (j+1)(\mu_{j+1}-\mu_j)^2. \tag{36}$$

To find the variance of s^2 itself, remember that

$$\operatorname{var}(s^2) \equiv \left\langle s^4 \right\rangle - \left\langle s^2 \right\rangle^2, \tag{37}$$

and

$$\left\langle s^2 \right\rangle = \frac{N-1}{N}\mu_2. \tag{38}$$

Now find $\left\langle s^4 \right\rangle$.

$$\begin{aligned} \left\langle s^4 \right\rangle &= \left\langle (s^2)^2 \right\rangle = \left\langle (\left\langle x^2 \right\rangle - \left\langle x \right\rangle^2)^2 \right\rangle \\ &= \left\langle \left[\frac{1}{N}\sum x_i{}^2 - \left(\frac{1}{N}\sum x_i\right)^2\right]^2 \right\rangle \\ &= \frac{1}{N^2}\left\langle \left(\sum x_i\right)^2 \right\rangle - \frac{2}{N^3}\left\langle \sum x_i{}^2 \left(\sum x_i\right)^2 \right\rangle \\ &\quad + \frac{1}{N^4}\left\langle \left(\sum x_i\right)^4 \right\rangle. \end{aligned} \tag{39}$$

Working on the first term of (39),

$$\begin{aligned} \left\langle \left(\sum x_i{}^2\right)^2 \right\rangle &= \left\langle \sum x_i{}^4 + \sum x_i{}^2 x_j{}^2 \right\rangle \\ &= \left\langle \sum x_i{}^4 \right\rangle + \left\langle \sum x_i{}^2 x_j{}^2 \right\rangle \\ &= N\left\langle x_i{}^4 \right\rangle + N(N-1)\left\langle x_i{}^2 \right\rangle \left\langle x_j{}^2 \right\rangle \\ &= N\mu_4' + N(N-1)\mu_2'^2. \end{aligned} \tag{40}$$

The second term of (39) is known from k-STATISTICS,

$$\left\langle \sum x_i{}^2 \left(\sum x_j\right)^2 \right\rangle = N\mu_4' + N(N-1)\mu_2'^2, \tag{41}$$

as is the third term,

$$\begin{aligned} \left\langle \left(\sum x_i\right)^4 \right\rangle &= N\left\langle \sum x_i{}^4 \right\rangle + 3N(N-1)\left\langle \sum x_i{}^2 x_j{}^2 \right\rangle \\ &= N\mu_4' + 3N(N-1)\mu_2'^2. \end{aligned} \tag{42}$$

Combining (39)-(42) gives

$$\begin{aligned} \left\langle s^4 \right\rangle &= \frac{1}{N^2}[N\mu_4' + N(N-1)\mu_2'^2] \\ &\quad - \frac{2}{N^3}[N\mu_4' + N(N-1)\mu_2'^2] \\ &\quad + \frac{1}{N^4}[N\mu_4' + 3N(N-1)\mu_2'^2] \\ &= \left(\frac{1}{N} - \frac{2}{N^2} + \frac{1}{N^3}\right)\mu_4' \\ &\quad + \left[\frac{N-1}{N} - \frac{2(N-1)}{N^2} + \frac{3(N-1)}{N^3}\right]\mu_2'^2 \\ &= \left(\frac{N^2-2N+1}{N^3}\right)\mu_4' \\ &\quad + \frac{(N-1)(N^2-2N+3)}{N^3}\mu_2'^2 \\ &= \frac{(N-1)[(N-1)\mu_4' + (N^2-2N+3)\mu_2'^2]}{N^3}, \end{aligned} \tag{43}$$

so plugging in (38) and (43) gives

$$\begin{aligned} \operatorname{var}(s^2) &= \left\langle s^4 \right\rangle - \left\langle s^2 \right\rangle^2 \\ &= \frac{(N-1)[(N-1)\mu_4' + (N^2-2N+3)\mu_2'^2]}{N^3} \\ &\quad - \frac{(N-1)^2 N}{N^3}\mu_2'^2 \\ &= \frac{N-1}{N^3}\{(N-1)\mu_4' + [(N^2-2N+3) \\ &\quad - N(N-1)]\mu_2'^2\} \\ &= \frac{(N-1)[(N-1)\mu_4' - (N-3)\mu_2'^2]}{N^3}. \end{aligned} \tag{44}$$

Student calculated the SKEWNESS and KURTOSIS of the distribution of s^2 as

$$\gamma_1 = \sqrt{\frac{8}{N-1}} \tag{45}$$

$$\gamma_2 = \frac{12}{N-1} \tag{46}$$

and conjectured that the true distribution is PEARSON TYPE III DISTRIBUTION

$$f(s^2) = C(s^2)^{(N-3)/2} e^{-Ns^2/2\sigma^2}, \tag{47}$$

where

$$\sigma^2 = \frac{Ns^2}{N-1} \tag{48}$$

$$C = \frac{\left(\frac{N}{2\sigma^2}\right)^{(N-1)/2}}{\Gamma\left(\frac{N-1}{2}\right)}. \tag{49}$$

This was proven by R. A. Fisher.

The distribution of s itself is given by

$$f(s) = 2\frac{\left(\frac{N}{2\sigma^2}\right)^{(N-1)/2}}{\Gamma\left(\frac{N-1}{2}\right)} e^{-ns^2/2\sigma^2} s^{N-2} \tag{50}$$

$$\langle s\rangle = \sqrt{\frac{2}{N}}\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)}\sigma \equiv b(N)\sigma, \tag{51}$$

where

$$b(N) \equiv \sqrt{\frac{2}{N}}\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)}. \tag{52}$$

The MOMENTS are given by

$$\mu_r = \left(\frac{2}{N}\right)^{r/2}\frac{\Gamma\left(\frac{N-1+r}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)}\sigma^r, \tag{53}$$

and the variance is

$$\begin{aligned}\operatorname{var}(s) &= \nu_2 - {\nu_1}^2 = \frac{N-1}{N}\sigma^2 - [b(N)\sigma]^2 \\ &= \frac{1}{N}\left[N - 1 - \frac{2\Gamma^2\left(\frac{N}{2}\right)}{\Gamma^2\left(\frac{N-1}{2}\right)}\sigma^2\right].\end{aligned} \tag{54}$$

An UNBIASED ESTIMATOR of σ is $s/b(N)$. Romanovsky showed that

$$b(N) = 1 - \frac{3}{4N} - \frac{7}{32N^2} - \frac{139}{51849N^3} + \ldots. \tag{55}$$

see also CORRELATION (STATISTICAL), COVARIANCE, COVARIANCE MATRIX, k-STATISTIC, MEAN, SAMPLE VARIANCE

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Variate

A RANDOM VARIABLE in statistics.

Variation

The Δ-variation is a variation in which the varied path over which an integral is evaluated may end at different times than the correct path, and there may be variation in the coordinates at the endpoints.

The δ-variation is a variation in which the varied path in configuration space terminates at the endpoints representing the system configuration at the same time t_1 and t_2 as the correct path; i.e., the varied path always returns to the same endpoints in configuration space, so

$$\delta q_i(t_1) = \delta q_i(t_2) = 0.$$

see also CALCULUS OF VARIATIONS, VARIATION OF ARGUMENT, VARIATION OF PARAMETERS

Variation of Argument

Let $[\arg f(z)]$ denote the change in argument of a function $f(z)$ around a closed loop γ. Also let N denote the number of ROOTS of $f(z)$ in γ and P denote the number of POLES of $f(z)$ in γ. Then

$$[\arg f(z)] = \frac{1}{2\pi}(N - P). \tag{1}$$

To find $[\arg f(z)]$ in a given region R, break R into paths and find $[\arg f(z)]$ for each path. On a circular ARC

$$z = Re^{i\theta}, \tag{2}$$

let $f(z)$ be a POLYNOMIAL $P(z)$ of degree n. Then

$$\begin{aligned}[\arg P(z)] &= \left[\arg\left(z^n\frac{P(z)}{z^n}\right)\right] \\ &= [\arg z^n] + \left[\arg\left(\frac{P(z)}{z^n}\right)\right].\end{aligned} \tag{3}$$

Plugging in $z = Re^{i\theta}$ gives

$$[\arg P(z)] = [\arg Re^{i\theta n}] + \left[\arg\frac{P(Re^{i\theta})}{Re^{i\theta n}}\right] \tag{4}$$

$$\lim_{R\to\infty}\frac{P(Re^{i\theta})}{Re^{i\theta n}} = [\text{constant}], \tag{5}$$

so

$$\left[\frac{P(Re^{i\theta})}{Re^{i\theta n}}\right] = 0, \tag{6}$$

and

$$[\arg P(z)] = [\arg e^{i\theta n}] = n(\theta_2 - \theta_1). \tag{7}$$

For a REAL segment $z = x$,

$$[\arg f(x)] = \tan^{-1}\left[\frac{0}{f(x)}\right] = 0. \tag{8}$$

For an IMAGINARY segment $z = iy$,

$$[\arg f(iy)] = \left\{\tan^{-1}\frac{\Im[P(iy)]}{\Re[P(iy)]}\right\}_{\theta_1}^{\theta_2}. \tag{9}$$

Note that the ARGUMENT must change continuously, so "jumps" occur across inverse tangent asymptotes.

Variation Coefficient

If s_x is the STANDARD DEVIATION of a set of samples x_i and $\bar{x}$ its MEAN, then

$$V \equiv \frac{s_x}{\bar{x}}.$$

Variation of Parameters

For a second-order ORDINARY DIFFERENTIAL EQUATION,

$$y'' + p(x)y' + q(x)y = g(x). \tag{1}$$

Assume that linearly independent solutions $y_1(x)$ and $y_2(x)$ are known. Find v_1 and v_2 such that

$$y^*(x) = v_1(x)y_1(x) + v_2(x)y_2(x) \tag{2}$$
$$y^{*\prime}(x) = (v_1' + v_2'y_2) + (v_1y_1' + v_2y_2'). \tag{3}$$

Now, impose the additional condition that

$$v_1'y_1 + v_2'y_2 = 0 \tag{4}$$

so that

$$y^{*\prime}(x) = (v_1y_1' + v_2y_2') \tag{5}$$
$$y^{*\prime\prime}(x) = v_1'y_1' + v_2'y_2' + v_1y_1'' + v_2y_2'. \tag{6}$$

Plug y^*, $y^{*\prime}$, and $y^{*\prime\prime}$ back into the original equation to obtain

$$v_1(y_1''+py_1'+qy_1)+v_2(y_2''+py_2'+qy_2)+v_1'y_1'+v_2'y_2' = g(x) \tag{7}$$
$$v_1'y_1' + v_2'y_2' = g(x). \tag{8}$$

Therefore,

$$v_1'y_1 + v_2'y_2 = 0 \tag{9}$$
$$v_1'y_1' + v_2'y_2' = g(x). \tag{10}$$

Generalizing to an nth degree ODE, let y_1, ..., y_n be the solutions to the homogeneous ODE and let $v_1'(x)$, ..., $v_n'(x)$ be chosen such that

$$\begin{cases} y_1v_1' + y_2v_2' + \ldots + y_nv_n' = 0 \\ y_1'v_1' + y_2'v_2' + \ldots + y_n'v_n' = 0 \\ \vdots \\ y_1^{(n-1)}v_1' + y_2^{(n-1)}v_2' + \ldots + y_n^{(n-1)}v_n' = g(x). \end{cases} \tag{11}$$

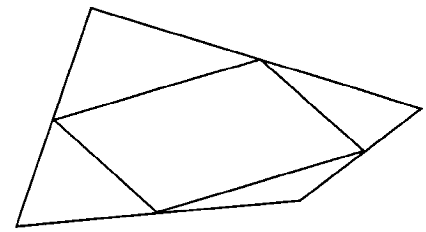
Then the particular solution is

$$y^*(x) = v_1(x)y_1(x) + \ldots + v_n(x)y_n(x). \tag{12}$$

Variety

see ALGEBRAIC VARIETY

Varignon Parallelogram



The figure formed when the BIMEDIANS (MIDPOINTS of the sides) of a convex QUADRILATERAL are joined. VARIGNON'S THEOREM demonstrated that this figure is a PARALLELOGRAM. The center of the Varignon parallelogram is the CENTROID if four point masses are placed on the VERTICES of the QUADRILATERAL.

see also MIDPOINT, PARALLELOGRAM, QUADRILATERAL, VARIGNON'S THEOREM

Varignon's Theorem

The figure formed when the BIMEDIANS (MIDPOINTS of the sides) of a convex QUADRILATERAL are joined in order is a PARALLELOGRAM. Equivalently, the BIMEDIANS bisect each other. The AREA of this VARIGNON PARALLELOGRAM is half that of the QUADRILATERAL. The PERIMETER is equal to the sum of the diagonals of the original QUADRILATERAL.

see also BIMEDIAN, MIDPOINT, QUADRILATERAL, VARIGNON PARALLELOGRAM

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Vassiliev Polynomial

Vassiliev (1990) introduced a radically new way of looking at KNOTS by considering a multidimensional space in which each point represents a possible 3-D knot configuration. If two KNOTS are equivalent, a path then exists in this space from one to the other. The paths can be associated with polynomial invariants.

Birman and Lin (1993) subsequently found a way to translate this scheme into a set of rules and list of potential starting points, which makes analysis of Vassiliev polynomials much simpler. Bar-Natan (1995) and Birman and Lin (1993) proved that JONES POLYNOMIALS and several related expressions are directly connected (Peterson 1992). In fact, substituting the POWER series for e^x as the variable in the JONES POLYNOMIAL yields a POWER SERIES whose COEFFICIENTS are Vassiliev polynomials (Birman and Lin 1993). Bar-Natan (1995) also discovered a link with Feynman diagrams (Peterson 1992).

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Vault

Let a vault consist of two equal half-CYLINDERS of length and diameter $2a$ which intersect at RIGHT ANGLES so that the lines of their intersections (the "groins") terminate in the VERTICES of a SQUARE. Then the SURFACE AREA of the vault is given by

$$A = 4(\pi - 2)a^2.$$

see also DOME

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Vector

A vector is a set of numbers $A_0, \ldots, A_n$ that transform as

$$A_i' = a_{ij} A_j. \tag{1}$$

This makes a vector a TENSOR of RANK 1. Vectors are invariant under TRANSLATION, and they reverse sign upon inversion.

A vector is uniquely specified by giving its DIVERGENCE and CURL within a region and its normal component over the boundary, a result known as HELMHOLTZ'S THEOREM (Arfken 1985, p. 79). A vector from a point A to a point B is denoted $\overrightarrow{AB}$, and a vector v may be denoted $\vec{v}$, or more commonly, $\mathbf{v}$.

A vector with unit length is called a UNIT VECTOR and is denoted with a HAT. An arbitrary vector may be converted to a UNIT VECTOR by dividing by its NORM, i.e.,

$$\hat{\mathbf{v}} \equiv \frac{\mathbf{v}}{|\mathbf{v}|}. \tag{2}$$

Let $\hat{\mathbf{n}}$ be the UNIT VECTOR defined by

$$\hat{\mathbf{n}} \equiv \begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix}. \tag{3}$$

Then the vectors $\hat{\mathbf{n}}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ satisfy the identities

$$\begin{aligned} \langle n_x \rangle &= \int_0^{2\pi}\int_o^{\pi} (\cos\theta\sin\phi)\sin\phi\,d\theta\,d\phi \\ &= [\sin\theta]_0^{2\pi}\int_0^{2\pi}\sin^2\phi\,d\phi = 0 \qquad (4) \\ \langle n_i \rangle &= 0 \qquad (5) \\ \langle n_i n_j \rangle &= \tfrac{1}{3}\delta_{ij} \qquad (6) \\ \langle n_i n_k n_k \rangle &= 0 \qquad (7) \\ \langle n_i n_k n_l n_m \rangle &= \tfrac{1}{15}(\delta_{ik}\delta_{lm} + \delta_{il}\delta_{km} + \delta_{im}\delta_{kl}) \qquad (8) \\ \left\langle (\mathbf{a}\cdot\hat{\mathbf{n}})^2 \right\rangle &= \tfrac{1}{3}a^2 \qquad (9) \\ \langle (\mathbf{a}\cdot\hat{\mathbf{n}})(\mathbf{b}\cdot\hat{\mathbf{n}}) \rangle &= \tfrac{1}{3}\mathbf{a}\cdot\mathbf{b} \qquad (10) \\ \langle (\mathbf{a}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} \rangle &= \tfrac{1}{3}a \qquad (11) \\ \left\langle (\mathbf{a}\times\hat{\mathbf{n}})^2 \right\rangle &= \tfrac{2}{3}a^2 \qquad (12) \\ \langle (\mathbf{a}\times\hat{\mathbf{n}})\cdot(\mathbf{b}\times\hat{\mathbf{n}}) \rangle &= \tfrac{2}{3}\mathbf{a}\cdot\mathbf{b}, \qquad (13) \end{aligned}$$

and

$$\begin{aligned} &\langle (\mathbf{a}\cdot\hat{\mathbf{n}})(\mathbf{b}\cdot\hat{\mathbf{n}})(\mathbf{c}\cdot\hat{\mathbf{n}})(\mathbf{d}\cdot\hat{\mathbf{n}}) \rangle \\ &= \tfrac{1}{15}[(bfa\cdot\mathbf{b})(bfc\cdot\mathbf{d}) + (bfa\cdot\mathbf{c})(bfb\cdot\mathbf{d}) + (bfa\cdot\mathbf{d})(bfb\cdot\mathbf{c})] \qquad (14) \end{aligned}$$

where δ_{ij} is the KRONECKER DELTA, $\mathbf{a}\cdot\mathbf{b}$ is a DOT PRODUCT, and EINSTEIN SUMMATION has been used.

see also FOUR-VECTOR, HELMHOLTZ'S THEOREM, NORM, PSEUDOVECTOR, SCALAR, TENSOR, UNIT VECTOR, VECTOR FIELD

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Vector Bundle

A special class of FIBER BUNDLE in which the FIBER is a VECTOR SPACE. Technically, a little more is required; namely, if $f : E \to B$ is a BUNDLE with FIBER $\mathbb{R}^n$, to be a vector bundle, all of the FIBERS $f^{-1}(x)$ for

$x \in B$ need to have a coherent VECTOR SPACE structure. One way to say this is that the "trivializations" $h : f^{-1}(U) \to U \times \mathbb{R}^n$, are FIBER-for-FIBER VECTOR SPACE ISOMORPHISMS.

see also BUNDLE, FIBER, FIBER BUNDLE, LIE ALGEBROID, STABLE EQUIVALENCE, TANGENT MAP, VECTOR SPACE, WHITNEY SUM

Vector Derivative

The basic types of derivatives operating on a VECTOR FIELD are the CURL $\nabla\times$, DIVERGENCE $\nabla\cdot$, and GRADIENT ∇.

Vector derivative identities involving the CURL include

$$\nabla \times (k\mathbf{A}) = k\nabla \times \mathbf{A} \tag{1}$$
$$\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) + (\nabla f) \times \mathbf{A} \tag{2}$$
$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) \tag{3}$$
$$\nabla \times \left(\frac{\mathbf{A}}{f}\right) = \frac{f(\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla f)}{f^2} \tag{4}$$
$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}. \tag{5}$$

In SPHERICAL COORDINATES,

$$\nabla \times \mathbf{r} = \mathbf{0} \tag{6}$$
$$\nabla \times \hat{\mathbf{r}} = \mathbf{0} \tag{7}$$
$$\nabla \times [\mathbf{r}f(r)] = f(r)(\nabla \times \mathbf{r}) + [\nabla f(r)] \times \mathbf{r} = f(r)(\mathbf{0}) + \frac{df}{dr}\hat{\mathbf{r}} \times \mathbf{r} = \mathbf{0} + \mathbf{0} = \mathbf{0}. \tag{8}$$

Vector derivative identities involving the DIVERGENCE include

$$\nabla \cdot (k\mathbf{A}) = k\nabla \cdot \mathbf{A} \tag{9}$$
$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + (\nabla f) \cdot \mathbf{A} \tag{10}$$
$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \tag{11}$$
$$\nabla \cdot \left(\frac{\mathbf{A}}{f}\right) = \frac{f(\nabla \cdot \mathbf{A}) - (\nabla f) \cdot \mathbf{A}}{f^2} \tag{12}$$
$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \tag{13}$$
$$\nabla(\mathbf{uv}) = \mathbf{u}\nabla \cdot \mathbf{v} + (\nabla \mathbf{u}) \cdot \mathbf{v}. \tag{14}$$

In SPHERICAL COORDINATES,

$$\nabla \cdot \mathbf{r} = 3 \tag{15}$$
$$\nabla \cdot \hat{\mathbf{r}} = \frac{2}{r} \tag{16}$$
$$\nabla \cdot [\mathbf{r}f(r)] = \frac{\partial}{\partial x}[xf(r)] + \frac{\partial}{\partial y}[yf(r)] + \frac{\partial}{\partial z}[zf(r)] \tag{17}$$
$$\frac{\partial}{\partial x}[xf(r)] = x\frac{\partial f}{\partial x} + f = x\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + f \tag{18}$$
$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2} = x(x^2+y^2+z^2)^{-1/2} = \frac{x}{r} \tag{19}$$
$$\frac{\partial}{\partial x}[xf(r)] = \frac{x^2}{r}\frac{df}{dr} + f. \tag{20}$$

By symmetry,

$$\nabla \cdot [\mathbf{r}f(r)] = 3f(r) + \frac{1}{r}(x^2+y^2+z^2)\frac{df}{dr} = 3f(r) + r\frac{df}{dr} \tag{21}$$
$$\nabla \cdot (\hat{\mathbf{r}}f(r)) = \frac{3}{r}f(r) + \frac{df}{dr} \tag{22}$$
$$\nabla \cdot (\hat{\mathbf{r}}r^n) = 3r^{n-1} + (n-1)r^{n-1} = (n+2)r^{n-1}. \tag{23}$$

Vector derivative identities involving the GRADIENT include

$$\nabla(kf) = k\nabla f \tag{24}$$
$$\nabla(fg) = f\nabla g + g\nabla f \tag{25}$$
$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} \tag{26}$$
$$\nabla(\mathbf{A} \cdot \nabla f) = \mathbf{A} \times (\nabla \times \nabla f) + \nabla f \times (\nabla \times \mathbf{A}) + \mathbf{A} \cdot \nabla(\nabla f) + \nabla f \cdot \nabla \mathbf{A} = \nabla f \times (\nabla \times \mathbf{A}) + \mathbf{A} \cdot \nabla(\nabla f) + \nabla f \cdot \nabla \mathbf{A} \tag{27}$$
$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2} \tag{28}$$
$$\nabla(f+g) = \nabla f + \nabla g \tag{29}$$
$$\nabla(\mathbf{A} \cdot \mathbf{A}) = 2\mathbf{A} \times (\nabla \times \mathbf{A}) + 2(\mathbf{A} \cdot \nabla)\mathbf{A} \tag{30}$$
$$(\mathbf{A} \cdot \nabla)\mathbf{A} = \nabla(\tfrac{1}{2}\mathbf{A}^2) - \mathbf{A} \times (\nabla \times \mathbf{A}). \tag{31}$$

Vector second derivative identities include

$$\nabla^2 t \equiv \nabla \cdot (\nabla t) = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} \tag{32}$$
$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}). \tag{33}$$

This very important second derivative is known as the LAPLACIAN.

$$\nabla \times (\nabla t) = \mathbf{0} \tag{34}$$
$$\nabla(\nabla \cdot \mathbf{A}) = \nabla^2\mathbf{A} + \nabla \times (\nabla \times \mathbf{A}) \tag{35}$$
$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{36}$$
$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$
$$\nabla \times (\nabla^2\mathbf{A}) = \nabla \times [\nabla(\nabla \cdot \mathbf{A})] - \nabla \times [\nabla \times (\nabla \times \mathbf{A})] = -\nabla \times [\nabla \times (\nabla \times \mathbf{A})] = -\{\nabla[\nabla \cdot (\nabla \times \mathbf{A})] - \nabla^2(\nabla \times \mathbf{A})]\} = \nabla^2(\nabla \times \mathbf{A}) \tag{37}$$
$$\nabla^2(\nabla \cdot \mathbf{A}) = \nabla \cdot [\nabla(\nabla \cdot \mathbf{A})] = \nabla \cdot [\nabla^2\mathbf{A} + \nabla \times (\nabla \times \mathbf{A})] = \nabla \cdot (\nabla^2\mathbf{A}) \tag{38}$$
$$\nabla^2[\nabla \times (\nabla \times \mathbf{A})] = \nabla^2[\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}] = \nabla^2[\nabla(\nabla \cdot \mathbf{A})] - \nabla^4\mathbf{A} \tag{39}$$
$$\nabla \times [\nabla^2(\nabla \times \mathbf{A})] = \nabla^2[\nabla(\nabla \cdot \mathbf{A})] - \nabla^4\mathbf{A} \tag{40}$$
$$\nabla^4\mathbf{A} = -\nabla^2[\nabla \times (\nabla \times \mathbf{A})] + \nabla^2[\nabla(\nabla \cdot \mathbf{A})] = \nabla \times [\nabla^2(\nabla \times \mathbf{A})] - \nabla^2[\nabla \times (\nabla \times \mathbf{A})]. \tag{41}$$

Combination identities include

$$\mathbf{A} \times (\nabla \mathbf{A}) = \tfrac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{A} \tag{42}$$

$$\nabla \times (\phi \nabla \phi) = \phi \nabla \times (\nabla \phi) + (\nabla \phi) \times (\nabla \phi) = \mathbf{0} \tag{43}$$

$$(\mathbf{A} \cdot \nabla)\hat{\mathbf{r}} = \frac{\mathbf{A} - \hat{\mathbf{r}}(\mathbf{A} \cdot \hat{\mathbf{r}})}{r} \tag{44}$$

$$\nabla f \cdot \mathbf{A} = \nabla \cdot (f\mathbf{A}) - f(\nabla \cdot \mathbf{A}) \tag{45}$$

$$f(\nabla \cdot \mathbf{A}) = \nabla \cdot (f\mathbf{A}) - \mathbf{A}\nabla f, \tag{46}$$

where (45) and (46) follow from divergence rule (2).

see also CURL, DIVERGENCE, GRADIENT, LAPLACIAN, VECTOR INTEGRAL, VECTOR QUADRUPLE PRODUCT, VECTOR TRIPLE PRODUCT

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Vector Direct Product

Given VECTORS $\mathbf{u}$ and $\mathbf{v}$, the vector direct product is

$$\mathbf{uv} \equiv \mathbf{u} \otimes \mathbf{v}^{\mathrm{T}},$$

where $\otimes$ is the MATRIX DIRECT PRODUCT and $\mathbf{v}^{\mathrm{T}}$ is the matrix TRANSPOSE. For 3×3 vectors

$$\mathbf{uv} = \begin{bmatrix} u_1\mathbf{v}^{\mathrm{T}} \\ u_2\mathbf{v}^{\mathrm{T}} \\ u_3\mathbf{v}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix}.$$

Note that if $\mathbf{u} = \hat{\mathbf{x}}_i$, then $u_j = \delta_{ij}$, where δ_{ij} is the KRONECKER DELTA.

see also MATRIX DIRECT PRODUCT, SHERMAN-MORRISON FORMULA, WOODBURY FORMULA

Vector Division

There is no unique solution A to the MATRIX equation $\mathbf{y} = \mathsf{A}\mathbf{x}$ unless $\mathbf{x}$ is PARALLEL to $\mathbf{y}$, in which case A is a SCALAR. Therefore, vector division is not defined.

see also MATRIX, SCALAR

Vector Field

A MAP $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ which assigns each $\mathbf{x}$ a VECTOR FUNCTION $\mathbf{f}(\mathbf{x})$. FLOWS are generated by vector fields and vice versa. A vector field is a SECTION of its TANGENT BUNDLE.

see also FLOW, SCALAR FIELD, SEIFERT CONJECTURE, TANGENT BUNDLE, VECTOR, WILSON PLUG

References

Gray, A. "Vector Fields $\mathbb{R}^n$" and "Derivatives of Vector Fields $\mathbb{R}^n$." §9.4–9.5 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 171–174 and 175–178, 1993.

Morse, P. M. and Feshbach, H. "Vector Fields." §1.2 in *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 8-21, 1953.

Vector Function

A function of one or more variables whose RANGE is 3-dimensional, as compared to a SCALAR FUNCTION, whose RANGE is 1-dimensional.

see also COMPLEX FUNCTION, REAL FUNCTION, SCALAR FUNCTION

Vector Harmonic

see VECTOR SPHERICAL HARMONIC

Vector Integral

The following vector integrals are related to the CURL THEOREM. If

$$\mathbf{F} \equiv \mathbf{c} \times \mathbf{P}(x, y, z), \tag{1}$$

then

$$\int_C d\mathbf{s} \times \mathbf{P} = \int_S (d\mathbf{a} \times \nabla) \times \mathbf{P}. \tag{2}$$

If

$$\mathbf{F} \equiv \mathbf{c}F, \tag{3}$$

then

$$\int_C F\, d\mathbf{s} = \int_S d\mathbf{a} \times \nabla \mathbf{F}. \tag{4}$$

The following are related to the DIVERGENCE THEOREM. If

$$\mathbf{F} \equiv \mathbf{c} \times \mathbf{P}(x, y, z), \tag{5}$$

then

$$\int_V \nabla \times \mathbf{F}\, dV = \int_S d\mathbf{a} \times \mathbf{F}. \tag{6}$$

Finally, if

$$\mathbf{F} \equiv \mathbf{c}F, \tag{7}$$

then

$$\int_V \nabla F\, dV = \int_S F\, d\mathbf{a}. \tag{8}$$

see also CURL THEOREM, DIVERGENCE THEOREM, GRADIENT THEOREM, GREEN'S FIRST IDENTITY, GREEN'S SECOND IDENTITY, LINE INTEGRAL, SURFACE INTEGRAL, VECTOR DERIVATIVE, VOLUME INTEGRAL

Vector Norm

Given an n-D VECTOR

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

a vector norm $||\mathbf{x}||$ (sometimes written simply $|\mathbf{x}|$) is a NONNEGATIVE number satisfying

1. $||x|| > 0$ when $\mathbf{x} \neq \mathbf{0}$ and $||\mathbf{x}|| = 0$ IFF $\mathbf{x} = \mathbf{0}$,
2. $||k\mathbf{x}|| = |k|\,||\mathbf{x}||$ for any SCALAR k,
3. $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.

see also COMPATIBLE, MATRIX NORM, NATURAL NORM, NORM

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1114, 1980.

Vector Ordering

If the first NONZERO component of the vector difference $\mathbf{A} - \mathbf{B}$ is > 0, then $\mathbf{A} \succ \mathbf{B}$. If the first NONZERO component of $\mathbf{A} - \mathbf{B}$ is < 0, then $\mathbf{A} \prec \mathbf{B}$.

see also PRECEDES, SUCCEEDS

Vector Potential

A function $\mathbf{A}$ such that

$$\mathbf{B} \equiv \nabla \times \mathbf{A}.$$

The most common use of a vector potential is the representation of a magnetic field. If a VECTOR FIELD has zero DIVERGENCE, it may be represented by a vector potential.

see also DIVERGENCE, HELMHOLTZ'S THEOREM, POTENTIAL FUNCTION, SOLENOIDAL FIELD, VECTOR FIELD

Vector Quadruple Product

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \tag{1}$$

$$\begin{aligned}(\mathbf{A} \times \mathbf{B})^2 &\equiv (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) \\ &= (\mathbf{A} \cdot \mathbf{A})(\mathbf{B} \cdot \mathbf{B}) - (\mathbf{A} \cdot \mathbf{B})(\mathbf{B} \cdot \mathbf{A}) \\ &= \mathbf{A}^2\mathbf{B}^2 - (\mathbf{A} \cdot \mathbf{B})^2 \end{aligned} \tag{2}$$

$$\mathbf{A} \times (\mathbf{B} \times (\mathbf{C} \times \mathbf{D})) = \mathbf{B}(\mathbf{A} \cdot (\mathbf{C} \times \mathbf{D})) - (\mathbf{A} \cdot \mathbf{B})(\mathbf{C} \times \mathbf{D}) \tag{3}$$

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) &= [\mathbf{A}, \mathbf{B}, \mathbf{D}]\mathbf{C} - [\mathbf{A}, \mathbf{B}, \mathbf{C}]\mathbf{D} \\ &= (\mathbf{C} \times \mathbf{D}) \times (\mathbf{B} \times \mathbf{A}) = [\mathbf{C}, \mathbf{D}, \mathbf{A}]\mathbf{D} - [\mathbf{C}, \mathbf{D}, \mathbf{B}]\mathbf{A}, \end{aligned} \tag{4}$$

where $[\mathbf{A}, \mathbf{B}, \mathbf{D}]$ denotes the VECTOR TRIPLE PRODUCT. Equation (1) is known as LAGRANGE'S IDENTITY.

see also LAGRANGE'S IDENTITY, VECTOR TRIPLE PRODUCT

Vector Space

A vector space over $\mathbb{R}^n$ is a set of VECTORS for which any VECTORS $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z} \in \mathbb{R}^n$ and any SCALARS r, $s \in \mathbb{R}$ have the following properties:

1. COMMUTATIVITY:
$$\mathbf{X} + \mathbf{Y} = \mathbf{Y} + \mathbf{X}.$$
2. ASSOCIATIVITY of vector addition:
$$(\mathbf{X} + \mathbf{Y}) + \mathbf{Z} = \mathbf{X} + (\mathbf{Y} + \mathbf{Z}).$$
3. Additive identity: For all $\mathbf{X}$,
$$\mathbf{0} + \mathbf{X} = \mathbf{X} + \mathbf{0} = \mathbf{X}.$$
4. Existence of additive inverse: For any $\mathbf{X}$, there exists a $-\mathbf{X}$ such that
$$\mathbf{X} + (-\mathbf{X}) = \mathbf{0}.$$
5. ASSOCIATIVITY of scalar multiplication:
$$r(s\mathbf{X}) = (rs)\mathbf{X}.$$
6. DISTRIBUTIVITY of scalar sums:
$$(r + s)\mathbf{X} = r\mathbf{X} + s\mathbf{X}.$$
7. DISTRIBUTIVITY of vector sums:
$$r(\mathbf{X} + \mathbf{Y}) = r\mathbf{X} + r\mathbf{Y}.$$
8. Scalar multiplication identity:
$$1\mathbf{X} = \mathbf{X}.$$

An n-D vector space of characteristic two has

$$S(k, n) = (2^n - 2^0)(2^n - 2^1) \cdots (2^n - 2^{k-1})$$

distinct SUBSPACES of DIMENSION k.

A MODULE is abstractly similar to a vector space, but it uses a RING to define COEFFICIENTS instead of the FIELD used for vector spaces. MODULES have COEFFICIENTS in much more general algebraic objects.

see also BANACH SPACE, FIELD, FUNCTION SPACE, HILBERT SPACE, INNER PRODUCT SPACE, MODULE, RING, TOPOLOGICAL VECTOR SPACE

References

Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 530–534, 1985.

Vector Spherical Harmonic

The SPHERICAL HARMONICS can be generalized to vector spherical harmonics by looking for a SCALAR FUNCTION ψ and a constant VECTOR $\mathbf{c}$ such that

$$\begin{aligned}\mathbf{M} &\equiv \nabla \times (\mathbf{c}\psi) = \psi(\nabla \times \mathbf{c}) + (\nabla\psi) \times \mathbf{c} \\ &= (\nabla\psi) \times \mathbf{c} = -\mathbf{c} \times \nabla, \psi \end{aligned} \tag{1}$$

so

$$\nabla \cdot \mathbf{M} = 0. \tag{2}$$

Now use the vector identities

$$\begin{aligned}\nabla^2\mathbf{M} &= \nabla^2(\nabla \times \mathbf{M}) = \nabla \times (\nabla^2\mathbf{M}) \\ &= \nabla \times (\nabla^2\mathbf{c}\psi) = \nabla \times (\mathbf{c}\nabla^2\psi) \end{aligned} \tag{3}$$

$$k^2\mathbf{M} = k^2\nabla \times (\mathbf{c}\psi) = \nabla \times (\mathbf{c}\nabla^2\psi), \tag{4}$$

so

$$\nabla^2\mathbf{M} + k^2\mathbf{M} = \nabla\times[\mathbf{c}(\nabla^2\psi + k^2\psi)], \quad (5)$$

and $\mathbf{M}$ satisfies the vector HELMHOLTZ DIFFERENTIAL EQUATION if ψ satisfies the scalar HELMHOLTZ DIFFERENTIAL EQUATION

$$\nabla^2\psi + k^2\psi = 0. \quad (6)$$

Construct another vector function

$$\mathbf{N} \equiv \frac{\nabla\times\mathbf{M}}{k}, \quad (7)$$

which also satisfies the vector HELMHOLTZ DIFFERENTIAL EQUATION since

$$\begin{aligned}\nabla^2\mathbf{N} &= \frac{1}{k}\nabla^2(\nabla\times\mathbf{M}) = \frac{1}{k}\nabla\times(\nabla^2\mathbf{M}) \\ &= \frac{1}{k}\nabla\times(-k^2\mathbf{M}) = -k\nabla\times\mathbf{M} = -k^2\mathbf{N}, \quad (8)\end{aligned}$$

which gives

$$\nabla^2\mathbf{N} + k^2\mathbf{N} = 0. \quad (9)$$

We have the additional identity

$$\begin{aligned}\nabla\times\mathbf{N} &= \frac{1}{k}\nabla\times(\nabla\times\mathbf{M}) = \frac{1}{k}\nabla(\nabla\cdot\mathbf{M}) \\ &= \frac{1}{k}\nabla^2\mathbf{M} - \frac{1}{k}\nabla^2\mathbf{M} = \frac{-\nabla^2\mathbf{M}}{k} = k\mathbf{M}. \quad (10)\end{aligned}$$

In this formalism, ψ is called the generating function and $\mathbf{c}$ is called the PILOT VECTOR. The choice of generating function is determined by the symmetry of the scalar equation, i.e., it is chosen to solve the desired scalar differential equation. If $\mathbf{M}$ is taken as

$$\mathbf{M} = \nabla\times(\mathbf{r}\psi), \quad (11)$$

where $\mathbf{r}$ is the radius vector, then $\mathbf{M}$ is a solution to the vector wave equation in spherical coordinates. If we want vector solutions which are tangential to the radius vector,

$$\mathbf{M}\cdot\mathbf{r} = \mathbf{r}\cdot(\nabla\psi\times\mathbf{c}) = (\nabla\psi)(\mathbf{c}\times\mathbf{r}) = 0, \quad (12)$$

so

$$\mathbf{c}\times\mathbf{r} = \mathbf{0} \quad (13)$$

and we may take

$$\mathbf{c} = \mathbf{r} \quad (14)$$

(Arfken 1985, pp. 707–711; Bohren and Huffman 1983, p. 88).

A number of conventions are in use. Hill (1954) defines

$$\begin{aligned}\mathbf{V}_l^m &\equiv -\sqrt{\frac{l+1}{2l+1}}Y_l^m\hat{\mathbf{r}} + \frac{1}{\sqrt{(l+1)(2l+1)}}\frac{\partial Y_l^m}{\partial\theta}\hat{\boldsymbol{\theta}} \\ &\quad + iM\sqrt{(l+1)(2l+1)}\sin\theta Y_l^m\hat{\boldsymbol{\phi}} \quad (15)\\ \mathbf{W}_l^m &= \sqrt{\frac{l}{2l+1}}Y_l^m\hat{\mathbf{r}} + \frac{1}{\sqrt{l(2l+1)}}\frac{\partial Y_l^m}{\partial\theta}\hat{\boldsymbol{\theta}} \\ &\quad + \frac{iM}{\sqrt{l(2l+1)}\sin\theta}Y_l^m\hat{\boldsymbol{\phi}} \quad (16)\\ \mathbf{X}_l^m &= -\frac{M}{\sqrt{l(l+1)}\sin\theta}Y_l^m\hat{\boldsymbol{\theta}} - \frac{i}{\sqrt{l(l+1)}}\frac{\partial Y_l^m}{\partial\theta}\hat{\boldsymbol{\phi}}. \\ &\quad (17)\end{aligned}$$

Morse and Feshbach (1953) define vector harmonics called $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{P}$ using rather complicated expressions.

References

Arfken, G. "Vector Spherical Harmonics." §12.11 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 707–711, 1985.

Blatt, J. M. and Weisskopf, V. "Vector Spherical Harmonics." Appendix B, §1 in *Theoretical Nuclear Physics.* New York: Wiley, pp. 796–799, 1952.

Bohren, C. F. and Huffman, D. R. *Absorption and Scattering of Light by Small Particles.* New York: Wiley, 1983.

Hill, E. H. "The Theory of Vector Spherical Harmonics." *Amer. J. Phys.* **22**, 211–214, 1954.

Jackson, J. D. *Classical Electrodynamics, 2nd ed.* New York: Wiley, pp. 744–755, 1975.

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part II.* New York: McGraw-Hill, pp. 1898–1901, 1953.

Vector Transformation Law

The set of n quantities v_j are components of an n-D VECTOR $\mathbf{v}$ IFF, under ROTATION,

$$v_i' = a_{ij}v_j$$

for $i = 1, 2, \ldots, n$. The DIRECTION COSINES between x_i' and x_j are

$$a_{ij} \equiv \frac{\partial x_i'}{\partial x_j} = \frac{\partial x_j}{\partial x_i'}.$$

They satisfy the orthogonality condition

$$a_{ij}a_{ik} = \frac{\partial x_j}{\partial x_i'}\frac{\partial x_i'}{\partial x_k} = \frac{\partial x_j}{\partial x_k} = \delta_{jk},$$

where δ_{jk} is the KRONECKER DELTA.

see also TENSOR, VECTOR

Vector Triple Product

The triple product can be written in terms of the LEVI-CIVITA SYMBOL ϵ_{ijk} as

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \epsilon_{ijk} A^i B^j C^k. \tag{1}$$

The BAC-CAB RULE can be written in the form

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{2}$$

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= -\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) \\ &= -\mathbf{A}(\mathbf{B} \cdot \mathbf{C}) + \mathbf{B}(\mathbf{A} \cdot \mathbf{C}).\end{aligned} \tag{3}$$

Addition identities are

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \tag{4}$$

$$[\mathbf{A}, \mathbf{B}, \mathbf{C}]\mathbf{D} = [\mathbf{D}, \mathbf{B}, \mathbf{C}]\mathbf{A} + [\mathbf{A}, \mathbf{D}, \mathbf{C}]\mathbf{B} + [\mathbf{A}, \mathbf{B}, \mathbf{D}]\mathbf{C} \tag{5}$$

$$[\mathbf{q}, \mathbf{q}', \mathbf{q}''][\mathbf{r}, \mathbf{r}', \mathbf{r}''] = \begin{vmatrix} \mathbf{q} \cdot \mathbf{r} & \mathbf{q} \cdot \mathbf{r}' & \mathbf{q} \cdot \mathbf{r}'' \\ \mathbf{q}' \cdot \mathbf{r} & \mathbf{q}' \cdot \mathbf{r}' & \mathbf{q}' \cdot \mathbf{r}'' \\ \mathbf{q}'' \cdot \mathbf{r} & \mathbf{q}'' \cdot \mathbf{r}' & \mathbf{q}'' \cdot \mathbf{r}'' \end{vmatrix}. \tag{6}$$

see also BAC-CAB RULE, CROSS PRODUCT, DOT PRODUCT, LEVI-CIVITA SYMBOL, SCALAR TRIPLE PRODUCT, VECTOR QUADRUPLE PRODUCT

References

Arfken, G. "Triple Scalar Product, Triple Vector Product." §1.5 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 26–33, 1985.

Vee

The symbol $\vee$ variously means "disjunction" (in LOGIC) or "join" (for a LATTICE).

see also WEDGE

Velocity

$$\mathbf{v} \equiv \frac{d\mathbf{r}}{dt},$$

where $\mathbf{r}$ is the POSITION VECTOR and d/dt is the derivative with respect to time. Expressed in terms of the ARC LENGTH,

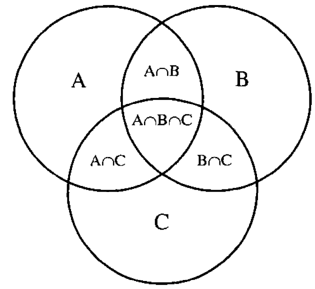
$$\mathbf{v} = \frac{ds}{dt}\hat{\mathbf{T}},$$

where $\hat{\mathbf{T}}$ is the unit TANGENT VECTOR, so the SPEED (which is the magnitude of the velocity) is

$$v \equiv |\mathbf{v}| = \frac{ds}{dt} = |\mathbf{r}'(t)|.$$

see also ANGULAR VELOCITY, POSITION VECTOR, SPEED

Venn Diagram



The simplest Venn diagram consists of three symmetrically placed mutually intersecting CIRCLES. It is used in LOGIC theory to represent collections of sets. The region of intersection of the three CIRCLES $A \cap B \cap C$, in the special case of the center of each being located at the intersection of the other two, is called a REULEAUX TRIANGLE.

In general, an order n Venn diagram is a collection of n simple closed curves in the PLANE such that

1. The curves partition the PLANE into 2^n connected regions, and
2. Each SUBSET S of $\{1, 2, \ldots, n\}$ corresponds to a unique region formed by the intersection of the interiors of the curves in S (Ruskey).

see also CIRCLE, FLOWER OF LIFE, LENS, MAGIC CIRCLES, REULEAUX TRIANGLE, SEED OF LIFE

References

Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., pp. 255–256, 1989.

Ruskey, F. "A Survey of Venn Diagrams." *Elec. J. Combin.* **4**, DS#5, 1997. `http://www.combinatorics.org/Surveys/ds5/VennEJC.html`.

Ruskey, F. "Venn Diagrams." `http://sue.csc.uvic.ca/~cos/inf/comb/SubsetInfo.html#Venn`.

Verging Construction

see NEUSIS CONSTRUCTION

Verhulst Model

see LOGISTIC MAP

Veronese Surface

A smooth 2-D surface given by embedding the PROJECTIVE PLANE into projective 5-space by the homogeneous parametric equations

$$v(x, y, z) = (x^2, y^2, z^2, xy, xz, yz).$$

The surface can be projected smoothly into 4-space, but all 3-D projections have singularities (Coffman). The projections of these surfaces in 3-D are called STEINER SURFACES. The VOLUME of the Veronese surface is $2\pi^2$.

see also STEINER SURFACE

References

Coffman, A. "Steiner Surfaces." `http://www.ipfw.edu/math/Coffman/steinersurface.html`.

Veronese Variety

see VERONESE SURFACE

Versed Sine

see VERSINE

Versiera

see WITCH OF AGNESI

Versine

$$\operatorname{vers}(z) \equiv 1 - \cos z,$$

where $\cos z$ is the COSINE. Using a trigonometric identity, the versine is equal to

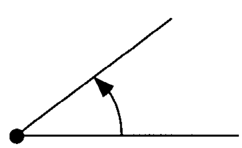
$$\operatorname{vers}(z) = 2\sin^2(\tfrac{1}{2}z).$$

see also COSINE, COVERSINE, EXSECANT, HAVERSINE

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 78, 1972.

Vertex Angle



The point about which an ANGLE is measured is called the angle's vertex, and the angle associated with a given vertex is called the vertex angle.

see also ANGLE

Vertex Coloring

BRELAZ'S HEURISTIC ALGORITHM can be used to find a good, but not necessarily minimal, VERTEX coloring of a GRAPH.

see also BRELAZ'S HEURISTIC ALGORITHM, COLORING

Vertex Connectivity

The minimum number of VERTICES whose deletion from a GRAPH disconnects it.

see also EDGE CONNECTIVITY

Vertex Cover

see HITTING SET

Vertex Degree

The degree of a VERTEX of a GRAPH is the number of EDGES which touch the VERTEX, also called the LOCAL DEGREE. The VERTEX degree of a point A in a GRAPH, denoted $\rho(A)$, satisfies

$$\sum_{i=1}^{n} \rho(A_i) = 2E,$$

where E is the total number of EDGES. DIRECTED GRAPHS have two types of degrees, known as the INDEGREE and the OUTDEGREE.

see also DIRECTED GRAPH, INDEGREE, LOCAL DEGREE, OUTDEGREE

Vertex Enumeration

A CONVEX POLYHEDRON is defined as the set of solutions to a system of linear inequalities

$$\mathsf{m}\mathbf{x} \leq \mathbf{b},$$

where m is a REAL $s \times d$ MATRIX and $\mathbf{b}$ is a REAL s-VECTOR. Given m and $\mathbf{b}$, vertex enumeration is the determination of the polyhedron's VERTICES.

see also CONVEX POLYHEDRON, POLYHEDRON

References

Avis, D. and Fukuda, K. "A Pivoting Algorithm for Convex Hulls and Vertex Enumeration of Arrangements and Polyhedra." In *Proceedings of the 7th ACM Symposium on Computational Geometry, North Conway, NH, 1991,* pp. 98–104, 1991.

Fukada, K. and Mizukosh, I. "Vertex Enumeration Package for Convex Polytopes and Arrangements, Version 0.41 Beta." http://www.mathsource.com/cgi-bin/MathSource/Applications/Mathematics/0202-633.

Vertex Figure

The line joining the MIDPOINTS of adjacent sides in a POLYGON is called the polygon's vertex figure. For a regular n-gon with side length s,

$$v = s\cos\left(\frac{\pi}{n}\right).$$

For a POLYHEDRON, the faces that join at a VERTEX form a solid angle whose section by the plane is the vertex figure.

see also TRUNCATION

Vertex (Graph)

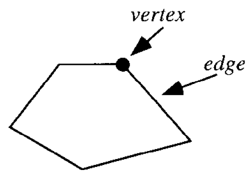
A point of a GRAPH, also called a NODE.

see also EDGE (GRAPH), NULL GRAPH, TAIT COLORING, TAIT CYCLE, TAIT'S HAMILTONIAN GRAPH CONJECTURE, VERTEX (POLYGON)

Vertex (Parabola)

For a PARABOLA oriented vertically and opening upwards, the vertex is the point where the curve reaches a minimum.

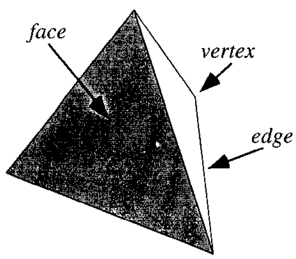
Vertex (Polygon)



A point at which two EDGES of a POLYGON meet.

see also PRINCIPAL VERTEX, VERTEX (GRAPH), VERTEX (POLYHEDRON)

Vertex (Polyhedron)



A point at which three of more EDGES of a POLYHEDRON meet. The concept can also be generalized to a POLYTOPE.

see also VERTEX (GRAPH), VERTEX (POLYGON)

Vertex (Polytope)

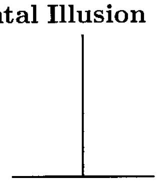
The vertex of a POLYTOPE is a point where edges of the POLYTOPE meet.

Vertical

Oriented in an up-down position.

see also HORIZONTAL

Vertical-Horizontal Illusion



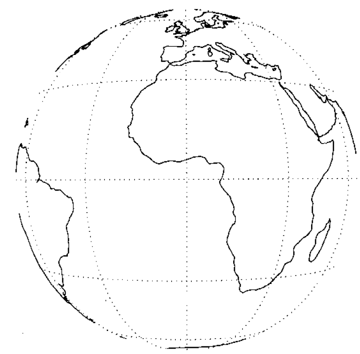
The HORIZONTAL line segment in the above figure appears to be shorter than the VERTICAL line segment, despite the fact that it has the same length.

see also ILLUSION, MÜLLER-LYER ILLUSION, POGGENDORFF ILLUSION, PONZO'S ILLUSION

References

Fineman, M. *The Nature of Visual Illusion.* New York: Dover, p. 153, 1996.

Vertical Perspective Projection



A MAP PROJECTION given by the transformation equations

$$x = k' \cos\phi \sin(\lambda - \lambda_0) \tag{1}$$
$$y = k'[\cos\phi_1 \sin\phi - \sin\phi_1 \cos\phi \cos(\lambda - \lambda_0)], \tag{2}$$

where P is the distance of the point of perspective in units of SPHERE RADII and

$$k' = \frac{P-1}{P - \cos c} \tag{3}$$
$$\cos c = \sin\phi_1 \sin\phi + \cos\phi_1 \cos\phi \cos(\lambda - \lambda_0). \tag{4}$$

References

Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, pp. 173–178, 1987.

Vertical Tangent

A function $f(x)$ has a vertical tangent line at x_0 if f is continuous at x_0 and

$$\lim_{x \to x_0} f'(x) = \pm\infty.$$

Vesica Piscis

see LENS

Vibration Problem

Solution of a system of second-order homogeneous ordinary differential equations with constant COEFFICIENTS of the form

$$\frac{d^2\mathbf{x}}{dt^2} + \mathsf{B}\mathbf{x} = 0,$$

where B is a POSITIVE DEFINITE MATRIX. To solve the vibration problem,

1. Solve the CHARACTERISTIC EQUATION of B to get EIGENVALUES $\lambda_1, \ldots, \lambda_n$. Define $\omega_i \equiv \sqrt{\lambda_i}$.
2. Compute the corresponding EIGENVECTORS $\mathbf{e}_1, \ldots, \mathbf{e}_n$.
3. The normal modes of oscillation are given by $\mathbf{x}_1 = A_1 \sin(\omega_1 t + \alpha_1)\mathbf{e}_1, \ldots, \mathbf{x}_n = A_n \sin(\omega_n t + \alpha_n)\mathbf{e}_n$, where $A_1, \ldots, A_n$ and $\alpha_1, \ldots, \alpha_n$ are arbitrary constants.
4. The general solution is $\mathbf{x} = \sum_{i=1}^n \mathbf{x}_i$.

Vickery Auction

An AUCTION in which the highest bidder wins but pays only the second-highest bid. This variation over the normal bidding procedure is supposed to encourage bidders to bid the largest amount they are willing to pay.

see also AUCTION

Viergruppe

The mathematical group $Z_4 \otimes Z_4$, also denoted D_2. Its multiplication table is

V	I	V_1	V_2	V_3
I	V_1	V_2	V_3	V_4
V_1	V_1	I	V_3	V_2
V_2	V_2	V_3	I	V_1
V_3	V_3	V_2	V_1	I

see also DIHEDRAL GROUP, FINITE GROUP—Z_4

Vieta's Substitution

The substitution of

$$x = w - \frac{p}{3w}$$

into the standard form CUBIC EQUATION

$$x^3 + px = q,$$

which reduces the cubic to a QUADRATIC EQUATION in w^3,

$$(w^3)^2 - \tfrac{1}{27}p^3(w^3) - q = 0.$$

see also CUBIC EQUATION

Vigesimal

The base-20 notational system for representing REAL NUMBERS. The digits used to represent numbers using vigesimal NOTATION are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F, G, H, I, and J. A base-20 number system was used by the Aztecs and Mayans. The Mayans compiled extensive observations of planetary positions in base-20 notation.

see also BASE (NUMBER), BINARY, DECIMAL, HEXADECIMAL, OCTAL, QUATERNARY, TERNARY

References

Weisstein, E. W. "Bases." `http://www.astro.virginia.edu/~eww6n/math/notebooks/Bases.m`.

Vigintillion

In the American system, 10^{63}.

see also LARGE NUMBER

Villarceau Circles

Given an arbitrary point on a TORUS, four CIRCLES can be drawn through it. The first is in the plane of the torus and the second is PERPENDICULAR to it. The third and fourth CIRCLES are called Villarceau circles.

see also TORUS

References

Melzak, Z. A. *Invitation to Geometry.* New York: Wiley, pp. 63–72, 1983.

Villarceau, M. "Théorème sur le tore." *Nouv. Ann. Math.* **7**, 345–347, 1848.

Vinculum

A horizontal line placed above multiple quantities to indicate that they form a unit. It is most commonly used to denote ROOTS ($\sqrt{12345}$) and repeating decimals ($0.\overline{111}$).

Vinogradov's Theorem

Every sufficiently large ODD number is a sum of three PRIMES. Proved in 1937.

see also GOLDBACH CONJECTURE

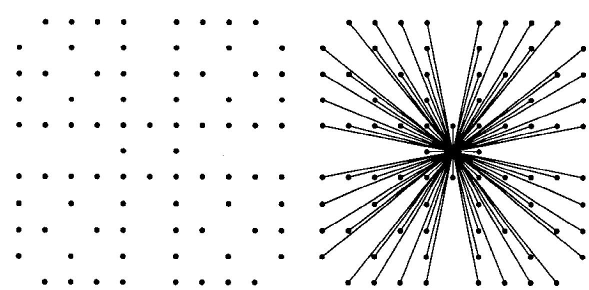
Virtual Group

see GROUPOID

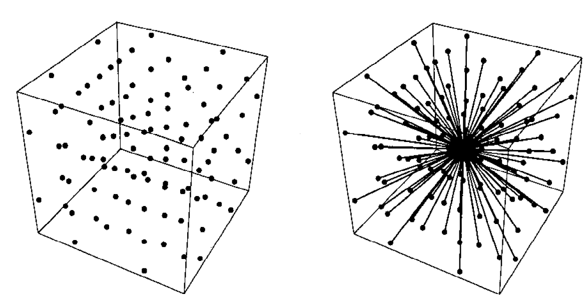
Visibility

see VISIBLE POINT

Visible Point



Two LATTICE POINTS (x, y) and (x', y') are mutually visible if the line segment joining them contains no further LATTICE POINTS. This corresponds to the requirement that $(x' - x, y' - y) = 1$, where (m, n) denotes the GREATEST COMMON DIVISOR. The plots above show the first few points visible from the ORIGIN.



If a LATTICE POINT is selected at random in 2-D, the probability that it is visible from the origin is $6/\pi^2$. This is also the probability that two INTEGERS picked at random are RELATIVELY PRIME. If a LATTICE POINT is picked at random in n-D, the probability that it is visible from the ORIGIN is $1/\zeta(n)$, where $\zeta(n)$ is the RIEMANN ZETA FUNCTION.

An invisible figure is a POLYGON all of whose corners are invisible. There are invisible sets of every finite shape. The lower left-hand corner of the invisible squares with smallest x coordinate of AREAS 2 and 3 are (14, 20) and (104, 6200).

see also LATTICE POINT, ORCHARD VISIBILITY PROBLEM, RIEMANN ZETA FUNCTION

References

Apostol, T. §3.8 in *Introduction to Analytic Number Theory.* New York: Springer-Verlag, 1976.

Baake, M.; Grimm, U.; and Warrington, D. H. "Some Remarks on the Visible Points of a Lattice." *J. Phys. A: Math. General* **27**, 2669–2674, 1994.

Beeler, M.; Gosper, R. W.; and Schroeppel, R. *HAKMEM.* Cambridge, MA: MIT Artificial Intelligence Laboratory, Memo AIM-239, Feb. 1972.

Herzog, F. and Stewart, B. M. "Patterns of Visible and Nonvisible Lattice Points." *Amer. Math. Monthly* **78**, 487–496, 1971.

Mosseri, R. "Visible Points in a Lattice." *J. Phys. A: Math. Gen.* **25**, L25–L29, 1992.

Schroeder, M. R. "A Simple Function and Its Fourier Transform." *Math. Intell.* **4**, 158–161, 1982.

Schroeder, M. R. *Number Theory in Science and Communication, 2nd ed.* New York: Springer-Verlag, 1990

Visible Point Vector Identity

A set of identities involving n-D visible lattice points was discovered by Campbell (1994). Examples include

$$\prod_{\substack{(a,b)=1\\ a\geq 0,b\leq 1}} (1-y^a z^b)^{-1/b} = (1-z)^{-1/(1-y)}$$

for $|yz|, |z| < 1$ and

$$\prod_{\substack{(a,b,c)=1\\ a,b\geq 0,c\leq 1}} (1-x^a y^b z^c)^{-1/c} = (1-z)^{-1/[(1-x)(1-y)]}$$

for $|xyz|, |xz|, |yz|, |z| < 1$.

References

Campbell, G. B. "Infinite Products Over Visible Lattice Points." *Internat. J. Math. Math. Sci.* **17**, 637–654, 1994.

Campbell, G. B. "Visible Point Vector Identities." `http://www.geocities.com/CapeCanaveral/Launchpad/9416/vpv.html`.

Vitali's Convergence Theorem

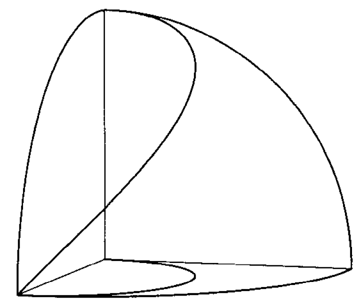
Let $f_n(z)$ be a sequence of functions, each regular in a region D, let $|f_n(z)| \leq M$ for every n and z in D, and let $f_n(z)$ tend to a limit as $n \to \infty$ at a set of points having a LIMIT POINT inside D. Then $f_n(z)$ tends uniformly to a limit in any region bounded by a contour interior to D, the limit therefore being an analytic function of z.

see also MONTEL'S THEOREM

References

Titchmarsh, E. C. *The Theory of Functions, 2nd ed.* Oxford, England: Oxford University Press, p. 168, 1960.

Viviani's Curve



The SPACE CURVE giving the intersection of the CYLINDER

$$(x-a)^2 + y^2 = a^2 \tag{1}$$

and the SPHERE

$$x^2 + y^2 + z^2 = 4^2. \tag{2}$$

It is given by the parametric equations

$$x = a(1+\cos t) \tag{3}$$
$$y = a\sin t \tag{4}$$
$$z = 2a\sin(\tfrac{1}{2}t). \tag{5}$$

The CURVATURE and TORSION are given by

$$\kappa(t) = \frac{\sqrt{13+3\cos t}}{a(3+\cos t)^{3/2}} \tag{6}$$
$$\tau(t) = \frac{6\cos(\frac{1}{2}t)}{a(13+3\cos t)}. \tag{7}$$

see also CYLINDER, SPHERE, STEINMETZ SOLID

References

Gray, A. "Viviani's Curve." §7.6 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 140–142, 1993.

von Seggern, D. *CRC Standard Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 270, 1993.

Viviani's Theorem

For a point P inside an EQUILATERAL TRIANGLE ΔABC, the sum of the perpendiculars p_i from P to the sides of the TRIANGLE is equal to the ALTITUDE h. This result is simply proved as follows,

$$\Delta ABC = \Delta PBC + \Delta PCA + \Delta PAB. \tag{1}$$

With s the side length,

$$\tfrac{1}{2}sh = \tfrac{1}{2}sp_a + \tfrac{1}{2}sp_b + \tfrac{1}{2}sp_c, \tag{2}$$

so

$$h = p_a + p_b + p_c. \tag{3}$$

see also ALTITUDE, EQUILATERAL TRIANGLE

Vojta's Conjecture

A conjecture which treats the heights of points relative to a canonical class of a curve defined over the INTEGERS.

References

Cox, D. A. "Introduction to Fermat's Last Theorem." *Amer. Math. Monthly* **101**, 3–14, 1994.

Volterra Integral Equation of the First Kind

An INTEGRAL EQUATION of the form

$$f(x) = \int_a^x k(x,t)\phi(t)\,dt.$$

see also FREDHOLM INTEGRAL EQUATION OF THE FIRST KIND, FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND, INTEGRAL EQUATION, VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND

References

Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, p. 865, 1985.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Volterra Equations." §18.2 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 786–788, 1992.

Volterra Integral Equation of the Second Kind

An INTEGRAL EQUATION of the form

$$\phi(x) = f(x) + \int_a^x k(x,t)\phi(t)\,dt.$$

see also FREDHOLM INTEGRAL EQUATION OF THE FIRST KIND, FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND, INTEGRAL EQUATION, VOLTERRA INTEGRAL EQUATION OF THE FIRST KIND

References

Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, p. 865, 1985.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Volterra Equations." §18.2 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 786–788, 1992.

Volume

The volume of a solid body is the amount of "space" it occupies. Volume has units of LENGTH cubed (i.e., cm^3, m^3, in^3, etc.) For example, the volume of a box (RECTANGULAR PARALLELEPIPED) of LENGTH L, WIDTH W, and HEIGHT H is given by

$$V = L \times W \times H.$$

The volume can also be computed for irregularly-shaped and curved solids such as the CYLINDER and CUBE. The volume of a SURFACE OF REVOLUTION is particularly simple to compute due to its symmetry.

The following table gives volumes for some common SURFACES. Here r denotes the RADIUS, h the height, A the base AREA, and s the SLANT HEIGHT (Beyer 1987).

Surface	V
cone	$\frac{1}{3}\pi r^2 h$
conical frustum	$\frac{1}{3}\pi h({R_1}^2 + {R_2}^2 + R_1R_2)$
cube	a^3
cylinder	$\pi r^2 h$
ellipsoid	$\frac{4}{3}\pi abc$
oblate spheroid	$\frac{4}{3}\pi a^2 b$
prolate spheroid	$\frac{4}{3}\pi ab^2$
pyramid	$\frac{1}{3}Ah$
pyramidal frustum	$\frac{1}{3}h(A_1 + A_2 + \sqrt{A_1A_2})$
sphere	$\frac{4}{3}\pi r^3$
spherical sector	$\frac{2}{3}\pi r^2 h$
spherical segment	$\frac{1}{3}\pi h^2 r(3r - h)$
torus	$2\pi^2 Rr^2$

Even simple SURFACES can display surprisingly counterintuitive properties. For instance, the SURFACE OF REVOLUTION of $y = 1/x$ around the x-axis for $x \geq 1$ is called GABRIEL'S HORN, and has finite volume, but infinite SURFACE AREA.

The generalization of volume to n DIMENSIONS for $n \geq 4$ is known as CONTENT.

see also ARC LENGTH, AREA, CONTENT, HEIGHT, LENGTH (SIZE), SURFACE AREA, SURFACE OF REVOLUTION, VOLUME ELEMENT, WIDTH (SIZE)

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 127–132, 1987.

Volume Element

A volume element is the differential element dV whose VOLUME INTEGRAL over some range in a given coordinate system gives the VOLUME of a solid,

$$V = \iiint_G dx\,dy\,dz. \tag{1}$$

In $\mathbb{R}^n$, the volume of the infinitesimal n-HYPERCUBE bounded by $dx_1, \ldots, dx_n$ has volume given by the WEDGE PRODUCT

$$dV = dx_1 \wedge \ldots \wedge dx_n \tag{2}$$

(Gray 1993).

The use of the antisymmetric WEDGE PRODUCT instead of the symmetric product $dx_1 \ldots dx_n$ is a technical refinement often omitted in informal usage. Dropping the wedges, the volume element for CURVILINEAR COORDINATES in $\mathbb{R}^3$ is given by

$$dV = |(h_1 \hat{\mathbf{u}}_1\, du_1) \cdot (h_2 \hat{\mathbf{u}}_2\, du_2) \times (h_3 \hat{\mathbf{u}}_3\, du_3)| \quad (3)$$
$$= h_1 h_2 h_3\, du_1\, du_2\, du_3 \quad (4)$$
$$= \left| \frac{\partial \mathbf{r}}{\partial u_1} \cdot \frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} \right| du_1\, du_2\, du_3 \quad (5)$$
$$= \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} & \frac{\partial x}{\partial u_3} \\ \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} & \frac{\partial y}{\partial u_3} \\ \frac{\partial z}{\partial u_1} & \frac{\partial z}{\partial u_2} & \frac{\partial z}{\partial u_3} \end{vmatrix} du_1\, du_2\, du_3 \quad (6)$$
$$= \left| \frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)} \right| du_1\, du_2\, du_3, \quad (7)$$

where the latter is the JACOBIAN and the h_i are SCALE FACTORS.

see also AREA ELEMENT, JACOBIAN, LINE ELEMENT, RIEMANNIAN METRIC, SCALE FACTOR, SURFACE INTEGRAL, VOLUME INTEGRAL

References
Gray, A. "Isometries of Surfaces." §13.2 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 255–258, 1993.

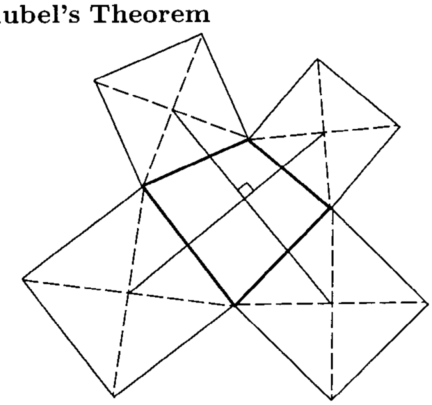
Volume Integral

A triple integral over three coordinates giving the VOLUME within some region R,

$$V = \iiint_G dx\, dy\, dz.$$

see also INTEGRAL, LINE INTEGRAL, MULTIPLE INTEGRAL, SURFACE INTEGRAL, VOLUME, VOLUME ELEMENT

von Aubel's Theorem



Given an arbitrary QUADRILATERAL, place a SQUARE outwardly on each side, and connect the centers of opposite SQUARES. Then the two lines are of equal length and cross at a RIGHT ANGLE.

see also QUADRILATERAL, RIGHT ANGLE, SQUARE

References
Kitchen, E. "Dörrie Tiles and Related Miniatures." *Math. Mag.* **67**, 128–130, 1994.

von Dyck's Theorem

Let a GROUP G have a presentation

$$G = (x_1, \ldots, x_n | r_j(x_1, \ldots, x_n), j \in J)$$

so that $G = F/R$, where F is the FREE GROUP with basis $\{x_1, \ldots, x_n\}$ and R is the NORMAL SUBGROUP generated by the r_j. If H is a GROUP with $H = \langle y_1, \ldots, y_n \rangle$ and if $r_j(y_1, \ldots, y_n) = 1$ for all j, then there is a surjective homomorphism $G \to H$ with $x_i \mapsto y_i$ for all i.

see also FREE GROUP, NORMAL SUBGROUP

References
Rotman, J. J. *An Introduction to the Theory of Groups, 4th ed.* New York: Springer-Verlag, p. 346, 1995.

von Mangoldt Function

see MANGOLDT FUNCTION

von Neumann Algebra

A GROUP "with bells and whistles." It was while studying von Neumann algebras that Jones discovered the amazing and highly unexpected connections with KNOT THEORY which led to the formulation of the JONES POLYNOMIAL.

References
Iyanaga, S. and Kawada, Y. (Eds.). "Von Neumann Algebras." §430 in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, pp. 1358–1363, 1980.

von Staudt-Clausen Theorem

$$B_{2n} = A_n - \sum_{\substack{p_k \\ p_k - 1 | 2n}} \frac{1}{p_k},$$

where B_{2n} is a BERNOULLI NUMBER, A_n is an INTEGER, and the p_ks are the PRIMES satisfying $p_k - 1 | 2k$. For example, for $k = 1$, the primes included in the sum are 2 and 3, since $(2-1)|2$ and $(3-1)|2$. Similarly, for $k = 6$, the included primes are (2, 3, 5, 7, 13), since (1, 2, 3, 6, 12) divide $12 = 2 \cdot 6$. The first few values of A_n for $n = 1, 2, \ldots$ are 1, 1, 1, 1, 1, 1, 2, -6, 56, -528, ... (Sloane's A000164).

The theorem was rediscovered by Ramanujan (Hardy 1959, p. 11) and can be proved using p-ADIC NUMBERS.

see also BERNOULLI NUMBER, p-ADIC NUMBER

References
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, p. 109, 1996.
Hardy, G. H. *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work, 3rd ed.* New York: Chelsea, 1959.
Hardy, G. H. and Wright, E. M. "The Theorem of von Staudt" and "Proof of von Staudt's Theorem." §7.9–7.10 in *An Introduction to the Theory of Numbers, 5th ed.* Oxford, England: Clarendon Press, pp. 90–93, 1979.
Sloane, N. J. A. Sequence A000146/M1717 in "An On-Line Version of the Encyclopedia of Integer Sequences."
Staudt. "Beweis eines Lehrsatzes, die Bernoullischen Zahlen betreffend." *J. reine angew. Math.* **21**, 372–374, 1840.

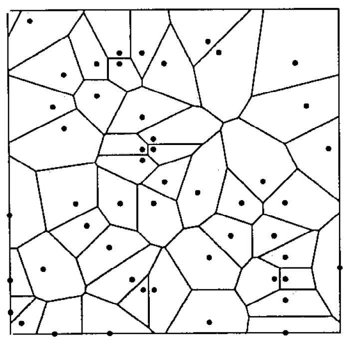
von Staudt Theorem

see VON STAUDT-CLAUSEN THEOREM

Voronoi Cell

The generalization of a VORONOI POLYGON to n-D, for $n > 2$.

Voronoi Diagram



The partitioning of a plane with n points into n convex POLYGONS such that each POLYGON contains exactly one point and every point in a given POLYGON is closer to its central point than to any other. A Voronoi diagram is sometimes also known as a DIRICHLET TESSELLATION. The cells are called DIRICHLET REGIONS, THIESSEN POLYTOPES, or VORONOI POLYGONS.

see also DELAUNAY TRIANGULATION, MEDIAL AXIS, VORONOI POLYGON

References

Eppstein, D. "Nearest Neighbors and Voronoi Diagrams." `http://www.ics.uci.edu/~eppstein/junkyard/nn.html`.

Voronoi Polygon

A POLYGON whose interior consists of all points in the plane which are closer to a particular LATTICE POINT than to any other. The generalization to n-D is called a DIRICHLET REGION, THIESSEN POLYTOPE, or VORONOI CELL.

Voting

It is possible to conduct a secret ballot even if the votes are sent in to a central polling station (Lipton and Widgerson, Honsberger 1985).

see also ARROW'S PARADOX, BALLOT PROBLEM, MAY'S THEOREM, QUOTA SYSTEM, SOCIAL CHOICE THEORY

References

Honsberger, R. *Mathematical Gems III.* Washington, DC: Math. Assoc. Amer., pp. 157–162, 1985.

Lipton, R. G.; and Widgerson, A. "Multi-Party Cryptographic Protocols."

VR Number

A "visual representation" number which is a sum of some simple function of its digits. For example,

$$1233 = 12^2 + 33^2$$

$$2661653 = 1653^2 - 266^2$$

$$221859 = 22^3 + 18^3 + 59^3$$

$$40585 + 4! + 0! + 5! + 8! + 5!$$

$$148349 = !1 + !4 + !8 + !3 + !4 + !9$$

$$4913 = (4 + 9 + 1 + 3)^3$$

are all VR numbers given by Madachy (1979).

References

Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, pp. 165–171, 1979.

Vulgar Series

see FAREY SERIES

W

W2-Constant

$$W_2 = 1.529954037\ldots.$$

References
Plouffe, S. "W2 Constant." http://lacim.uqam.ca/piDATA/w2.txt.

W-Function

see LAMBERT'S W-FUNCTION

Wada Basin

A BASIN OF ATTRACTION in which every point on the common boundary of that basin and another basin is also a boundary of a third basin. In other words, no matter how closely a boundary point is zoomed into, all three basins appear in the picture.

see also BASIN OF ATTRACTION

References
Nusse, H. E. and Yorke, J. A. "Basins of Attraction." *Science* **271**, 1376–1380, 1996.

Walk

A sequence of VERTICES and EDGES such that the VERTICES and EDGES are adjacent. A walk is therefore equivalent to a graph CYCLE, but with the VERTICES along the walk enumerated as well as the EDGES.

see also CIRCUIT, CYCLE (GRAPH), PATH, RANDOM WALK

Wallace-Bolyai-Gerwein Theorem

Two POLYGONS are congruent by DISSECTION IFF they have the same AREA. In particular, any POLYGON is congruent by DISSECTION to a SQUARE of the same AREA. Laczkovich (1988) also proved that a CIRCLE is congruent by DISSECTION to a SQUARE (furthermore, the DISSECTION can be accomplished using TRANSLATIONS only).

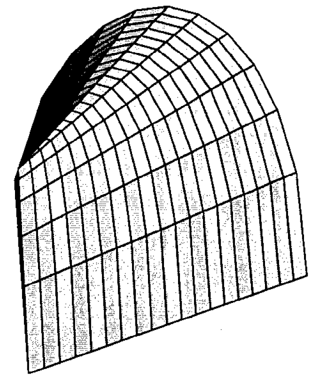
see also DISSECTION

References
Klee, V. and Wagon, S. *Old and New Unsolved Problems in Plane Geometry and Number Theory.* Washington, DC: Math. Assoc. Amer., pp. 50–51, 1991.
Laczkovich, M. "Von Neumann's Paradox with Translation." *Fund. Math.* **131**, 1–12, 1988.

Wallace-Simson Line

see SIMSON LINE

Wallis's Conical Edge



The RIGHT CONOID surface given by the parametric equations

$$x(u,v) = v\cos u$$
$$y(u,v) = v\sin u$$
$$z(u,v) = c\sqrt{a^2 - b^2\cos^2 u}.$$

see also RIGHT CONOID

References
Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 354–355, 1993.

Wallis Cosine Formula

$$\int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \frac{\pi}{2}\frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n} & \text{for } n = 2, 4, \ldots \\ \frac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots n} & \text{for } n = 3, 5, \ldots. \end{cases}$$

see also WALLIS FORMULA, WALLIS SINE FORMULA

Wallis Formula

The Wallis formula follows from the INFINITE PRODUCT representation of the SINE

$$\sin x = x\prod_{n=1}^\infty \left(1 - \frac{x^2}{\pi^2 n^2}\right). \tag{1}$$

Taking $x = \pi/2$ gives

$$1 = \frac{\pi}{2}\prod_{n=1}^\infty \left[1 - \frac{1}{(2n)^2}\right] = \frac{\pi}{2}\prod_{n=1}^\infty \left[\frac{(2n)^2 - 1}{(2n)^2}\right], \tag{2}$$

so

$$\frac{\pi}{2} = \prod_{n=1}^\infty \left[\frac{(2n)^2}{(2n-1)(2n+1)}\right] = \frac{2\cdot 2}{1\cdot 3}\frac{4\cdot 4}{3\cdot 5}\frac{6\cdot 6}{5\cdot 7}\cdots. \tag{3}$$

A derivation due to Y. L. Yung uses the RIEMANN ZETA FUNCTION. Define

$$F(s) \equiv -\mathrm{Li}_s(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$$
$$= (1-2^{1-s})\zeta(s) \quad (4)$$
$$F'(s) = \sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n^s}, \quad (5)$$

so

$$F'(0) = \sum_{n=1}^{\infty} (-1)^n \ln n = -\ln 1 + \ln 2 - \ln 3 + \ldots$$
$$= \ln\left(\frac{2\cdot 4\cdot 6\cdots}{1\cdot 3\cdot 5\cdots}\right). \quad (6)$$

Taking the derivative of the zeta function expression gives

$$\frac{d}{ds}(1-2^{1-s})\zeta(s) = 2^{1-s}(\ln 2)\zeta(s) + (1-2^{1-s})\zeta'(s) \quad (7)$$

$$\left[\frac{d}{ds}(1-2^{1-s})\zeta(s)\right]_{s=0} = -\ln 2 - \zeta'(0)$$
$$= -\ln 2 + \tfrac{1}{2}\ln(2\pi) = \ln\left(\frac{\sqrt{2\pi}}{2}\right) = \ln\left(\sqrt{\frac{\pi}{2}}\right). \quad (8)$$

Equating and squaring then gives the Wallis formula, which can also be expressed

$$\frac{\pi}{2} = \left[4^{\zeta(0)} e^{-\zeta'(0)}\right]^2. \quad (9)$$

The q-ANALOG of the Wallis formula for $q = 2$ is

$$\prod_{k=1}^{\infty} (1-q^{-k})^{-1} = 3.4627466194\ldots \quad (10)$$

(Finch).

see also WALLIS COSINE FORMULA, WALLIS SINE FORMULA

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 258, 1972.

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/dig/dig.html`.

Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics, Pt. 2, 2nd ed.* Princeton, NJ: Van Nostrand, pp. 63–64, 1951.

Wallis's Problem

Find solutions to $\sigma(x^2) = \sigma(y^2)$ other than $(x, y) = (4, 5)$, where σ is the DIVISOR FUNCTION.

see also FERMAT'S SIGMA PROBLEM

Wallis Sieve

A compact set W_∞ with AREA

$$\mu(W_\infty) = \frac{8}{9}\frac{24}{25}\frac{48}{49}\cdots = \frac{\pi}{4}$$

created by punching a square hole of length 1/3 in the center of a square. In each of the eight squares remaining, punch out another hole of length $1/(3\cdot 5)$, and so on.

Wallis Sine Formula

$$\int_0^{\pi/2} \sin^n x\,dx = \begin{cases} \frac{\pi}{2}\frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n} & \text{for } n = 2,\ 4,\ \ldots \\ \frac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots n} & \text{for } n = 3,\ 5,\ \ldots. \end{cases}$$

see also WALLIS COSINE FORMULA, WALLIS FORMULA

Wallpaper Groups

The 17 PLANE SYMMETRY GROUPS. Their symbols are p1, p2, pm, pg, cm, pmm, pmg, pgg, cmm, p4, p4m, p4g, p3, p31m, p3m1, p6, and p6m. For a description of the symmetry elements present in each space group, see Coxeter (1969, p. 413).

References

Coxeter, H. S. M. *Introduction to Geometry, 2nd ed.* New York: Wiley, 1969.

Hilbert, D. and Cohn-Vossen, S. *Geometry and the Imagination.* New York: Chelsea, 1952.

Joyce, D. E. "Wallpaper Groups (Plane Symmetry Groups)." `http://aleph0.clarku.edu/~djoyce/wallpaper/`.

Lee, X. "The Discontinuous Groups of Rotation and Translation in the Plane." `http://www.best.com/~xah/Wallpaper_dir/c0_WallPaper.html`.

Schattschneider, D. "The Plane Symmetry Groups: Their Recognition and Notation." *Amer. Math. Monthly* **85**, 439–450, 1978.

Weyl, H. *Symmetry.* Princeton, NJ: Princeton University Press, 1952.

Walsh Function

Functions consisting of a number of fixed-amplitude square pulses interposed with zeros. Following Harmuth (1969), designate those with EVEN symmetry $\mathrm{Cal}(k,t)$ and those with ODD symmetry $\mathrm{Sal}(k,t)$. Define the SEQUENCY k as half the number of zero crossings in the time base. Walsh functions with nonidentical SEQUENCIES are ORTHOGONAL, as are the functions $\mathrm{Cal}(k,t)$ and $\mathrm{Sal}(k,t)$. The product of two Walsh functions is also a Walsh function. The Walsh functions

$$\mathrm{Wal}(k,t) = \begin{cases} \mathrm{Cal}(k/2,t) & \text{for } k = 0,\ 2,\ 4,\ \ldots \\ \mathrm{Sal}((k+1)/2,t) & \text{for } k = 1,\ 3,\ 5,\ \ldots. \end{cases}$$

The Walsh functions $\mathrm{Cal}(k,t)$ for $k = 0, 1, \ldots, n/2-1$ and $\mathrm{Sal}(k,t)$ for $k = 1, 2, \ldots, n/2$ are given by the rows of the HADAMARD MATRIX H_n.

see also HADAMARD MATRIX, SEQUENCY

References
Beauchamp, K. G. *Walsh Functions and Their Applications.* London: Academic Press, 1975.
Harmuth, H. F. "Applications of Walsh Functions in Communications." *IEEE Spectrum* **6**, 82–91, 1969.
Thompson, A. R.; Moran, J. M.; and Swenson, G. W. Jr. *Interferometry and Synthesis in Radio Astronomy.* New York: Wiley, p. 204, 1986.
Tzafestas, S. G. *Walsh Functions in Signal and Systems Analysis and Design.* New York: Van Nostrand Reinhold, 1985.
Walsh, J. L. "A Closed Set of Normal Orthogonal Functions." *Amer. J. Math.* **45**, 5–24, 1923.

Walsh Index

The statistical INDEX

$$P_W \equiv \frac{\sum \sqrt{q_0 q_n}\, p_n}{\sum \sqrt{q_0 q_n}\, p_0},$$

where p_n is the price per unit in period n and q_n is the quantity produced in period n.

see also INDEX

References
Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics, Pt. 1, 3rd ed.* Princeton, NJ: Van Nostrand, p. 66, 1962.

Wang's Conjecture

Wang's conjecture states that if a set of tiles can tile the plane, then they can always be arranged to do so periodically (Wang 1961). The CONJECTURE was refuted when Berger (1966) showed that an aperiodic set of tiles existed. Berger used 20,426 tiles, but the number has subsequently been greatly reduced.

see also TILING

References
Adler, A. and Holroyd, F. C. "Some Results on One-Dimensional Tilings." *Geom. Dedicata* **10**, 49–58, 1981.
Berger, R. "The Undecidability of the Domino Problem." *Mem. Amer. Math. Soc. No.* **66**, 1–72, 1966.
Grünbaum, B. and Sheppard, G. C. *Tilings and Patterns.* New York: W. H. Freeman, 1986.
Hanf, W. "Nonrecursive Tilings of the Plane. I." *J. Symbolic Logic* **39**, 283–285, 1974.
Mozes, S. "Tilings, Substitution Systems, and Dynamical Systems Generated by Them." *J. Analyse Math.* **53**, 139–186, 1989.
Myers, D. "Nonrecursive Tilings of the Plane. II." *J. Symbolic Logic* **39**, 286–294, 1974.
Robinson, R. M. "Undecidability and Nonperiodicity for Tilings of the Plane." *Invent. Math.* **12**, 177–209, 1971.
Wang, H. *Bell Systems Tech. J.* **40**, 1–41, 1961.

Ward's Primality Test

Let N be an ODD INTEGER, and assume there exists a LUCAS SEQUENCE $\{U_n\}$ with associated SYLVESTER CYCLOTOMIC NUMBERS $\{Q_n\}$ such that there is an $n > \sqrt{N}$ (with n and N RELATIVELY PRIME) for which N DIVIDES Q_n. Then N is a PRIME unless it has one of the following two forms:

1. $N = (n-1)^2$, with $n-1$ PRIME and $n > 4$, or
2. $N = n^2 - 1$, with $n-1$ and $n+1$ PRIME.

see also LUCAS SEQUENCE, SYLVESTER CYCLOTOMIC NUMBER

References
Ribenboim, P. *The Book of Prime Number Records, 2nd ed.* New York: Springer-Verlag, pp. 69–70, 1989.

Waring's Conjecture

see WARING'S PRIME CONJECTURE, WARING'S SUM CONJECTURE

Waring Formula

$$A^n + B^n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (AB)^j (A+B)^{n-2j},$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION and $\binom{n}{k}$ is a BINOMIAL COEFFICIENT.

see also FERMAT'S LAST THEOREM

Waring's Prime Conjecture

Every ODD INTEGER is a PRIME or the sum of three PRIMES.

Waring's Problem

Waring proposed a generalization of LAGRANGE'S FOUR-SQUARE THEOREM, stating that every RATIONAL INTEGER is the sum of a fixed number $g(n)$ of nth POWERS of INTEGERS, where n is any given POSITIVE INTEGER and $g(n)$ depends only on n. Waring originally speculated that $g(2) = 4$, $g(3) = 9$, and $g(4) = 19$. In 1909, Hilbert proved the general conjecture using an identity in 25-fold multiple integrals (Rademacher and Toeplitz 1957, pp. 52–61).

In LAGRANGE'S FOUR-SQUARE THEOREM, Lagrange proved that $g(2) = 4$, where 4 may be reduced to 3 except for numbers of the form $4^n(8k+7)$ (as proved by Legendre). In the early twentieth century, Dickson, Pillai, and Niven proved that $g(3) = 9$. Hilbert, Hardy, and Vinogradov proved $g(4) \leq 21$, and this was subsequently reduced to $g(4) = 19$ by Balasubramanian *et al.* (1986). Liouville proved (using LAGRANGE'S FOUR-SQUARE THEOREM and LIOUVILLE POLYNOMIAL IDENTITY) that $g(5) \leq 53$, and this was improved to 47, 45, 41, 39, 38, and finally $g(5) \leq 37$ by Wieferich. See Rademacher and Toeplitz (1957, p. 56) for a simple proof. J.-J. Chen (1964) proved that $g(5) = 37$.

Dickson, Pillai, and Niven also conjectured an explicit formula for $g(s)$ for $s > 6$ (Bell 1945), based on the relationship

$$\left(\frac{3}{2}\right)^n - \left\lfloor \left(\frac{3}{2}\right)^n \right\rfloor = 1 - \left(\frac{1}{2}\right)^n \left\{ \left\lfloor \left(\frac{3}{2}\right)^n + 2 \right\rfloor \right\}. \quad (1)$$

If the DIOPHANTINE (i.e., n is restricted to being an INTEGER) inequality

$$\left\{\left(\frac{3}{2}\right)^n\right\} \leq 1 - \left(\frac{3}{4}\right)^n \tag{2}$$

is true, then

$$g(n) = 2^n + \left\lfloor\left(\frac{3}{2}\right)^n\right\rfloor - 2. \tag{3}$$

This was given as a lower bound by Euler, and has been verified to be correct for $6 \leq n \leq 200,000$. Since 1957, it has been known that at most a FINITE number of k exceed Euler's lower bound.

There is also a related problem of finding the least INTEGER n such that every POSITIVE INTEGER beyond a certain point (i.e., all but a FINITE number) is the SUM of $G(n)$ nth POWERS. From 1920–1928, Hardy and Littlewood showed that

$$G(n) \leq (n-2)2^{n-1} + 5 \tag{4}$$

and conjectured that

$$G(k) < \begin{cases} 2k+1 & \text{for } k \text{ not a power of 2} \\ 4k & \text{for } k \text{ a power of 2.} \end{cases} \tag{5}$$

The best currently known bound is

$$G(k) < ck\ln k \tag{6}$$

for some constant c. Heilbronn (1936) improved Vinogradov's results to obtain

$$G(n) \leq 6n\ln n + \left[4 + 3\ln\left(3 + \frac{2}{n}\right)\right] n + 3. \tag{7}$$

It has long been known that $G(2) = 4$. Dickson and Landau proved that the only INTEGERS requiring nine CUBES are 23 and 239, thus establishing $G(3) \leq 8$. Wieferich proved that only 15 INTEGERS require eight CUBES: 15, 22, 50, 114, 167, 175, 186, 212, 213, 238, 303, 364, 420, 428, and 454, establishing $G(3) \leq 7$. The largest number known requiring seven CUBES is 8042. In 1933, Hardy and Littlewood showed that $G(4) \leq 19$, but this was improved in 1936 to 16 or 17, and shown to be exactly 16 by Davenport (1939b). Vaughan (1986) greatly improved on the method of Hardy and Littlewood, obtaining improved results for $n \geq 5$. These results were then further improved by Brüdern (1990), who gave $G(5) \leq 18$, and Wooley (1992), who gave $G(n)$ for $n = 6$ to 20. Vaughan and Wooley (1993) showed $G(8) \leq 42$.

Let $G^+(n)$ denote the smallest number such that *almost all* sufficiently large INTEGERS are the sum of $G^+(n)$ nth POWERS. Then $G^+(3) = 4$ (Davenport 1939a), $G^+(4) = 15$ (Hardy and Littlewood 1925), $G^+(8) = 32$ (Vaughan 1986), and $G^+(16) = 64$ (Wooley 1992). If the negatives of POWERS are permitted in addition to the powers themselves, the largest number of nth POWERS needed to represent an aribtrary integer are denoted $eg(n)$ and $EG(n)$ (Wright 1934, Hunter 1941, Gardner 1986). In general, these values are much harder to calculate than are $g(n)$ and $G(n)$.

The following table gives $g(n)$, $G(n)$, $G^+(n)$, $eg(n)$, and $EG(n)$ for $n \leq 20$. The sequence of $g(n)$ is Sloane's A002804.

n	$g(n)$	$G(n)$	$G^+(n)$	$eg(n)$	$EG(n)$
2	4	4		3	3
3	9	≤ 7	≤ 4	[4, 5]	
4	19	16	≤ 15	[9, 10]	
5	37	≤ 18			
6	73	≤ 27			
7	143	≤ 36			
8	279	≤ 42	≤ 32		
9	548	≤ 55			
10	1079	≤ 63			
11	2132	≤ 70			
12	4223	≤ 79			
13	8384	≤ 87			
14	16673	≤ 95			
15	33203	≤ 103			
16	66190	≤ 112	≤ 64		
17	132055	≤ 120			
18	263619	≤ 129			
19	526502	≤ 138			
20	1051899	≤ 146			

see also EULER'S CONJECTURE, SCHNIRELMANN'S THEOREM, VINOGRADOV'S THEOREM

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Waring's Sum Conjecture

see WARING'S PROBLEM

Waring's Theorem

If each of two curves meets the LINE AT INFINITY in distinct, nonsingular points, and if all their intersections are finite, then if to each common point there is attached a weight equal to the number of intersections absorbed therein, the CENTER OF MASS of these points is the center of gravity of the intersections of the asymptotes.

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Watchman Theorem

see ART GALLERY THEOREM

Watson's Formula

Let $J_\nu(z)$ be a BESSEL FUNCTION OF THE FIRST KIND, $Y_\nu(z)$ a BESSEL FUNCTION OF THE SECOND KIND, and $K_\nu(z)$ a MODIFIED BESSEL FUNCTION OF THE FIRST KIND. Also let $\Re[z] > 0$ and require $\Re[\mu-\nu] < 1$. Then

$$J_\mu(z)Y_\nu(z) - J_\nu(z)Y_\mu(z) = \frac{4\sin[(\mu-\nu)\pi]}{\pi^2}\int_0^\infty K_{\nu-\mu}(2z\sinh t)e^{-(\mu+\nu)t}\,dt.$$

The fourth edition of Gradshteyn and Ryzhik (1979), Iyanaga and Kawada (1980), and Ito (1987) erroneously give the exponential with a PLUS SIGN. A related integral is given by

$$J_\nu(z)\frac{\partial Y_\nu(z)}{\partial\nu} - Y_\nu(z)\frac{\partial J_\nu(z)}{\partial\nu} = -\frac{4}{\pi}\int_0^\infty K_0(2z\sinh t)e^{-2\nu t}\,dt$$

for $\Re[z] > 0$.

see also DIXON-FERRAR FORMULA, NICHOLSON'S FORMULA

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Watson-Nicholson Formula

Let $H_\nu^{(\iota)}$ be a HANKEL FUNCTION OF THE FIRST or SECOND KIND, let $x, \nu > 0$, and define

$$w = \sqrt{\left(\frac{x}{\nu}\right)^2 - 1}.$$

Then

$$H_\nu^{(\iota)}(x) = 3^{-1/2}w\exp\{(-1)^{\iota+1}i[\pi/6 + \nu(w - \tfrac{1}{3}w^3 - \tan^{-1}w)]\}H_{1/3}^{(\iota)}(\tfrac{1}{3}\nu w) + \mathcal{O}|\nu^{-1}|.$$

References
Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1475, 1980.

Watson Quintuple Product Identity

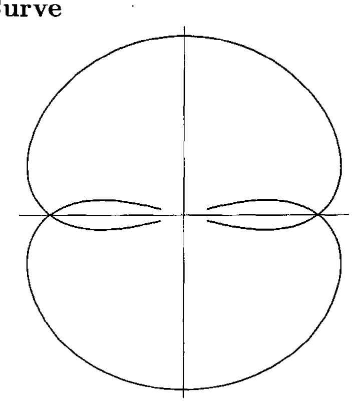
see QUINTUPLE PRODUCT IDENTITY

Watson's Theorem

$$ {}_3F_2\left[\begin{matrix} a, b, c \\ \frac{1}{2}(a+b+1), c\end{matrix}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+c)\Gamma[\frac{1}{2}(1+a+b)]\Gamma(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c)}{\Gamma[\frac{1}{2}(1+a)]\Gamma[\frac{1}{2}(1+b)]\Gamma(\frac{1}{2}-\frac{1}{2}a+c)\Gamma(\frac{1}{2}-\frac{1}{2}b+c)},$$

where ${}_3F_2(a, b, c; d, e; z)$ is a GENERALIZED HYPERGEOMETRIC FUNCTION and $\Gamma(z)$ is the GAMMA FUNCTION.

Watt's Curve



A curve named after James Watt (1736–1819), the Scottish engineer who developed the steam engine (MacTutor Archive). The curve is produced by a LINKAGE of rods connecting two wheels of equal diameter. Let the two wheels have RADIUS b and let their centers be located a distance $2a$ apart. Further suppose that a rod of length $2c$ is fixed at each end to the CIRCUMFERENCE of the two wheels. Let P be the MIDPOINT of the rod. Then Watt's curve C is the LOCUS of P.

The POLAR equation of Watt's curve is

$$r^2 = b^2 - (a\sin\theta \pm \sqrt{c^2 - a^2\cos^2\theta}\,)^2.$$

If $a = c$, then C is a CIRCLE of RADIUS b with a figure of eight inside it.

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MacTutor History of Mathematics Archive. "Watt's Curve." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Watts.html`.

Watt's Parallelogram

A LINKAGE used in the original steam engine to turn back-and-forth motion into approximately straight-line motion.

see also LINKAGE

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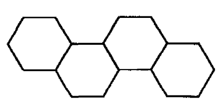
Wave



A 4-POLYHEX.

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Wave Equation

The wave equation is

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}, \tag{1}$$

where ∇^2 is the LAPLACIAN.

The 1-D wave equation is

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}. \tag{2}$$

In order to specify a wave, the equation is subject to boundary conditions

$$\psi(0,t) = 0 \tag{3}$$

$$\psi(L,t) = 0, \tag{4}$$

and initial conditions

$$\psi(x,0) = f(x) \tag{5}$$

$$\frac{\partial\psi}{\partial t}(x,0) = g(x). \tag{6}$$

The wave equation can be solved using the so-called d'Alembert's solution, a FOURIER TRANSFORM method, or SEPARATION OF VARIABLES.

d'Alembert devised his solution in 1746, and Euler subsequently expanded the method in 1748. Let

$$\xi \equiv x - at \tag{7}$$

$$\eta \equiv x + at. \tag{8}$$

By the CHAIN RULE,

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\psi}{\partial\xi^2} + 2\frac{\partial^2\psi}{\partial\xi\partial\eta} + \frac{\partial 2\psi}{\partial\eta^2} \tag{9}$$

$$\frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial\xi^2} - 2\frac{\partial^2\psi}{\partial\xi\partial\eta} + \frac{\partial^2\psi}{\partial\eta^2}. \tag{10}$$

The wave equation then becomes

$$\frac{\partial^2\psi}{\partial\xi\partial\eta} = 0. \tag{11}$$

Any solution of this equation is of the form

$$\psi(\xi,\eta) = f(\eta) + g(\xi) = f(x+vt) + g(x-vt), \tag{12}$$

where f and g are *any* functions. They represent two waveforms traveling in opposite directions, f in the NEGATIVE x direction and g in the POSITIVE x direction.

The 1-D wave equation can also be solved by applying a FOURIER TRANSFORM to each side,

$$\int_{-\infty}^{\infty} \frac{\partial^2\psi(x,t)}{\partial x^2} e^{-2\pi ikx}\,dx = \frac{1}{v^2}\int_{-\infty}^{\infty} \frac{\partial^2\psi(x,t)}{\partial t^2} e^{-2\pi ikx}\,dx, \tag{13}$$

which is given, with the help of the FOURIER TRANSFORM DERIVATIVE identity, by

$$(2\pi ik)^2\Psi(k,t) = \frac{1}{v^2}\frac{\partial^2\Psi(k,t)}{\partial t^2}, \tag{14}$$

where

$$\Psi(k,t) \equiv \mathcal{F}[\psi(x,t)] = \int_{-\infty}^{\infty} \psi(x,t)e^{-2\pi ikx}\,dx. \tag{15}$$

This has solution

$$\Psi(k,t) = A(k)e^{2\pi ikvt} + B(k)e^{-2\pi ikvt}. \tag{16}$$

Taking the inverse FOURIER TRANSFORM gives

$$\begin{aligned}\psi(x,t) &\equiv \int_{-\infty}^{\infty} \Psi(k,t)e^{2\pi ikx}\,dx \\ &= \int_{-\infty}^{\infty} [A(k)e^{2\pi ikvt} \\ &\quad + B(k)e^{-2\pi ikvt}]e^{-2\pi ikx}\,dk \\ &= \int_{-\infty}^{\infty} A(k)e^{-2\pi ik(x-vt)}\,dk \\ &\quad + \int_{-\infty}^{\infty} B(k)e^{-2\pi ik(x+vt)}\,dk \\ &= f_1(x-vt) + b(k)f_2(x+vt),\end{aligned} \tag{17}$$

where

$$f_1(u) \equiv \mathcal{F}[A(k)] = \int_{-\infty}^{\infty} A(k)e^{-2\pi iku}\,dk \tag{18}$$

$$f_2(u) \equiv \mathcal{F}[B(k)] = \int_{-\infty}^{\infty} B(k)e^{-2\pi iku}\,dk. \tag{19}$$

This solution is still subject to all other initial and boundary conditions.

The 1-D wave equation can be solved by SEPARATION OF VARIABLES using a trial solution

$$\psi(x,t) = X(x)T(t). \tag{20}$$

This gives

$$T\frac{d^2X}{dx^2} = \frac{1}{v^2}X\frac{d^2T}{dt^2} \tag{21}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2} = -k^2. \tag{22}$$

So the solution for X is

$$X(x) = C\cos(kx) + D\sin(kx). \tag{23}$$

Rewriting (22) gives

$$\frac{1}{T}\frac{d^2T}{dt^2} = -v^2k^2 \equiv -\omega^2, \tag{24}$$

so the solution for T is

$$T(t) = E\cos(\omega t) + F\sin(\omega t), \tag{25}$$

where $v \equiv \omega/k$. Applying the boundary conditions $\psi(0,t) = \psi(L,t) = 0$ to (23) gives

$$C = 0 \qquad kL = m\pi, \tag{26}$$

where m is an INTEGER. Plugging (23), (25) and (26) back in for ψ in (21) gives, for a particular value of m,

$$\begin{aligned}\psi_m(x,t) &= [E_m\sin(\omega_m t) + F_m\cos(\omega_m t)]D_m\sin\left(\frac{m\pi x}{L}\right) \\ &\equiv [A_m\cos(\omega_m t) + B_m\sin(\omega_m t)]\sin\left(\frac{m\pi x}{L}\right).\end{aligned} \tag{27}$$

The initial condition $\dot{\psi}(x,0) = 0$ then gives $B_m = 0$, so (27) becomes

$$\psi_m(x,t) = A_m\cos(\omega_m t)\sin\left(\frac{m\pi x}{L}\right). \tag{28}$$

The general solution is a sum over all possible values of m, so

$$\psi(x,t) = \sum_{m=1}^{\infty} A_m\cos(\omega_m t)\sin\left(\frac{m\pi x}{L}\right). \tag{29}$$

Using ORTHOGONALITY of sines again,

$$\int_0^L \sin\left(\frac{l\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \tfrac{1}{2}L\delta_{lm}, \tag{30}$$

where δ_{lm} is the KRONECKER DELTA defined by

$$\delta_{mn} \equiv \begin{cases}1 & m = n \\ 0 & m \neq n\end{cases}, \tag{31}$$

gives

$$\begin{aligned}&\int_0^L \psi(x,0)\sin\left(\frac{m\pi x}{L}\right)dx \\ &\qquad = \sum_{l=1}^{\infty} A_l\sin\left(\frac{l\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx \\ &\qquad\qquad = \sum_{l=1}^{\infty} A_l\tfrac{1}{2}L\delta_{lm} = \tfrac{1}{2}LA_m,\end{aligned} \tag{32}$$

so we have

$$A_m = \frac{2}{L}\int_0^L \psi(x,0)\sin\left(\frac{m\pi x}{L}\right)dx. \tag{33}$$

The computation of A_ms for specific initial distortions is derived in the FOURIER SINE SERIES section. We already have found that $B_m = 0$, so the equation of motion for the string (29), with

$$\omega_m \equiv vk_m = \frac{vm\pi}{L}, \tag{34}$$

is

$$\psi(x,t) = \sum_{m=1}^{\infty} A_m\cos\left(\frac{vm\pi t}{L}\right)\sin\left(\frac{m\pi x}{L}\right), \tag{35}$$

where the A_m COEFFICIENTS are given by (33).

A damped 1-D wave

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} + b\frac{\partial\psi}{\partial t}, \tag{36}$$

given boundary conditions

$$\psi(0,t) = 0 \tag{37}$$
$$\psi(L,t) = 0, \tag{38}$$

initial conditions

$$\psi(x,0) = f(x) \tag{39}$$
$$\frac{\partial\psi}{\partial t}(x,0) = g(x), \tag{40}$$

and the additional constraint

$$0 < b < \frac{2\pi}{Lv}, \tag{41}$$

can also be solved as a FOURIER SERIES.

$$\psi(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) e^{-v^2bt/2}[a_n \sin(\mu_n t) + b_n \cos(\mu_n t)], \tag{42}$$

where

$$\mu_n \equiv \frac{\sqrt{4v^2n^2\pi^2 - b^2L^2v^4}}{2L} = \frac{v\sqrt{4n^2\pi^2 - b^2L^2v^2}}{2L} \tag{43}$$
$$b_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx \tag{44}$$
$$a_n = \frac{2}{L\mu_n}\left\{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\left[g(x) + \frac{v^2b}{2}f(x)\right]\right\}dx. \tag{45}$$

To find the motion of a rectangular membrane with sides of length L_x and L_y (in the absence of gravity), use the 2-D wave equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}, \tag{46}$$

where $z(x,y,t)$ is the vertical displacement of a point on the membrane at position (x,y) and time t. Use SEPARATION OF VARIABLES to look for solutions of the form

$$z(x,y,t) = X(x)Y(y)T(t). \tag{47}$$

Plugging (47) into (46) gives

$$YT\frac{d^2X}{dx^2} + XT\frac{d^2Y}{dy^2} = \frac{1}{v^2}XY\frac{d^2T}{dt^2}, \tag{48}$$

where the partial derivatives have now become complete derivatives. Multiplying (48) by v^2/XYT gives

$$\frac{v^2}{X}\frac{d^2X}{dx^2} + \frac{v^2}{Y}\frac{d^2Y}{dy^2} = \frac{1}{T}\frac{d^2T}{dt^2}. \tag{49}$$

The left and right sides must both be equal to a constant, so we can separate the equation by writing the right side as

$$\frac{1}{T}\frac{d^2T}{dt^2} = -\omega^2. \tag{50}$$

This has solution

$$T(t) = C_\omega \cos(\omega t) + D_\omega \sin(\omega t). \tag{51}$$

Plugging (50) back into (49),

$$\frac{v^2}{X}\frac{d^2X}{dx^2} + \frac{v^2}{Y}\frac{d^2Y}{dy^2} = -\omega^2, \tag{52}$$

which we can rewrite as

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{\omega^2}{v^2} = -k_x{}^2 \tag{53}$$

since the left and right sides again must both be equal to a constant. We can now separate out the $Y(y)$ equation

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = k_x{}^2 - \frac{\omega^2}{v^2} \equiv -k_y{}^2, \tag{54}$$

where we have defined a new constant k_y satisfying

$$k_x{}^2 + k_y{}^2 = \frac{\omega^2}{v^2}. \tag{55}$$

Equations (53) and (54) have solutions

$$X(x) = E\cos(k_x x) + F\sin(k_x x) \tag{56}$$
$$Y(y) = G\cos(k_y y) + H\sin(k_y y). \tag{57}$$

We now apply the boundary conditions to (56) and (57). The conditions $z(0,y,t) = 0$ and $z(x,0,t) = 0$ mean that

$$E = 0 \qquad G = 0. \tag{58}$$

Similarly, the conditions $z(L_x,y,t) = 0$ and $z(x,L_y,t) = 0$ give $\sin(k_xL_x) = 0$ and $\sin(k_yL_y) = 0$, so $L_xk_x = p\pi$ and $L_yk_y = q\pi$, where p and q are INTEGERS. Solving for the allowed values of k_x and k_y then gives

$$k_x = \frac{p\pi}{L_x} \qquad k_y = \frac{q\pi}{L_y}. \tag{59}$$

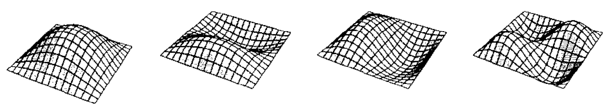
Plugging (52), (56), (57), (58), and (59) back into (22) gives the solution for particular values of p and q,

$$z_{pq}(x,y,t) = [C_\omega\cos(\omega t) + D_\omega\sin(\omega t)] \times \left[F_p \sin\left(\frac{p\pi x}{L_x}\right)\right]\left[H_q \sin\left(\frac{q\pi y}{L_y}\right)\right]. \tag{60}$$

Lumping the constants together by writing $A_{pq} \equiv C_\omega F_p H_q$ (we can do this since ω is a function of p and q, so C_ω can be written as C_{pq}) and $B_{pq} \equiv D_\omega F_p H_q$, we obtain

$$z_{pq}(x,y,t) = [A_{pq}\cos(\omega_{pq}t) + B_{pq}\sin(\omega_{pq}t)] \times \sin\left(\frac{p\pi x}{L_x}\right)\sin\left(\frac{q\pi y}{L_y}\right). \quad (61)$$

Plots of the spatial part for modes (1, 1), (1, 2), (2, 1), and (2, 2) follow.



The general solution is a sum over all possible values of p and q, so the final solution is

$$z(x,y,t) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}[A_{pq}\cos(\omega_{pq}t) + B_{pq}\sin(\omega_{pq}t)] \times \sin\left(\frac{p\pi x}{L_x}\right)\sin\left(\frac{q\pi y}{L_y}\right), \quad (62)$$

where ω is defined by combining (55) and (59) to yield

$$\omega_{pq} \equiv \pi v\sqrt{\left(\frac{p}{L_x}\right)^2 + \left(\frac{q}{L_y}\right)^2}. \quad (63)$$

Given the initial conditions $z(x,y,0)$ and $\frac{\partial z}{\partial t}(x,y,0)$, we can compute the A_{pq}s and B_{pq}s explicitly. To accomplish this, we make use of the orthogonality of the SINE function in the form

$$I \equiv \int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \tfrac{1}{2}L\delta_{mn}, \quad (64)$$

where δ_{mn} is the KRONECKER DELTA. This can be demonstrated by direct INTEGRATION. Let $u \equiv \pi x/L$ so $du = (\pi/L)\,dx$ in (64), then

$$I = \frac{L}{\pi}\int_0^\pi \sin(mu)\sin(nu)\,du. \quad (65)$$

Now use the trigonometric identity

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \quad (66)$$

to write

$$I = \frac{L}{2\pi}\int_0^\pi \cos[(m-n)u]\,du + \int_0^\pi \cos[(m+n)u]\,du. \quad (67)$$

Note that for an INTEGER $l \neq 0$, the following INTEGRAL vanishes

$$\int_0^\pi \cos(lu)\,du = \frac{1}{l}[\sin(lu)]_0^\pi = \frac{1}{l}[\sin(l\pi) - \sin 0] = \frac{1}{l}\sin(l\pi) = 0, \quad (68)$$

since $\sin(l\pi) = 0$ when l is an INTEGER. Therefore, $I = 0$ when $l \equiv m - n \neq 0$. However, I does *not* vanish when $l = 0$, since

$$\int_0^\pi \cos(0\cdot u)\,du = \int_0^\pi du = \pi. \quad (69)$$

We therefore have that $I = L\delta_{mn}/2$, so we have derived (64). Now we multiply $z(x,y,0)$ by two sine terms and integrate between 0 and L_x and between 0 and L_y,

$$I = \int_0^{L_y}\left[\int_0^{L_x} z(x,y,0)\sin\left(\frac{p\pi x}{L_x}\right)dx\right] \times \sin\left(\frac{q\pi y}{L_y}\right)dy. \quad (70)$$

Now plug in $z(x,y,t)$, set $t = 0$, and prime the indices to distinguish them from the p and q in (70),

$$I = \sum_{q'=1}^{\infty}\int_0^{L_y}\left[\sum_{p'=1}^{\infty}A_{p'q'}\int_0^{L_x}\sin\left(\frac{p\pi x}{L_x}\right)\sin\left(\frac{p'\pi x}{L_x}\right)dx\right] \times \sin\left(\frac{q\pi y}{L_y}\right)\sin\left(\frac{q'\pi y}{L_y}\right)dy. \quad (71)$$

Making use of (64) in (71),

$$I = \sum_{q'=1}^{\infty}\int_0^{L_y}\sum_{p'=1}^{\infty}A_{p'q'}\frac{L_x}{2}\delta_{p,p'} \times \sin\left(\frac{q\pi y}{L_y}\right)\sin\left(\frac{q'\pi y}{L_y}\right)dy, \quad (72)$$

so the sums over p' and q' collapse to a single term

$$I = \frac{L_x}{2}\sum_{q=1}^{\infty}A_{pq'}\frac{L_y}{2}\delta_{q,q'} = \frac{L_xL_y}{4}A_{pq}. \quad (73)$$

Equating (72) and (73) and solving for A_{pq} then gives

$$A_{pq} = \frac{4}{L_xL_y}\int_0^{L_y}\left[\int_0^{L_x} z(x,y,0)\sin\left(\frac{p\pi x}{L_x}\right)dx\right] \times \sin\left(\frac{q\pi y}{L_y}\right)dy. \quad (74)$$

An analogous derivation gives the B_{pq}s as

$$B_{pq} = \frac{4}{\omega_{pq} L_x L_y} \int_0^{L_y} \left[\int_0^{L_x} \frac{\partial z}{\partial t}(x, y, 0) \sin\left(\frac{p\pi x}{L_x}\right) dx \right] \times \sin\left(\frac{q\pi y}{L_y}\right) dy. \quad (75)$$

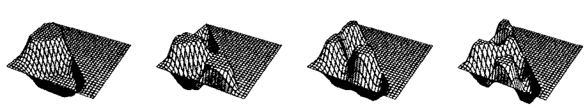
The equation of motion for a membrane shaped as a RIGHT ISOSCELES TRIANGLE of length c on a side and with the sides oriented along the POSITIVE x and y axes is given by

$$\psi(x, y, t) = [C_{pq} \cos(\omega_{pq} t) + D_{pq} \sin(\omega_{pq} t)] \times \left[\sin\left(\frac{p\pi x}{c}\right) \sin\left(\frac{q\pi y}{c}\right) - \sin\left(\frac{q\pi x}{c}\right) \sin\left(\frac{p\pi y}{c}\right) \right], \quad (76)$$

where

$$\omega_{pq} = \frac{\pi v}{c} \sqrt{p^2 + q^2} \quad (77)$$

and p, q INTEGERS with $p > q$. This solution can be obtained by subtracting two wave solutions for a square membrane with the indices reversed. Since points on the diagonal which are equidistant from the center must have the same wave equation solution (by symmetry), this procedure gives a wavefunction which will vanish along the diagonal *as long as p and q are both* EVEN *or* ODD. We must further restrict the modes since those with $p < q$ give wavefunctions which are just the NEGATIVE of (q, p) and (p, p) give an identically zero wavefunction. The following plots show (3, 1), (4, 2), (5, 1), and (5,3).



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Wave Operator

An OPERATOR relating the asymptotic state of a DYNAMICAL SYSTEM governed by the Schrödinger equation

$$i \frac{d}{dt} \psi(t) = H \psi(t)$$

to its original asymptotic state.

see also SCATTERING OPERATOR

Wave Surface

A SURFACE represented parametrically by ELLIPTIC FUNCTIONS.

Wavelet

Wavelets are a class of a functions used to localize a given function in both space and scaling. A family of wavelets can be constructed from a function $\psi(x)$, sometimes known as a "mother wavelet," which is confined in a finite interval. "Daughter wavelets" $\psi^{a,b}(x)$ are then formed by translation (b) and contraction (a). Wavelets are especially useful for compressing image data, since a WAVELET TRANSFORM has properties which are in some ways superior to a conventional FOURIER TRANSFORM.

An individual wavelet can be defined by

$$\psi^{a,b}(x) = |a|^{-1/2} \psi\left(\frac{x - b}{a}\right). \quad (1)$$

Then

$$W_\psi(f)(a, b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \psi\left(\frac{t - b}{a}\right) dt, \quad (2)$$

and CALDERÓN'S FORMULA gives

$$f(x) = C_\psi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle f, \psi^{a,b} \rangle \psi^{a,b}(x) a^{-2} \, da \, db. \quad (3)$$

A common type of wavelet is defined using HAAR FUNCTIONS.

see also FOURIER TRANSFORM, HAAR FUNCTION, LEMARIÉ'S WAVELET, WAVELET TRANSFORM

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Wavelet Matrix

A MATRIX composed of HAAR FUNCTIONS which is used in the WAVELET TRANSFORM. The fourth-order wavelet matrix is given by

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

A wavelet matrix can be computed in $\mathcal{O}(n)$ steps, compared to $\mathcal{O}(n \lg 2)$ for the FOURIER MATRIX.

see also FOURIER MATRIX, WAVELET, WAVELET TRANSFORM

Wavelet Transform

A transform which localizes a function both in space and scaling and has some desirable properties compared to the FOURIER TRANSFORM. The transform is based on a WAVELET MATRIX, which can be computed more quickly than the analogous FOURIER MATRIX.

see also DAUBECHIES WAVELET FILTER, LEMARIE'S WAVELET

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Strang, G. "Wavelet Transforms Versus Fourier Transforms." *Bull. Amer. Math. Soc.* **28**, 288–305, 1993.

Weak Convergence

Weak convergence is usually either denoted $x_n \xrightarrow{w} x$ or $x_n \rightharpoonup x$. A SEQUENCE $\{x_n\}$ of VECTORS in an INNER PRODUCT SPACE E is called weakly convergent to a VECTOR in E if

$$\langle x_n, y\rangle \to \langle x, y\rangle \quad \text{as } n \to \infty, \quad \text{for all } y \in E.$$

Every STRONGLY CONVERGENT sequence is also weakly convergent (but the opposite does not usually hold). This can be seen as follows. Consider the sequence $\{x_n\}$ that converges strongly to x, i.e., $\|x_n - x\| \to 0$ as $n \to \infty$. SCHWARZ'S INEQUALITY now gives

$$|\langle x_n - x, y\rangle| \leq \|x_n - x\|\,\|y\| \quad \text{as } n \to \infty.$$

The definition of weak convergence is therefore satisfied.

see also INNER PRODUCT SPACE, SCHWARZ'S INEQUALITY, STRONG CONVERGENCE

Weak Law of Large Numbers

Also known as BERNOULLI'S THEOREM. Let $x_1, \ldots, x_n$ be a sequence of independent and identically distributed random variables, each having a MEAN $\langle xi\rangle = \mu$ and STANDARD DEVIATION σ. Define a new variable

$$x \equiv \frac{x_1 + \ldots + x_n}{n}. \tag{1}$$

Then, as $n \to \infty$, the sample mean $\langle x\rangle$ equals the population MEAN μ of each variable.

$$\langle x\rangle = \left\langle \frac{x_1 + \ldots + x_n}{n} \right\rangle = \frac{1}{n}(\langle x_1\rangle + \ldots + \langle x_n\rangle) = \frac{n\mu}{n} = \mu \tag{2}$$

$$\begin{aligned} \operatorname{var}(x) &= \operatorname{var}\left(\frac{x_1 + \ldots + x_2}{n}\right) \\ &= \operatorname{var}\left(\frac{x_1}{n}\right) + \ldots + \operatorname{var}\left(\frac{x_n}{n}\right) \\ &= \frac{\sigma^2}{n^2} + \ldots + \frac{\sigma^2}{n^2} = \frac{\sigma^2}{n}. \end{aligned} \tag{3}$$

Therefore, by the CHEBYSHEV INEQUALITY, for all $\epsilon > 0$,

$$P(|x - \mu| \geq \epsilon) \leq \frac{\operatorname{var}(x)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}. \tag{4}$$

As $n \to \infty$, it then follows that

$$\lim_{n\to\infty} P(|x-\mu| \geq \epsilon) = 0 \tag{5}$$

for ϵ arbitrarily small; i.e., as $n \to \infty$, the sample MEAN is the same as the population MEAN.

Stated another way, if an event occurs x times in s TRIALS and if p is the probability of success in a single TRIAL, then the probability that the relative frequency of successes is x/s differs from p by less than any arbitrary POSITIVE quantity ϵ which approaches 1 as $s \to \infty$.

see also LAW OF TRULY LARGE NUMBERS, STRONG LAW OF LARGE NUMBERS

Weakly Binary Tree

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

A ROOTED TREE for which the ROOT is adjacent to at most two VERTICES, and all nonroot VERTICES are adjacent to at most three VERTICES. Let $b(n)$ be the number of weakly binary trees of order n, then $b(5) = 6$. Let

$$g(z) = \sum_{i=0}^{\infty} g_i z^i, \tag{1}$$

where

$$g_0 = 0 \tag{2}$$

$$g_1 = g_2 = g_3 = 1 \tag{3}$$

$$g_{2i+1} = \sum_{j=1}^{i} g_{2i+1-j} g_j \tag{4}$$

$$g_{2i} = \tfrac{1}{2} g_i(g_i+1) + \sum_{j=1}^{i-1} g_{2i-j} g_j. \tag{5}$$

Otter (Otter 1948, Harary and Palmer 1973, Knuth 1969) showed that

$$\lim_{n\to\infty} \frac{b(n)n^{3/2}}{\xi^n} = \eta, \tag{6}$$

where

$$\xi = 2.48325\ldots \tag{7}$$

is the unique POSITIVE ROOT of

$$g\left(\frac{1}{x}\right) = 1,$$

and

$$\eta = 0.7916032\ldots. \tag{8}$$

ξ is also given by

$$\xi = \lim_{n\to\infty} (c_n)^{2^{-n}}, \tag{9}$$

where c_n is given by

$$c_0 = 2 \tag{10}$$

$$c_n = (c_{n-1})^2 + 2, \tag{11}$$

giving

$$\eta = \frac{1}{2}\sqrt{\frac{\xi}{\pi}}\sqrt{3 + \frac{1}{c_1} + \frac{1}{c_1c_2} + \frac{1}{c_1c_2c_3} + \ldots}. \tag{12}$$

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Otter, R. "The Number of Trees." *Ann. Math.* **49**, 583–599, 1948.

Weakly Complete Sequence

A SEQUENCE of numbers $V = \{\nu_n\}$ is said to be weakly complete if every POSITIVE INTEGER n beyond a certain point N is the sum of some SUBSEQUENCE of V (Honsberger 1985). Dropping two terms from the FIBONACCI NUMBERS produces a SEQUENCE which is not even weakly complete. However, the SEQUENCE

$$F_n' \equiv F_n - (-1)^n$$

is weakly complete, even with any finite subsequence deleted (Graham 1964).

see also COMPLETE SEQUENCE

References

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Honsberger, R. *Mathematical Gems III.* Washington, DC: Math. Assoc. Amer., p. 128, 1985.

Weakly Independent

An infinite sequence $\{a_i\}$ of POSITIVE INTEGERS is called weakly independent if any relation $\sum \epsilon_i a_i$ with $\epsilon_i = 0$ or ± 1 and $\epsilon_i = 0$, except finitely often, IMPLIES $\epsilon_i = 0$ for all i.

see also STRONGLY INDEPENDENT

References

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Weakly Triple-Free Set

see TRIPLE-FREE SET

Web Graph

A graph formed by connecting several concentric WHEEL GRAPHS along spokes.

see also WHEEL GRAPH

Weber Differential Equations

Consider the differential equation satisfied by

$$w = z^{-1/2} W_{k,-1/4}(\tfrac{1}{2}z^2), \tag{1}$$

where W is a WHITTAKER FUNCTION.

$$\frac{d}{z\,dz}\left[\frac{d(wz^{1/2})}{z\,dz}\right] + \left(-\frac{1}{4} + \frac{2k}{z^2} + \frac{3}{4z^4}\right) wz^{1/2} = 0 \tag{2}$$

$$\frac{d^2w}{dz^2} + (2k - \tfrac{1}{4}z^2)w = 0. \tag{3}$$

This is usually rewritten

$$\frac{d^2 D_n(z)}{dz^2} + (n + \tfrac{1}{2} - \tfrac{1}{4}z^2)D_n(z) = 0. \tag{4}$$

The solutions are PARABOLIC CYLINDER FUNCTIONS.

The equations

$$\frac{d^2U}{du^2} - (c + k^2u^2)U = 0 \tag{5}$$

$$\frac{d^2V}{dv^2} + (c - k^2v^2)V = 0, \tag{6}$$

which arise by separating variables in LAPLACE'S EQUATION in PARABOLIC CYLINDRICAL COORDINATES, are also known as the Weber differential equations. As above, the solutions are known as PARABOLIC CYLINDER FUNCTIONS.

Weber's Discontinuous Integrals

$$\int_0^\infty J_0(ax)\cos(cx)\,dx = \begin{cases} 0 & a < c \\ \frac{1}{\sqrt{a^2-c^2}} & a > c \end{cases}$$

$$\int_0^\infty J_0(ax)\sin(cx)\,dx = \begin{cases} \frac{1}{\sqrt{c^2-a^2}} & a < c \\ 0 & a > c, \end{cases}$$

where $J_0(z)$ is a zeroth order BESSEL FUNCTION OF THE FIRST KIND.

References

Bowman, F. *Introduction to Bessel Functions.* New York: Dover, pp. 59–60, 1958.

Weber's Formula

$$\frac{1}{2p^2} e^{-(a^2+b^2)/(4p^2)} I_\nu\left(\frac{ab}{2p^2}\right) = \int_0^\infty e^{-p^2t^2} J_\nu(at) J_\nu(bt)t\,dt,$$

where $\Re[\nu] > -1$, $|\arg p| < \pi/4$, and a, $b > 0$, $J_\nu(z)$ is a BESSEL FUNCTION OF THE FIRST KIND, and $I_\nu(z)$ is a MODIFIED BESSEL FUNCTION OF THE FIRST KIND.

see also BESSEL FUNCTION OF THE FIRST KIND, MODIFIED BESSEL FUNCTION OF THE FIRST KIND

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1476, 1980.

Weber Functions

Although BESSEL FUNCTIONS OF THE SECOND KIND are sometimes called Weber functions, Abramowitz and Stegun (1972) define a separate Weber function as

$$\mathcal{E}_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin(\nu\theta - z\sin\theta)\,d\theta. \tag{1}$$

Letting $\zeta_n = e^{2\pi i/m}$ be a ROOT OF UNITY, another set of Weber functions is defined as

$$f(z) = \frac{\eta(\frac{1}{2}(z+1))}{\zeta_{48}\eta(z)} \tag{2}$$

$$f_1(z) = \frac{\eta(\frac{1}{2}z)}{\eta(z)} \tag{3}$$

$$f_2(z) = \sqrt{2}\frac{\eta(2z)}{\eta(z)} \tag{4}$$

$$\gamma_2 = \frac{f^{24}(z) - 16}{f^8(z)} \tag{5}$$

$$\gamma_3 = \frac{[f^{24}(z) + 8][f_1{}^8(z) - f_2{}^8(z)]}{f^8(z)} \tag{6}$$

(Weber 1902, Atkin and Morain 1993), where $\eta(z)$ is the DEDEKIND ETA FUNCTION. The Weber functions satisfy the identities

$$f(z+1) = \frac{f_1(z)}{\zeta_{48}} \tag{7}$$

$$f_1(z+1) = \frac{f(z)}{\zeta_{48}} \tag{8}$$

$$f_2(z+1) = \zeta_{24} f_2(z) \tag{9}$$

$$f\left(-\frac{1}{z}\right) = f(z) \tag{10}$$

$$f_1\left(-\frac{1}{z}\right) = f_2(z) \tag{11}$$

$$f_2\left(-\frac{1}{z}\right) = f_1(z) \tag{12}$$

(Weber 1902, Atkin and Morain 1993).

see also ANGER FUNCTION, BESSEL FUNCTION OF THE SECOND KIND, DEDEKIND ETA FUNCTION, j-FUNCTION, JACOBI IDENTITIES, JACOBI TRIPLE PRODUCT, MODIFIED STRUVE FUNCTION, Q-FUNCTION, STRUVE FUNCTION

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Anger and Weber Functions." §12.3 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 498–499, 1972.

Atkin, A. O. L. and Morain, F. "Elliptic Curves and Primality Proving." *Math. Comput.* **61**, 29–68, 1993.

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Weber, H. *Lehrbuch der Algebra, Vols. I–II.* New York: Chelsea, pp. 113–114, 1902.

Weber-Sonine Formula

For $\Re[\mu+nu]>0$, $|\arg p|<\pi/4$, and $a>0$,

$$\int_0^\infty J_\nu(at)e^{-p^2t^2}t^{\mu-1}\,dt$$
$$\left(\frac{a}{2p}\right)^\nu \frac{\Gamma[\frac{1}{2}(\nu+\mu)]}{2p^\mu\Gamma(\nu+1)}\,{}_1F_1\left(\tfrac{1}{2}(\nu+\mu);\nu+1;-\frac{a^2}{2p^2}\right),$$

where $J_\nu(z)$ is a BESSEL FUNCTION OF THE FIRST KIND, $\Gamma(z)$ is the GAMMA FUNCTION, and ${}_1F_1(a;b;z)$ is a CONFLUENT HYPERGEOMETRIC FUNCTION.

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1474, 1980.

Weber's Theorem

If two curves of the same GENUS (CURVE) >1 are in rational correspondence, then that correspondence is BIRATIONAL.

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 135, 1959.

Wedderburn's Theorem

A FINITE DIVISION RING is a FIELD.

Weddle's Rule

$$\int_{x_1}^{x_{6n}} f(x)\,dx = \tfrac{3}{10}h(f_1+5f_2+f_3$$
$$+6f_4+5f_5+f_6+\ldots+5f_{6n-1}+f_{6n})$$

see also BODE'S RULE, HARDY'S RULE, NEWTON-COTES FORMULAS, SIMPSON'S 3/8 RULE, SIMPSON'S RULE, TRAPEZOIDAL RULE, WEDDLE'S RULE

Wedge

A right triangular PRISM turned so that it rests on one of its lateral faces.

see also CONICAL WEDGE, CYLINDRICAL WEDGE, PRISM

Wedge Product

An antisymmetric operation on DIFFERENTIAL FORMS (also called the EXTERIOR DERIVATIVE)

$$dx_i \wedge dx_j = -dx_j \wedge dx_i, \tag{1}$$

which IMPLIES

$$dx_i \wedge dx_i = 0 \tag{2}$$
$$b_i \wedge dx_j = dx_j \wedge b_i = b_i\,dx_j \tag{3}$$
$$dx_i \wedge (b_i\,dx_j) = b_i\,dx_i \wedge dx_j \tag{4}$$
$$\begin{aligned}\theta_1\wedge\theta_2 &= (b_1\,dx_1+b_2\,dx_2)\wedge(c_1\,dx_1+c_2\,dx_2)\\ &= (b_1c_2-b_2c_1)\,dx_1\wedge dx_2\\ &= -\theta_2\wedge\theta_1.\end{aligned} \tag{5}$$

The wedge product is ASSOCIATIVE

$$(s\wedge t)\wedge u = s\wedge(t\wedge u), \tag{6}$$

and BILINEAR

$$(\alpha_1s_1+\alpha_2s_2)\wedge t = \alpha_1(s_1\wedge t)+\alpha_2(s_2\wedge t) \tag{7}$$
$$s\wedge(\alpha_1t_1+\alpha_2t_2) = \alpha_1(s\wedge t_1)+\alpha_2(s\wedge t_2), \tag{8}$$

but not (in general) COMMUTATIVE

$$s\wedge t = (-1)^{pq}(t\wedge s), \tag{9}$$

where s is a p-form and t is a q-form. For a 0-form s and 1-form t,

$$(s\wedge t)_\mu = st_\mu. \tag{10}$$

For a 1-form s and 1-form t,

$$(s\wedge t)_{\mu\nu} = \tfrac{1}{2}(s_\mu t_\nu - s_\nu t_\mu). \tag{11}$$

The wedge product is the "correct" type of product to use in computing a VOLUME ELEMENT

$$dV = dx_1\wedge\ldots\wedge dx_n. \tag{12}$$

see also DIFFERENTIAL FORM, EXTERIOR DERIVATIVE, INNER PRODUCT, VOLUME ELEMENT

Weekday

The day of the week W for a given day of the month D, month M, and year $100C+Y$ can be determined from the simple equation

$$W \equiv D+\lfloor 2.6M-0.2\rfloor+\left\lfloor\tfrac{1}{4}Y\right\rfloor+\left\lfloor\tfrac{1}{4}C\right\rfloor-2C \pmod 7,$$

where *months are numbered beginning with March* and $W=0$ for Sunday, $W=1$ for Monday, etc. (Uspensky and Heaslet 1939, Vardi 1991).

A more complicated form is given by

$$W \equiv D+M+C+Y \pmod 7,$$

where $W=1$ for Sunday, $W=2$ for Monday, etc. and the numbers assigned to months, centuries, and years are given in the tables below (Kraitchik 1942, pp. 110–111).

Month	M
January	1
February	4
March	3
April	6
May	1
June	4
July	6
August	2
September	5
October	0
November	3
December	5

Gregorian Century	C
15, 19, 23	1
16, 20, 24	0
17, 21, 25	5
18, 22, 26	3

Julian Century	C
00, 07, 14	5
01, 08, 15	4
02, 09, 16	3
03, 10, 17	2
04, 11, 18	1
05, 12, 19	0
06, 13, 20	6

Year									Y
00	06		17	23	28	34		45	0
01	07	12	18		29	35	40	46	1
02		13	19	24	30		41	47	2
03	08	14		25	31	36	42		3
	09	15	20	26		37	43	48	4
04	10		21	27	32	38		49	5
05	11	16	22		33	39	44	50	6
51	56	62		73	79	84	90		0
	57	63	68	74		85	91	96	1
52	58		69	75	80	86		97	2
53	59	64	70		81	87	92	98	3
54		65	71	76	82		93	99	4
55	60	66		77	83	88	94		5
	61	67	72	78		89	95		6

see also FRIDAY THE THIRTEENTH

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Weibull Distribution

The Weibull distribution is given by

$$P(x) = \alpha\beta^{-\alpha}x^{\alpha-1}e^{-(x/\beta)^\alpha} \quad (1)$$

$$D(x) = 1 - e^{-(x/\beta)^\alpha} \quad (2)$$

for $x \in [0, \infty)$ (*Mathematica*® `Statistics'Continuous Distributions'WeibullDistribution[a,b]`, Wolfram Research, Champaign, IL). The MEAN, VARIANCE, SKEWNESS, and KURTOSIS of this distribution are

$$\mu = \beta\Gamma(1+\alpha^{-1}) \quad (3)$$

$$\sigma^2 = \beta^2[\Gamma(1+2\alpha^{-1}) - \Gamma^2(1+\alpha^{-1})] \quad (4)$$

$$\gamma_1 = \frac{2\Gamma^3(1+\alpha^{-1}) - 3\Gamma(1+\alpha^{-1})\Gamma(1+2\alpha^{-1})}{[\Gamma(1+2\alpha^{-1}) - \Gamma^2(1+\alpha^{-1})]^{3/2}} + \frac{\Gamma(1+3\alpha^{-1})}{[\Gamma(1+2\alpha^{-1}) - \Gamma^2(1+\alpha^{-1})]^{3/2}} \quad (5)$$

$$\gamma_2 = \frac{f(a)}{[\Gamma(1+2\alpha^{-1}) - \Gamma^2(1+\alpha^{-1})]^2}, \quad (6)$$

where $\Gamma(z)$ is the GAMMA FUNCTION and

$$\begin{aligned} f(a) \equiv {} & -6\Gamma^4(1+\alpha^{-1}) + 12\Gamma^2(1+\alpha^{-1})\Gamma(1+2\alpha^{-1}) \\ & -3\Gamma^2(1+2\alpha^{-1}) - 4\Gamma(1+\alpha^{-1})\Gamma(1+3\alpha^{-1}) \\ & +\Gamma(1+4\alpha^{-1}). \end{aligned} \quad (7)$$

A slightly different form of the distribution is

$$P(x) = \frac{\alpha}{\beta}x^{\alpha-1}e^{-x^\alpha/\beta} \quad (8)$$

$$D(x) = 1 - e^{-x^\alpha/\beta} \quad (9)$$

(Mendenhall and Sincich 1995). The MEAN and VARIANCE for this form are

$$\mu = \beta^{1/\alpha}\Gamma(1+\alpha^{-1}) \quad (10)$$

$$\sigma^2 = \beta^{2/\alpha}[\Gamma(1+2\alpha^{-1}) - \Gamma^2(1+\alpha^{-1})]. \quad (11)$$

The Weibull distribution gives the distribution of lifetimes of objects. It was originally proposed to quantify fatigue data, but it is also used in analysis of systems involving a "weakest link."

see also FISHER-TIPPETT DISTRIBUTION

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Weierstraß Approximation Theorem

If f is continuous on $[a, b]$, then there exists a POLYNOMIAL p on $[a, b]$ such that

$$|f(x) - P(x)| < \epsilon$$

for all $x \in [a, b]$ and $\epsilon > 0$. In words, any continuous function on a closed and bounded interval can be uniformly approximated on that interval by POLYNOMIALS to any degree of accuracy.

see also MÜNTZ'S THEOREM

Weierstraß-Casorati Theorem

An ANALYTIC FUNCTION approaches any given value arbitrarily closely in any ϵ-NEIGHBORHOOD of an ESSENTIAL SINGULARITY.

Weierstraß Constant

$$\sigma(\tfrac{1}{2}) = \tfrac{1}{2} \prod_{\substack{(m,n)\neq\\(0,0)}} \left[1 - \frac{1}{2(m+ni)}\right] \times e^{1/[2(m+ni)]+1/[8(m+ni)^2]} = \frac{2^{5/4}\sqrt{\pi}\, e^{\pi/8}}{\Gamma^2(\frac{1}{4})} = 0.4749493799\ldots.$$

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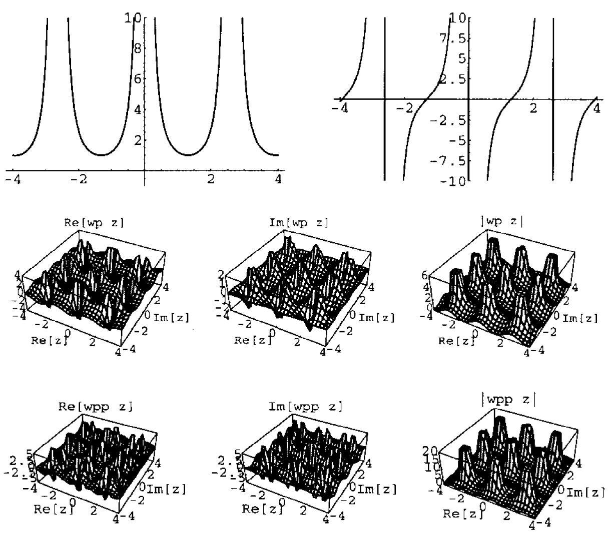
Weierstraß Elliptic Function



The Weierstraß elliptic functions are elliptic functions which, unlike the JACOBI ELLIPTIC FUNCTIONS, have a second-order POLE at $z = 0$. The above plots show the Weierstraß elliptic function $\wp(z)$ and its derivative $\wp'(z)$ for invariants (defined below) of $g_2 = 0$ and $g_3 = 0$. Weierstraß elliptic functions are denoted $\wp(z)$ and can be defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n=-\infty}^{\infty}{}' \left[\frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2}\right]. \quad (1)$$

Write $\Omega_{mn} \equiv 2m\omega_1 + 2n\omega_2$. Then this can be written

$$\wp(z) = z^{-2} + \sum_{m,n}{}' [(z - \Omega_{mn})^{-2} - \Omega_{mn}^{-2}]. \quad (2)$$

An equivalent definition which converges more rapidly is

$$\wp(z) = \left(\frac{\pi}{2\omega_1}\right)^2 \left[-\frac{1}{3} + \sum_{-\infty}^{\infty} \csc^2\left(\frac{z - 2n\omega_2}{2\omega_1}\pi\right) - \sum_{n=-\infty}^{\infty}{}' \csc^2\left(\frac{n\omega_2}{n_1}\pi\right)\right]. \quad (3)$$

$\wp(z)$ is an EVEN FUNCTION since $\wp(-z)$ gives the same terms in a different order. To specify $\wp$ completely, its periods or invariants, written $\wp(z|\omega_1,\omega_2)$ and $\wp(z;g_2,g_3)$, respectively, must also be specified.

The differential equation from which Weierstraß elliptic functions arise can be found by expanding about the origin the function $f(z) \equiv \wp(z) - z^{-2}$.

$$\wp(z) - z^{-2} = f(0) + f'(0)z + \tfrac{1}{2!}f''(0)z^2 + \tfrac{1}{3!}f'''(0)z^3 + \tfrac{1}{4}f^{(4)}(0)z^4 + \ldots. \quad (4)$$

But $f(0) = 0$ and the function is even, so $f'(0) = f'''(0) = 0$ and

$$f(z) = \wp(z) - z^{-2} = \tfrac{1}{2!}f''(0)z^2 + \tfrac{1}{4}f^{(4)}(0)z^4 + \ldots. \quad (5)$$

Taking the derivatives

$$f' = -2\Sigma'[(z - \Omega_{mn})^{-3}] \quad (6)$$
$$f'' = 6\Sigma'(z - \Omega_{mn})^{-4} \quad (7)$$
$$f''' = -24\Sigma'(z - \Omega_{mn})^{-5} \quad (8)$$
$$f^{(4)} = 120\Sigma'(z - \Omega_{mn})^{-6}. \quad (9)$$

So

$$f''(0) = 6\Sigma'\Omega_{mn}^{-4} \quad (10)$$
$$f^{(4)}(0) = 120\Sigma'\Omega_{mn}^{-6}. \quad (11)$$

Plugging in,

$$\wp(z) - z^{-2} = 3\Sigma'\Omega_{mn}^{-4}z^2 + 5\Sigma'\Omega_{mn}^{-6}z^4 + \mathcal{O}(z^6). \quad (12)$$

Define the INVARIANTS

$$g_2 \equiv 60\Sigma'\Omega_{mn}^{-4} \quad (13)$$
$$g_3 \equiv 140\Sigma'\Omega_{mn}^{-6}, \quad (14)$$

then

$$\wp(z) = z^{-2} + \tfrac{1}{20}g_2z^2 + \tfrac{1}{28}g_3z^4 + \mathcal{O}(z^6) \quad (15)$$

$$\wp'(z) = -2z^{-3} + \tfrac{1}{10}g_2z + \tfrac{1}{7}g_3z^3 + \mathcal{O}(z^5). \quad (16)$$

Now cube (15) and square (16)

$$\wp^3(z) = z^{-6} + \tfrac{3}{20}g_2z^{-2} + \tfrac{3}{28}g_3 + \mathcal{O}(z^2) \quad (17)$$

$$\wp'^2(z) = 4z^{-6} - \tfrac{2}{5}g_2 z^{-2} - \tfrac{4}{7}g_3 + \mathcal{O}(z^2). \qquad (18)$$

Taking (18) $- 4 \times$ (17) cancels out the z^{-6} term, giving

$$\begin{aligned}\wp'^2(z) &- 4\wp^3(z)\\ &= \left(-\tfrac{2}{5} - \tfrac{3}{5}\right) g_2 z^{-2} + \left(-\tfrac{4}{7} - \tfrac{3}{7}\right) g_3 + \mathcal{O}(z^2)\\ &= -g_2 z^{-2} - g_3 + \mathcal{O}(z^2) \qquad (19)\end{aligned}$$

$$\wp'^2(z) - 4\wp^3(z) + g_2 z^{-2} + g_3 = \mathcal{O}(z^2). \qquad (20)$$

But, from (5)

$$\wp(z) = z^{-2} + \tfrac{1}{2!}f''(0)z^2 + \tfrac{1}{4}f^{(4)}(0)z^4 + \ldots, \qquad (21)$$

so $\wp(z) = z^{-2} + \mathcal{O}(z^2)$ and (20) can be written

$$\wp'^2(z) - 4\wp^3(z) + g_2\wp(z) + g_3 = \mathcal{O}(z^2). \qquad (22)$$

The Weierstraß elliptic function is analytic at the origin and therefore at all points congruent to the origin. There are no other places where a singularity can occur, so this function is an ELLIPTIC FUNCTION with no SINGULARITIES. By LIOUVILLE'S ELLIPTIC FUNCTION THEOREM, it is therefore a constant. But as $z \to 0$, $\mathcal{O}(z^2) \to 0$, so

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3. \qquad (23)$$

The solution to the differential equation

$$y'^2 = 4y^3 - g_2 y - g_3 \qquad (24)$$

is therefore given by $y = \wp(z + \alpha)$, providing that numbers ω_1 and ω_2 exist which satisfy the equations defining the INVARIANTS. Writing the differential equation in terms of its roots e_1, e_2, and e_3,

$$y'^2 = 4y^3 - g_2 y - g_3 = 4(y - e_1)(y - e_2)(y - e_3) \qquad (25)$$

$$2\ln(y') = \ln 4 + \sum_{r=1}^{3} \ln(y - e_r) \qquad (26)$$

$$\frac{2y''}{y'} = y' \sum_{r=1}^{3} (y - e_r)^{-1} \qquad (27)$$

$$\frac{2y''}{y'^2} = \sum_{r=1}^{3} (y - e_r)^{-1} \qquad (28)$$

$$2\frac{y'^2 y''' - y''(2y'y'')}{y'^4} = -y' \sum_{r=1}^{3} (y - e_r)^{-2} \qquad (29)$$

$$\frac{2y'''}{y'^3} - \frac{4y''^2}{y'^4} = -\sum_{r=1}^{3} (y - e_r)^{-2}. \qquad (30)$$

Now take $(30)/4 + [(30)/4]^2$,

$$\begin{aligned}&\left[\frac{y'''}{2y'^3} - \frac{y''^2}{y'^4}\right] + \left[\frac{y''^2}{4y'^4}\right]\\ &\quad = -\frac{1}{4}\sum_{r=1}^{3}(y - e_r)^{-2} + \frac{1}{16}\left[\sum_{r=1}^{3}(y - e_r)^{-1}\right]^2 \qquad (31)\end{aligned}$$

$$\frac{3y''^2}{4y'^4} - \frac{y'''}{2y'^3} = \tfrac{3}{16}\sum_{r=1}^{3}(y - e_r)^{-2} - \tfrac{3}{8}y\prod_{r=1}^{3}(y - e_r)^{-1}. \qquad (32)$$

The term on the right is half the SCHWARZIAN DERIVATIVE.

The DERIVATIVE of the Weierstraß elliptic function is given by

$$\begin{aligned}\wp'(z) = \frac{d}{dz}\wp(z) &= -2\sum_{m,n}\frac{1}{(z - \Omega_{mn})^3}\\ &= -2z^{-3} - 2{\sum_{m,n}}'(z - \Omega_{mn})^{-3}. \qquad (33)\end{aligned}$$

This is an ODD FUNCTION which is itself an elliptic function with pole of order 3 at $z = 0$. The INTEGRAL is given by

$$z = \int_{\wp(z)}^{\infty} (4t^3 - g_2 t - g_3)^{-1/2}\, dt. \qquad (34)$$

A duplication formula is obtained as follows.

$$\begin{aligned}\wp(2z) &= \lim_{y\to z}\wp(y + z) = \frac{1}{4}\lim_{y\to z}\left[\frac{\wp'(z) - \wp'(y)}{\wp(z) - \wp(y)}\right]^2\\ &\quad - \wp(z) - \lim_{y\to z}\wp(y)\\ &= \frac{1}{4}\lim_{h\to 0}\left[\frac{\wp(z) - \wp'(z+h)}{\wp(z) - \wp(z+h)}\right]^2 - 2\wp(z)\\ &= \frac{1}{4}\left\{\left[\lim_{h\to 0}\frac{\wp'(z) - \wp'(z+h)}{h}\right]\left[\lim_{h\to 0}\frac{h}{\wp(z) - \wp(z+h)}\right]\right\}^2\\ &\quad - 2\wp(z)\\ &= \frac{1}{4}\left[\frac{\wp''(z)}{\wp'(z)}\right]^2 - 2\wp(z). \qquad (35)\end{aligned}$$

A general addition theorem is obtained as follows. Given

$$\wp'(z) = A\wp(z) + B \qquad (36)$$

$$\wp'(y) = A\wp(y) + B \qquad (37)$$

with zero y and z where $z \not\equiv \pm y \pmod{2\omega_1, 2\omega_2}$, find the third zero ζ. Consider $\wp'(\zeta) - A\wp(\zeta) - B$. This has a pole of order three at $\zeta = 0$, but the sum of zeros ($= 0$) equals the sum of poles for an ELLIPTIC FUNCTION, so $z + y + \zeta = 0$ and $\zeta = -z - y$.

$$\wp'(-z - y) = A\wp(-z - y) + B \qquad (38)$$

$$-\wp'(z+y) = A\wp(z+y) + B. \quad (39)$$

Combining (36), (37), and (39) gives

$$\begin{bmatrix} \wp(z) & \wp'(z) & 1 \\ \wp(y) & \wp'(y) & 1 \\ \wp(z+y) & -\wp(z+y) & 1 \end{bmatrix} \begin{bmatrix} A \\ -1 \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (40)$$

so

$$\begin{vmatrix} \wp(z) & \wp'(z) & 1 \\ \wp(y) & \wp'(y) & 1 \\ \wp(z+y) & -\wp(z+y) & 1 \end{vmatrix} = 0. \quad (41)$$

Defining $u+v+w=0$ where $u \equiv z$ and $v \equiv y$ gives the symmetric form

$$\begin{vmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp(w) & 1 \end{vmatrix} = 0. \quad (42)$$

To get the expression explicitly, start again with

$$\wp'(\zeta) - A\wp(\zeta) - B = 0, \quad (43)$$

where $\zeta = z, y, -z-y$.

$$\wp'^2(\zeta) - [A\wp(\zeta) + B]^2 = 0. \quad (44)$$

But $\wp^2(\zeta) = 4\wp^4(\zeta) - g_2\wp(\zeta) - g_3$, so

$$4\wp^3(\zeta) - A^2\wp^2(\zeta) - (2AB+g_2)\wp(\zeta) - (B^2+g_3) = 0. \quad (45)$$

The solutions $\wp(\zeta) \equiv z$ are given by

$$4z^3 - A^2z^2 - (2AB+g_2)z - (B^2+g_3) = 0. \quad (46)$$

But the sum of roots equals the COEFFICIENT of the squared term, so

$$\wp(z) + \wp(y) + \wp(z+y) = \tfrac{1}{4}A^2 \quad (47)$$

$$\wp'(z) - \wp'(y) = A[\wp(z) - \wp(y)] \quad (48)$$

$$A = \frac{\wp'(z) - \wp'(y)}{\wp(z) - \wp(y)} \quad (49)$$

$$\wp(z+y) = \frac{1}{4}\left[\frac{\wp(z) - \wp'(y)}{\wp(z) - \wp(y)}\right]^2 - \wp(z) - \wp(y). \quad (50)$$

Half-period identities include

$$x \equiv \wp(\tfrac{1}{2}\omega_1) = \wp(-h\omega_1 + \omega_1) = e_1 + \frac{(e_1-e_2)(e_1-e_3)}{\wp(-\frac{1}{2}\omega_1) - e_1}$$

$$= e_1 + \frac{(e_1-e_2)(e_1-e_3)}{x - e_1}. \quad (51)$$

Multiplying through,

$$x^2 - e_1x = e_1x - e_1{}^2 + (e_1-e_2)(e_1-e_3) \quad (52)$$

$$x^2 - 2e_1 + [e_1{}^2 - (e_1-e_2)(e_1-e_3)] = 0, \quad (53)$$

which gives

$$\wp(\tfrac{1}{2}\omega_1) = \tfrac{1}{2}\left\{2e_1 \pm \sqrt{4e_1{}^2 - 4[e_1{}^2 - (e_1-e_2)(e_1-e_3)]}\right\}$$

$$= e_1 \pm \sqrt{(e_1-e_2)(e_1-e_3)}. \quad (54)$$

From Whittaker and Watson (1990, p. 445),

$$\wp'(\tfrac{1}{2}\omega_1) = -2\sqrt{(e_1-e_2)(e_1-e_3)} \times (\sqrt{e_1-e_2} + \sqrt{e_1-e_3}). \quad (55)$$

The function is HOMOGENEOUS,

$$\wp(\lambda z|\lambda\omega_1, \lambda\omega_2) = \lambda^{-2}\wp(z|\omega_1, \omega_2) \quad (56)$$

$$\wp(\lambda z; \lambda^{-4}g_2, \lambda^{-6}g_3) = \lambda^{-2}\wp(z; g_2, g_3). \quad (57)$$

To invert the function, find $2\omega_1$ and $2\omega_2$ of $\wp(z|\omega_1, \omega_2)$ when given $\wp(z; g_2, g_3)$. Let e_1, e_2, and e_3 be the roots such that $(e_1-e_2)/(e_1-e_3)$ is not a REAL NUMBER > 1 or < 0. Determine the PARAMETER τ from

$$\frac{e_1-e_2}{e_1-e_3} = \frac{\vartheta_4{}^4(0|\tau)}{\vartheta_3{}^4(0|\tau)}. \quad (58)$$

Now pick

$$A \equiv \frac{\sqrt{e_1-e_2}}{\vartheta_4{}^2(0|\tau)}. \quad (59)$$

As long as $g_2{}^3 \neq 27g_3$, the periods are then

$$2\omega_1 = \pi A \quad (60)$$

$$2\omega_2 = \frac{\pi\tau}{A}. \quad (61)$$

Weierstraß elliptic functions can be expressed in terms of JACOBI ELLIPTIC FUNCTIONS by

$$\wp(u; g_2, g_3) = e_3 + (e_1-e_3) \times \mathrm{ns}^2\left(u\sqrt{e_1-e_3}, \sqrt{\frac{e_2-e_3}{e_1-e_3}}\right), \quad (62)$$

where

$$\wp(\omega_1) = e_1 \quad (63)$$

$$\wp(\omega_2) = e_2 \quad (64)$$

$$\wp(\omega_3) = -\wp(-\omega_1 - \omega_2) = e_3, \quad (65)$$

and the INVARIANTS are

$$g_2 \equiv 60\Sigma'\Omega_{mn}^{-4} \quad (66)$$

$$g_3 \equiv 140\Sigma'\Omega_{mn}^{-6}. \quad (67)$$

Here, $\Omega_{mn} \equiv 2m\omega_1 - 2n\omega_2$.

An addition formula for the Weierstraß elliptic function can be derived as follows.

$$\wp(z+\omega_1)+\wp(z)+\wp(\omega_1) = \frac{1}{4}\left[\frac{\wp'(z)-\wp'(\omega_1)}{\wp(z)-\wp(\omega_1)}\right]^2 = \frac{1}{4}\frac{\wp'^2(z)}{[\wp(z)-e_1]^2}. \tag{68}$$

Use

$$\wp'(z) = 4\prod_{r=1}^{3}[\wp(z)-e_r], \tag{69}$$

so

$$\begin{aligned}\wp(z+\omega_1) &= -\wp(z)-e_1+\frac{1}{4}\frac{4\prod_{r=1}^{3}[\wp(z)-e_r]}{[\wp(z)-e_1]^2}\\ &= -\wp(z)-e_1+\frac{[\wp(z)-e_2][\wp(z)-e_3]}{\wp(z)-e_1}.\end{aligned} \tag{70}$$

Use $\sum_{r=1}^{3} e_r = 0$,

$$\begin{aligned}\wp(z+\omega_1) &= e_1+\frac{[-2e_1-\wp(z)][\wp(z)-e_1]}{\wp(z)-e_1}\\ &\quad+\frac{\wp^2(z)-\wp(z)(e_2+e_3)+e_2e_3}{\wp(z)-e_1}\\ &= e_1+\frac{-\wp(z)(e_1+e_2+e_3)+e_2e_3+2{e_1}^2}{\wp(z)-e_1}.\end{aligned} \tag{71}$$

But $\sum_{r=1}^{3} e_r = 0$ and

$$2{e_1}^2+e_2e_3 = {e_1}^2-e_1(e_2+e_3)+e_2e_3 = (e_1-e_2)(e_1-e_3), \tag{72}$$

so

$$\wp(z+\omega_1) = e_1+\frac{(e_1-e_2)(e_1-e_3)}{\wp(z)-e_1}. \tag{73}$$

The periods of the Weierstraß elliptic function are given as follows. When g_2 and g_3 are REAL and ${g_2}^3-27{g_3}^2>0$, then e_1, e_2, and e_3 are REAL and defined such that $e_1>e_2>e_3$.

$$\omega_1 = \int_{e_1}^{\infty}(4t^3-g_2t-g_3)^{-1/2}\,dt \tag{74}$$

$$\omega_3 = -i\int_{-\infty}^{e_3}(g_3+g_2t-4t^3)^{-1/2}\,dt \tag{75}$$

$$\omega_2 = -\omega_1-\omega_3. \tag{76}$$

The roots of the Weierstraß elliptic function satisfy

$$e_1 = \wp(\omega_1) \tag{77}$$

$$e_2 = \wp(\omega_2) \tag{78}$$

$$e_3 = \wp(\omega_3), \tag{79}$$

where $\omega_3 \equiv -\omega_1-\omega_2$. The e_is are ROOTS of $4t^3-g_2t-g_3$ and are unequal so that $e_1 \neq e_2 \neq e_3$. They can be found from the relationships

$$e_1+e_2+e_3 = -a_2 = 0 \tag{80}$$

$$e_2e_3+e_3e_1+e_1e_2 = a_1 = -\tfrac{1}{4}g_2 \tag{81}$$

$$e_1e_2e_3 = -a_0 = \tfrac{1}{4}g_3. \tag{82}$$

see also EQUIANHARMONIC CASE, LEMNISCATE CASE, PSEUDOLEMNISCATE CASE

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Weierstraß-Erdman Corner Condition

In the CALCULUS OF VARIATIONS, the condition

$$f_{y'}(x,y,y'(x_-)) = f_{y'}(x,y,y'(x_+))$$

must hold at a corner (x,y) of a minimizing arc E_{12}.

Weierstraß Extreme Value Theorem

see EXTREME VALUE THEOREM

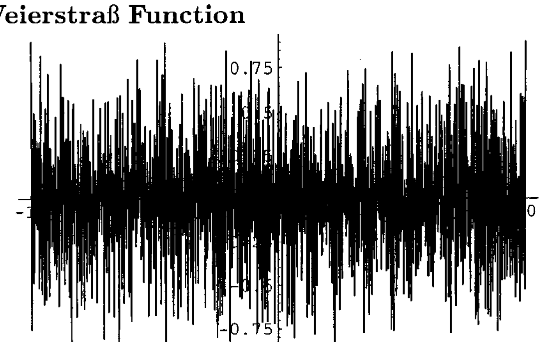
Weierstraß Form

A general form into which an ELLIPTIC CURVE over any FIELD K can be transformed is called the Weierstraß form, and is given by

$$y^2+ay = x^3+bx^2+cxy+dx+e,$$

where a, b, c, d, and e are elements of K.

Weierstraß Function



A CONTINUOUS FUNCTION which is nowhere DIFFERENTIABLE. It is given by

$$f(x) = \sum_{n=1}^{\infty} b^n \cos(a^n \pi x)$$

where n is an ODD INTEGER, $b \in (0,1)$, and $ab > 1 + 3\pi/2$. The above plot is for $a = 10$ and $b = 1/2$.

see also BLANCMANGE FUNCTION, CONTINUOUS FUNCTION, DIFFERENTIABLE

References

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du Bois-Reymond, P. "Versuch einer Klassification der willkürlichen Functionen reeller Argumente nach ihren Änderungen in den kleinsten Intervallen." *J. für Math.* **79**, 21–37, 1875.

Faber, G. "Einfaches Beispiel einer stetigen nirgends differentiierbaren Funktion." *Jahresber. Deutschen Math. Verein.* **16** 538–540, 1907.

Hardy, G. H. "Weierstrass's Non-Differentiable Function." *Trans. Amer. Math. Soc.* **17**, 301–325, 1916.

Landsberg, G. "Über Differentzierbarkeit stetiger Funktionen." *Jahresber. Deutschen Math. Verein.* **17**, 46–51, 1908.

Lerch, M. "Über die Nichtdifferentiirbarkeit gewisser Functionen." *J. reine angew. Math.* **13**, 126–138, 1888.

Pickover, C. A. *Keys to Infinity.* New York: W. H. Freeman, p. 190, 1995.

Weierstraß, K. *Abhandlungen aus der Functionenlehre.* Berlin: J. Springer, p. 97, 1886.

Weierstraß's Gap Theorem

Given a succession of nonsingular points which are on a nonhyperelliptic curve of GENUS p, but are not a group of the canonical series, the number of groups of the first k which cannot constitute the group of simple POLES of a RATIONAL FUNCTION is p. If points next to each other are taken, then the theorem becomes: Given a nonsingular point of a nonhyperelliptic curve of GENUS p, then the orders which it cannot possess as the single pole of a RATIONAL FUNCTION are p in number.

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 290, 1959.

Weierstraß Intermediate Value Theorem

If a continuous function defined on an interval is sometimes POSITIVE and sometimes NEGATIVE, it must be 0 at some point.

Weierstraß M-Test

Let $\sum_{k=1}^{\infty} u_n(x)$ be a SERIES of functions all defined for a set E of values of x. If there is a CONVERGENT series of constants

$$\sum_{n=1}^{\infty} M_n,$$

such that

$$|u_n(x)| \leq M_n$$

for all $x \in E$, then the series exhibits ABSOLUTE CONVERGENCE for each $x \in E$ as well as UNIFORM CONVERGENCE in E.

see also ABSOLUTE CONVERGENCE, UNIFORM CONVERGENCE

References

Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 301–303, 1985.

Weierstraß Point

A POLE of multiplicity less than $p + 1$.

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, pp. 290–291, 1959.

Weierstraß's Polynomial Theorem

A function, continuous in a finite close interval, can be approximated with a preassigned accuracy by POLYNOMIALS. A function of a REAL variable which is continuous and has period 2π can be approximated by trigonometric POLYNOMIALS.

References

Szegő, G. *Orthogonal Polynomials, 4th ed.* Providence, RI: Amer. Math. Soc., p. 5, 1975.

Weierstraß Product Inequality

If $0 \leq a, b, c, d \leq 1$, then

$$(1-a)(1-b)(1-c)(1-d) + a + b + c + d \geq 1.$$

References

Honsberger, R. *Mathematical Gems III.* Washington, DC: Math. Assoc. Amer., pp. 244–245, 1985.

Weierstraß Sigma Function

The QUASIPERIODIC FUNCTION defined by

$$\frac{d}{dz} \ln \sigma(z) = \zeta(z), \tag{1}$$

where $\zeta(z)$ is the WEIERSTRAß ZETA FUNCTION and

$$\lim_{z \to 0} \frac{\sigma(z)}{z} = 1. \tag{2}$$

Then

$$\sigma(z) = z \prod_{mn}{}' \left[\left(1 - \frac{z}{\Omega_{mn}}\right) \exp\left(\frac{z}{\Omega_{mn}} + \frac{z^2}{2\Omega_{mn}^2}\right) \right] \tag{3}$$

$$\sigma(z + 2\omega_1) = -e^{2\eta_1(z+\omega_1)}\sigma(z) \tag{4}$$

$$\sigma(z + 2\omega_2) = -e^{2\eta_2(z+\omega_2)}\sigma(z) \tag{5}$$

$$\sigma_r(z) = \frac{e^{-\eta_r z}\sigma(z+\omega_r)}{\sigma(\omega_r)} \tag{6}$$

for $r = 1, 2, 3$.

$$\sigma(z|\omega_1,\omega_2) = \frac{2\omega_1}{\pi\vartheta_1'} \exp\left(-\frac{\nu^2\vartheta_1'''}{6\vartheta_1'}\right)\vartheta_1\left(\nu\left|\frac{\omega_2}{\omega_1}\right.\right), \tag{7}$$

where $\nu \equiv \pi z/(2\omega_1)$, and

$$\eta_1 = -\frac{\pi^2\vartheta_1'''}{12\omega_1\vartheta_1'} \tag{8}$$

$$\eta_2 = -\frac{\pi^2\omega_2\vartheta_1'''}{12\omega_1{}^2\vartheta_1'} - \frac{\pi i}{2\omega_1}. \tag{9}$$

References
Abramowitz, M. and Stegun, C. A. (Eds.). "Weierstrass Elliptic and Related Functions." Ch. 18 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 627–671, 1972.

Weierstraß's Theorem

The only hypercomplex number systems with commutative multiplication and addition are the algebra with one unit such that $e = e^2$ and the GAUSSIAN INTEGERS.

see also GAUSSIAN INTEGER, PEIRCE'S THEOREM

Weierstraß Zeta Function

The QUASIPERIODIC FUNCTION defined by

$$\frac{d\zeta(z)}{dz} \equiv -\wp(z) \tag{1}$$

with

$$\lim_{z\to 0}[\zeta(z) - z^{-1}] = 0. \tag{2}$$

Then

$$\begin{aligned}\zeta(z) - z^{-1} &= -\int_0^z [\wp(z) - z^{-2}]\,dz \\ &= -\Sigma' \int_0^z [(z-\Omega_{mn})^{-2} - \Omega_{mn}^{-2}]\,dz\end{aligned} \tag{3}$$

$$\zeta(z) = z^{-1} + \sum_{m,n=-\infty}^{\infty}{}' [(z-\Omega_{mn})^{-1} + \Omega_{mn}^{-1} + z\Omega_{mn}^{-2}] \tag{4}$$

so $\zeta(z)$ is an ODD FUNCTION. Integrating $\wp(z+2\omega_1) = \wp(z)$ gives

$$\zeta(z+2\omega_1) = \zeta(z) + 2\eta_1. \tag{5}$$

Letting $z = -\omega_1$ gives $\zeta(-\omega_1) + 2\eta_1 = -\zeta(\omega_1) + 2\eta_1$, so $\eta_1 = \zeta(\omega_1)$. Similarly, $\eta_2 = \zeta(\omega_2)$. From Whittaker and Watson (1990),

$$\eta_1\omega_2 - \eta_2\omega_1 = \tfrac{1}{2}\pi i. \tag{6}$$

If $x + y + z = 0$, then

$$[\zeta(x) + \zeta(y) + \zeta(z)]^2 + \zeta'(x) + \zeta'(y)\zeta'(z) = 0. \tag{7}$$

Also,

$$2\frac{\begin{vmatrix} 1 & \wp(x) & \wp^2(x) \\ 1 & \wp(y) & \wp^2(y) \\ 1 & \wp(z) & \wp^2(z)\end{vmatrix}}{\begin{vmatrix} 1 & \wp(x) & \wp'(x) \\ 1 & \wp(y) & \wp'(y) \\ 1 & \wp(z) & \wp'(z)\end{vmatrix}} = \zeta(x+y+z) - \zeta(x) - \zeta(y) - \zeta(z) \tag{8}$$

(Whittaker and Watson 1990, p. 446).

References
Abramowitz, M. and Stegun, C. A. (Eds.). "Weierstrass Elliptic and Related Functions." Ch. 18 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 627–671, 1972.
Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis, 4th ed.* Cambridge, England: Cambridge University Press, 1990.

Weighings

n weighings are SUFFICIENT to find a bad COIN among $(3^n - 1)/2$ COINS. vos Savant (1993) gives an algorithm for finding a bad ball among 12 balls in three weighings (which, in addition, determines if the bad ball is heavier or lighter than the other 11).

Bachet's weights problem asks for the *minimum number* of weights (which can be placed in *either* pan of a two-arm balance) required to weigh any integral number of pounds from 1 to 40. The solution is 1, 3, 9, and 27: 1, $2 = -1 + 3$, 3, $4 = 1 + 3$, $5 = -1 - 3 + 9$, $6 = -3 + 9$, $7 = 1 - 3 + 9$, $8 = -1 + 9$, 9, $10 = 1 + 9$, $11 = -1 + 3 + 9$, $12 = 3 + 9$, $13 = 1 + 3 + 9$, $14 = -1 - 3 - 9 + 27$, $15 = -3 - 9 + 27$, $16 = 1 - 3 - 9 + 27$, $17 = -1 - 9 + 27$, and so on.

see also GOLOMB RULER, PERFECT DIFFERENCE SET, THREE JUG PROBLEM

References
Bachet, C. G. Problem 5, Appendix in *Problèmes plaisans et délectables, 2nd ed.* p. 215, 1624.
Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 50–52, 1987.
Kraitchik, M. *Mathematical Recreations.* New York: W. W. Norton, pp. 52–55, 1942.
Pappas, T. "Counterfeit Coin Puzzle." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 181, 1989.
Tartaglia. Book 1, Ch. 16, §32 in *Trattato de' numeri e misure, Vol. 2.* Venice, 1556.
vos Savant, M. *The World's Most Famous Math Problem.* New York: St. Martin's Press, pp. 39–42, 1993.

Weight

The word weight has many uses in mathematics. It can refer to a function $w(x)$ (also called a WEIGHTING FUNCTION or WEIGHT FUNCTION) used to normalize ORTHONORMAL FUNCTIONS. It can also be used to indicate one of a set of a multiplicative constants placed in front of terms in a MOVING AVERAGE, NEWTON-COTES FORMULAS, edge or vertex of a GRAPH or TREE, etc.

see also WEIGHTED TREE, WEIGHTING FUNCTION

Weight Function

see WEIGHTING FUNCTION

Weighted Tree

A TREE in which each branch is given a numerical WEIGHT (i.e., a labelled TREE).

see also LABELLED GRAPH, TAYLOR'S CONDITION, TREE

Weighting Function

A function $w(x)$ used to normalize ORTHONORMAL FUNCTIONS

$$\int [f_n(x)]^2\, w(x)\, dx = N_n.$$

see also WEIGHT

Weingarten Equations

The Weingarten equations express the derivatives of the NORMAL using derivatives of the position vector. Let $\mathbf{x}: U \to \mathbb{R}^3$ a REGULAR PATCH, then the SHAPE OPERATOR S of $\mathbf{x}$ is given in terms of the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ by

$$-S(\mathbf{x}_u) = \mathbf{N}_u = \frac{fF - eG}{EG - F^2}\mathbf{x}_u + \frac{eF - fE}{EG - F^2}\mathbf{x}_v \quad (1)$$

$$-S(\mathbf{x}_v) = \mathbf{N}_v = \frac{gF - fG}{EG - F^2}\mathbf{x}_u + \frac{fF - gE}{EG - F^2}\mathbf{x}_v, \quad (2)$$

where $\mathbf{N}$ is the NORMAL VECTOR, E, F, and G the coefficients of the first FUNDAMENTAL FORM

$$ds^2 = E\, du^2 + 2F\, du\, dv + G\, dv^2, \quad (3)$$

and e, f, and g the coefficients of the second FUNDAMENTAL FORM given by

$$e = -\mathbf{N}_u \cdot \mathbf{x}_u = \mathbf{N} \cdot \mathbf{x}_{uu} \quad (4)$$

$$f = -\mathbf{N}_v \cdot \mathbf{x}_u = \mathbf{N} \cdot \mathbf{x}_{uv}$$
$$= \mathbf{N}_{vu} \cdot \mathbf{x}_{vu} = -\mathbf{N}_u \cdot \mathbf{x}_v \quad (5)$$

$$g = -\mathbf{N}_v \cdot \mathbf{x}_v = \mathbf{N} \cdot \mathbf{x}_{vv}. \quad (6)$$

see also FUNDAMENTAL FORMS, SHAPE OPERATOR

References
Gray, A. "Calculation of the Shape Operator." §14.3 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 274–277, 1993.

Weingarten Map

see SHAPE OPERATOR

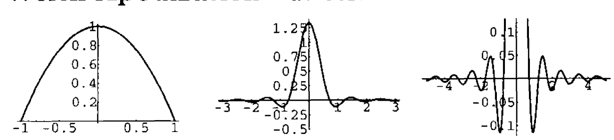
Weird Number

A number which is ABUNDANT without being SEMIPERFECT. (A SEMIPERFECT NUMBER is the sum of any set of its own DIVISORS.) The first few weird numbers are 70, 836, 4030, 5830, 7192, 7912, 9272, 10430, ... (Sloane's A006037). No ODD weird numbers are known, but an infinite number of weird numbers are known to exist. The SEQUENCE of weird numbers has POSITIVE SCHNIRELMANN DENSITY.

see also ABUNDANT NUMBER, SCHNIRELMANN DENSITY, SEMIPERFECT NUMBER

References
Benkoski, S. "Are All Weird Numbers Even?" *Amer. Math. Monthly* **79**, 774, 1972.
Benkoski, S. J. and Erdős, P. "On Weird and Pseudoperfect Numbers." *Math. Comput.* **28**, 617–623, 1974.
Guy, R. K. "Almost Perfect, Quasi-Perfect, Pseudoperfect, Harmonic, Weird, Multiperfect and Hyperperfect Numbers." §B2 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 45–53, 1994.
Sloane, N. J. A. Sequence A006037/M5339 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Welch Apodization Function



The APODIZATION FUNCTION

$$A(x) = 1 - \frac{x^2}{a^2}.$$

Its FULL WIDTH AT HALF MAXIMUM is $\sqrt{2}\,a$. Its INSTRUMENT FUNCTION is

$$I(k) = a2\sqrt{2\pi}\,\frac{J_{3/2}(2\pi ka)}{(2\pi ka)^{3/2}}$$
$$= a\frac{\sin(2\pi ka) - 2\pi ak\cos(2\pi ak)}{2a^3k^3\pi^3},$$

where $J_\nu(z)$ is a BESSEL FUNCTION OF THE FIRST KIND. It has a width of 1.59044, a maximum of $\frac{4}{3}$, maximum NEGATIVE sidelobe of -0.0861713 times the peak, and maximum POSITIVE sidelobe of 0.356044 times the peak.

see also APODIZATION FUNCTION, INSTRUMENT FUNCTION

References
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, p. 547, 1992.

Well-Defined

An expression is called well-defined (or UNAMBIGUOUS) if its definition assigns it a unique interpretation or value. Otherwise, the expression is said to not be well defined or to be AMBIGUOUS.

For example, the expression abc (the PRODUCT) is well-defined if a, b, and c are integers. Because integers are ASSOCIATIVE, abc has the same value whether it is interpreted to mean $(ab)c$ or $a(bc)$. However, if a, b, and c are MATRICES or CAYLEY NUMBERS, then the expression abc is *not* well-defined, since MATRICES and CAYLEY NUMBER are not, in general, ASSOCIATIVE, so that the two interpretations $(ab)c$ and $a(bc)$ can be different.

Sometimes, ambiguities are implicitly resolved by notational convention. For example, the conventional interpretation of $a \wedge b \wedge c = a^{b^c}$ is $a^{(b^c)}$, never $(a^b)^c$, so that the expression $a \wedge b \wedge c$ is well-defined even though exponentiation is nonassociative.

Well-Ordered Set

A SET having the property that every nonempty SUBSET has a least member.

see also AXIOM OF CHOICE, HILBERT'S PROBLEMS, SUBSET, WELL-ORDERING PRINCIPLE

Well-Ordering Principle

Every nonempty set of POSITIVE integers contains a smallest member.

see also WELL-ORDERED SET

References

Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, p. 149, 1993.

Werner Formulas

$$2\sin\alpha\cos\beta = \sin(\alpha-\beta)+\sin(\alpha+\beta) \quad (1)$$

$$2\cos\alpha\cos\beta = \cos(\alpha-\beta)+\cos(\alpha+\beta) \quad (2)$$

$$2\cos\alpha\sin\beta = \sin(\alpha+\beta)-\sin(\alpha-\beta) \quad (3)$$

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta)-\cos(\alpha+\beta). \quad (4)$$

see also TRIGONOMETRY

Weyl's Criterion

A SEQUENCE $\{x_1, x_2, \ldots\}$ is EQUIDISTRIBUTED IFF

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} e^{2\pi i m x_n} = 0$$

for each $m = 1, 2, \ldots$.

see also EQUIDISTRIBUTED SEQUENCE, RAMANUJAN'S SUM

References

Pólya, G. and Szegő, G. *Problems and Theorems in Analysis I.* New York: Springer-Verlag, 1972.

Vardi, I. *Computational Recreations in Mathematica.* Redwood City, CA: Addison-Wesley, pp. 155–156 and 254, 1991.

Weyl Tensor

The TENSOR

$$C^{ij}{}_{kl} = R^i j_{kl} - 2\delta^{[i}{}_{[k}R^{j]}{}_{l]} + \tfrac{1}{3}\delta^{[i}{}_{[k}\delta^{j]}{}_{l]}R,$$

where $R^i j_{kl}$ is the RIEMANN TENSOR and R is the CURVATURE SCALAR. The Weyl tensor is defined so that every CONTRACTION between indices gives 0. In particular, $C^\lambda{}_{\mu\lambda\kappa} = 0$. The number of independent components for a Weyl tensor in N-D is given by

$$C_N = \tfrac{1}{12}N(N+1)(N+2)(N-3).$$

see also CURVATURE SCALAR, RIEMANN TENSOR

References

Weinberg, S. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* New York: Wiley, p. 146, 1972.

Weyrich's Formula

Under appropriate constraints,

$$\frac{1}{2}i\int_{-\infty}^{\infty} H_0^{(1)}(r\sqrt{k^2-r^2})e^{i\tau x}\,d\tau = \frac{e^{ik\sqrt{r^2+k^2}}}{\sqrt{r^2+x^2}},$$

where $H_0^{(1)}(z)$ is a HANKEL FUNCTION OF THE FIRST KIND.

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1474, 1980.

Wheat and Chessboard Problem

Let one grain of wheat be placed on the first square of a CHESSBOARD, two on the second, three on the third, etc. How many grains total are placed on an 8×8 CHESSBOARD? Since this is a GEOMETRIC SERIES, the answer for n squares is

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1.$$

Plugging in $n = 8 \times 8 = 64$ then gives $2^{64} - 1 = 18446744073709551615$.

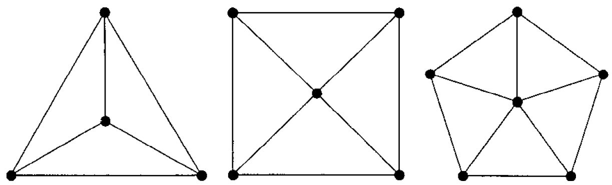
References

Pappas, T. "The Wheat and & Chessboard." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 17, 1989.

Wheel

see ARISTOTLE'S WHEEL PARADOX, BENHAM'S WHEEL, WHEEL GRAPH

Wheel Graph



W_4 W_5 W_6

A GRAPH W_n of order n which contains a CYCLE of order $n-1$, and for which every NODE in the cycle is connected to one other NODE (known as the HUB). In a wheel graph, the HUB has DEGREE $n-1$, and other nodes have degree 3. $W_4 = K_4$, where K_4 is the COMPLETE GRAPH of order four.

see also COMPLETE GRAPH, GEAR GRAPH, HUB, WEB GRAPH

Wheel Paradox

see ARISTOTLE'S WHEEL PARADOX

Whewell Equation

An INTRINSIC EQUATION which expresses a curve in terms of its ARC LENGTH s and TANGENTIAL ANGLE ϕ.

see also ARC LENGTH, CESÀRO EQUATION, INTRINSIC EQUATION, NATURAL EQUATION, TANGENTIAL ANGLE

References

Yates, R. C. "Intrinsic Equations." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 123–126, 1952.

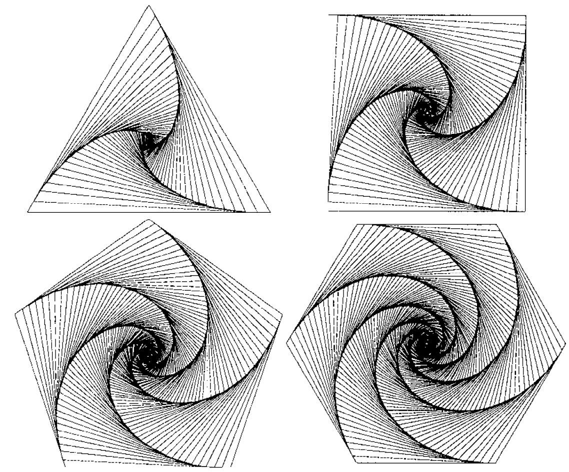
Whipple's Transformation

$$
{}_7F_6\left[\begin{matrix} a, 1+\tfrac{1}{2}a, b, c, d, e, -m \\ \tfrac{1}{2}a, 1+a-b, 1+a-c, \\ 1+a-d, 1+a-e, 1+a+m \end{matrix}\right]
= \frac{(1+a)_m(1+a-d-e)_m}{(1+a-d)_m(1+a-e)_m}
\times {}_4F_3\left[\begin{matrix} 1+a-b-c, d, e, -m \\ 1+a-b, 1+a-c, d+e-a-m \end{matrix}\right],
$$

where ${}_7F_6$ and ${}_4F_3$ are GENERALIZED HYPERGEOMETRIC FUNCTIONS and $\Gamma(z)$ is the GAMMA FUNCTION.

see also GENERALIZED HYPERGEOMETRIC FUNCTION

Whirl



Whirls are figures constructed by nesting a sequence of polygons (each having the same number of sides), each slightly smaller and rotated relative to the previous one.

see also DAISY, SWIRL

References

Lauwerier, H. *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, p. 66, 1991.

Pappas, T. "Spider & Spirals." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 228, 1989.

Weisstein, E. W. "Fractals." `http://www.astro.virginia.edu/~eww6n/math/notebooks/Fractal.m`.

Whisker Plot

see BOX-AND-WHISKER PLOT

Whitehead Double

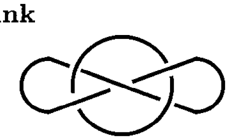
The SATELLITE KNOT of an UNKNOT twisted inside a TORUS.

see also SATELLITE KNOT, TORUS, UNKNOT

References

Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, pp. 115–116, 1994.

Whitehead Link



The LINK 5^{02}_{01}, illustrated above, with BRAID WORD ${\sigma_1}^2{\sigma_2}^2{\sigma_1}^{-1}{\sigma_2}^{-2}$ and JONES POLYNOMIAL

$$V(t) = t^{-3/2}(-1+t-2t^2+t^3-2t^4+t^5).$$

The Whitehead link has LINKING NUMBER 0.

Whitehead Manifold

An open 3-MANIFOLD which is simply connected but is topologically distinct from Euclidean 3-space.

References

Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, p. 82, 1976.

Whitehead's Theorem

MAPS between CW-COMPLEXES that induce ISOMORPHISMS on all HOMOTOPY GROUPS are actually HOMOTOPY equivalences.

see also CW-COMPLEX, HOMOTOPY GROUP, ISOMORPHISM

Whitney-Graustein Theorem

A 1937 theorem which classified planar regular closed curves up to regular HOMOTOPY by their WINDING NUMBERS. In his thesis, S. Smale generalized this result to regular closed curves on an n-MANIFOLD.

Whitney-Mikhlin Extension Constants

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let $B_n(r)$ be the n-D closed BALL of RADIUS $r > 1$ centered at the ORIGIN. A function which is defined on $B(r)$ is called an extension to $B(r)$ of a function f defined on $B(1)$ if

$$F(x) = f(x) \forall\, x \in B(1). \tag{1}$$

Given 2 BANACH SPACES of functions defined on $B(1)$ and $B(r)$, find the extension operator from one to the other of minimal norm. Mikhlin (1986) found the best constants χ such that this condition, corresponding to the Sobolev $W(1,2)$ integral norm, is satisfied,

$$\sqrt{\int_{B(1)}\left[[f(x)]^2+\sum_{j=1}^n\left(\frac{\partial f}{\partial x_j}\right)^2\right]dx} \leq \chi\sqrt{\int_{B(r)}\left[[F(x)]^2+\sum_{j=1}^n\left(\frac{\partial F}{\partial x_j}\right)^2\right]dx}. \tag{2}$$

$\chi(1,r) = 1$. Let

$$\nu = \tfrac{1}{2}(n-2), \tag{3}$$

then for $n > 2$,

$$\chi(n,r) = \sqrt{1+\frac{I_\nu(1)}{I_{\nu+1}(1)}\frac{I_\nu(r)K_{\nu+1}(1)+K_\nu(r)I_{\nu+1}(1)}{I_\nu(r)K_\nu(1)-K_\nu(r)I_\nu(1)}}, \tag{4}$$

where $I_\nu(z)$ is a MODIFIED BESSEL FUNCTION OF THE FIRST KIND and $K_\nu(z)$ is a MODIFIED BESSEL FUNCTION OF THE SECOND KIND. For $n = 2$,

$$\chi(2,r) = \max \left\{\sqrt{1+\frac{I_\nu(1)}{I_{\nu+1}(1)}\frac{I_\nu(r)K_{\nu+1}(1)+K_\nu(r)I_{\nu+1}(1)}{I_\nu(r)K_\nu(1)-K_\nu(r)I_\nu(1)}}\right. \left.\sqrt{1+\frac{I_1(1)}{I_1(1)+I_2(1)}\left[1+\frac{I_1(r)K_0(1)+K_1(r)I_0(1)}{I_1(r)K_1(1)-K_1(r)I_1(1)}\right]}\right\}. \tag{5}$$

For $r \to \infty$,

$$\chi(n,\infty) = \sqrt{1+\frac{I_\nu(1)}{I_{\nu+1}(1)}\frac{K_\nu(1)}{K_\nu(1)}}, \tag{6}$$

which is bounded by

$$n-1 < \chi(n,\infty) < \sqrt{(n-1)^2+4}. \tag{7}$$

For ODD n, the RECURRENCE RELATIONS

$$a_{k+1} = a_{k-1} - (2k-1)a_k \tag{8}$$

$$b_{k+1} = b_{k-1} + (2k-1)b_k \tag{9}$$

with

$$a_0 = e + e^{-1} \tag{10}$$

$$a_1 = e - e^{-1} \tag{11}$$

$$b_0 = e^{-1} \tag{12}$$

$$b_1 = e^{-1} \tag{13}$$

where e is the constant 2.71828..., give

$$\chi(2k+1,\infty) = \sqrt{1+\frac{a_k}{a_{k+1}}\frac{b_{k+1}}{b_k}}. \tag{14}$$

The first few are

$$\chi(3,\infty) = e \tag{15}$$

$$\chi(5,\infty) = \sqrt{\frac{e^2}{e^2-7}} \tag{16}$$

$$\chi(7,\infty) = \sqrt{\frac{2}{7}}\sqrt{\frac{e^2}{37-5e^2}} \tag{17}$$

$$\chi(9,\infty) = \frac{1}{\sqrt{37}}\sqrt{\frac{e^2}{18e^2-133}} \tag{18}$$

$$\chi(11,\infty) = \frac{1}{\sqrt{133}}\sqrt{\frac{e^2}{2431-329e^2}} \tag{19}$$

$$\chi(13,\infty) = \sqrt{\frac{2}{2431}}\sqrt{\frac{e^2}{3655e^2-27007}}. \tag{20}$$

Similar formulas can be given for even n in terms of $I_0(1)$, $I_1(1)$, $K_0(1)$, $K_1(1)$.

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Whitney Singularity

see PINCH POINT

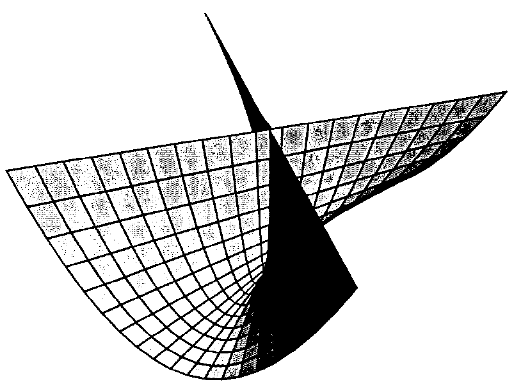
Whitney Sum

An operation that takes two VECTOR BUNDLES over a fixed SPACE and produces a new VECTOR BUNDLE over the same SPACE. If E_1 and E_2 are VECTOR BUNDLES over B, then the Whitney sum $E_1 \oplus E_2$ is the VECTOR BUNDLE over B such that each FIBER over B is naturally the direct sum of the E_1 and E_2 FIBERS over B.

The Whitney sum is therefore the FIBER for FIBER direct sum of the two BUNDLES E_1 and E_2. An easy formal definition of the Whitney sum is that $E_1 \oplus E_2$ is the pull-back BUNDLE of the diagonal map from B to $B \times B$, where the BUNDLE over $B \times B$ is $E_1 \times E_2$.

see also BUNDLE, FIBER, VECTOR BUNDLE

Whitney Umbrella



A surface which can be interpreted as a self-intersecting RECTANGLE in 3-D. It is given by the parametric equations

$$x = uv \tag{1}$$
$$y = u \tag{2}$$
$$z = v^2 \tag{3}$$

for $u, v \in [-1, 1]$. The center of the "plus" shape which is the end of the line of self-intersection is a PINCH POINT. The coefficients of the first FUNDAMENTAL FORM are

$$E = 0 \tag{4}$$
$$F = \frac{2v}{\sqrt{u^2 + 4v^2 + 4v^4}} \tag{5}$$
$$G = -\frac{2u}{\sqrt{u^2 + 4v^2 + 4v^4}}, \tag{6}$$

and the coefficients of the second FUNDAMENTAL FORM are

$$e = 1 + v^2 \tag{7}$$
$$f = uv \tag{8}$$
$$g = u^2 + 4v^2, \tag{9}$$

giving GAUSSIAN CURVATURE and MEAN CURVATURE

$$K = -\frac{4v^2}{(u^2 + 4v^2 + 4v^4)^2} \tag{10}$$
$$H = -\frac{u(1 + 3v^2)}{(u^2 + 4v^2 + 4v^4)^{3/2}}. \tag{11}$$

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Whittaker Differential Equation

$$\frac{d^2u}{dz^2} + \frac{du}{dz} + \left(\frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right) u = 0. \tag{1}$$

Let $u \equiv e^{-z/2} W_{k,m}(z)$, where $W_{k,m}(z)$ denotes a WHITTAKER FUNCTION. Then (1) becomes

$$\frac{d}{dz}(-\tfrac{1}{2}e^{-z/2}W + e^{-z/2}W') + (-\tfrac{1}{2}e^{-z/2}W + e^{-z/2}W') + \left(\frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right) e^{-z/2}W = 0. \tag{2}$$

Rearranging,

$$(\tfrac{1}{4}e^{-z/2}W - \tfrac{1}{2}e^{-z/2}W' - \tfrac{1}{2}e^{-z/2}W' + e^{-z/2}W'')p + (-\tfrac{1}{2}e^{-z/2}W + e^{-z/2}W') + \left(\frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right) e^{-z/2}W = 0 \tag{3}$$

$$-\tfrac{1}{4}e^{-z/2}W + e^{-z/2}W'' + \left(\frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right) e^{-z/2}W = 0, \tag{4}$$

so

$$W'' + \left(-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right) W = 0, \tag{5}$$

where $W' \equiv dW/dz$. The solutions are known as WHITTAKER FUNCTIONS.

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Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 505, 1972.

Whittaker Function

Solutions to the WHITTAKER DIFFERENTIAL EQUATION. The linearly independent solutions are

$$M_{k,m}(z) \equiv z^{1/2+m}e^{-z/2} \times \left(1 + \frac{\frac{1}{2}+m-k}{1!(2m+1)} + \frac{(\frac{1}{2}+m-k)(\frac{3}{2}+m-k)}{2!(2m+1)(2m+2)}z^2 + \ldots\right), \quad (1)$$

and $M_{k,-m}(z)$, where $M_{k,m}(z)$ is a CONFLUENT HYPERGEOMETRIC FUNCTION. In terms of CONFLUENT HYPERGEOMETRIC FUNCTIONS, the Whittaker functions are

$$M_{k,m}(z) = e^{-z/2}z^{m+1/2}{}_1F_1(\tfrac{1}{2}+m-k, 1+2m; z) \quad (2)$$

$$W_{k,m}(z) = e^{-z/2}z^{m+1/2}U(\tfrac{1}{2}+m-k, 1+2m; z) \quad (3)$$

(see Whittaker and Watson 1990, pp. 339–351). However, the CONFLUENT HYPERGEOMETRIC FUNCTION disappears when $2m$ is an INTEGER, so Whittaker functions are often defined instead. The Whittaker functions are related to the PARABOLIC CYLINDER FUNCTIONS. When $|\arg z| < 3\pi/2$ and $2m$ is not an INTEGER,

$$W_{k,m}(z) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)}M_{k,m}(z) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}M_{k,-m}(z). \quad (4)$$

When $|\arg(-z)| < 3\pi/2$ and $2m$ is not an INTEGER,

$$W_{-k,m}(-z) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)}M_{-k,m}(-z) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m+k)}M_{-k,-m}(-z). \quad (5)$$

Whittaker functions satisfy the RECURRENCE RELATIONS

$$W_{k,m}(z) = z^{1/2}W_{k-1/2,m-1/2}(z) + (\tfrac{1}{2}-k+m)W_{k-1,m}(z) \quad (6)$$

$$W_{k,m}(z) = z^{1/2}W_{k-1/2,m+1/2}(z) + (\tfrac{1}{2}-k-m)W_{k-1,m}(z) \quad (7)$$

$$zW'_{k,m}(z) = (k-\tfrac{1}{2}z)W_{k,m}(z) - [m^2-(k-\tfrac{1}{2})^2]W_{k-1,m}(z). \quad (8)$$

see also CONFLUENT HYPERGEOMETRIC FUNCTION, KUMMER'S FORMULAS, PEARSON-CUNNINGHAM FUNCTION, SCHLÖMILCH'S FUNCTION, SONINE POLYNOMIAL

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Whole Number

One of the numbers 1, 2, 3, ... (Sloane's A000027), also called the COUNTING NUMBERS or NATURAL NUMBERS. 0 is sometimes included in the list of "whole" numbers (Bourbaki 1968, Halmos 1974), but there seems to be no general agreement. Some authors also interpret "whole number" to mean "a number having FRACTIONAL PART of zero," making the whole numbers equivalent to the integers.

Due to lack of standard terminology, the following terms are recommended in preference to "COUNTING NUMBER," "NATURAL NUMBER," and "whole number."

Set	Name	Symbol
..., −2, −1, 0, 1, 2, ...	integers	$\mathbb{Z}$
1, 2, 3, 4, ...	positive integers	$\mathbb{Z}^+$
0, 1, 2, 3, 4 ...	nonnegative integers	$\mathbb{Z}^*$
−1, −2, −3, −4, ...	negative integers	$\mathbb{Z}^-$

see also COUNTING NUMBER, FRACTIONAL PART, INTEGER, $\mathbb{N}$, NATURAL NUMBER, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^+$, $\mathbb{Z}^*$

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Width (Partial Order)

For a PARTIAL ORDER, the size of the longest ANTICHAIN is called the width.

see also ANTICHAIN, LENGTH (PARTIAL ORDER), PARTIAL ORDER

Width (Size)

The width of a box is the horizontal distance from side to side (usually defined to be greater than the DEPTH, the horizontal distance from front to back).

see also DEPTH (SIZE), HEIGHT

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Wiedersehen Manifold

The only Wiedersehen manifolds are the standard round spheres, as was established by proof of the BLASCHKE CONJECTURE.

see also BLASCHKE CONJECTURE

Wieferich Prime

A Wieferich prime is a PRIME p which is a solution to the CONGRUENCE equation

$$2^{p-1} \equiv 1 \pmod{p^2}.$$

Note the similarity of this expression to the special case of FERMAT'S LITTLE THEOREM

$$2^{p-1} \equiv 1 \pmod{p},$$

which holds for *all* ODD PRIMES. However, the only Wieferich primes less than 4×10^{12} are $p = 1093$ and 3511 (Lehmer 1981, Crandall 1986, Crandall *et al.* 1997). Interestingly, one less than these numbers have suggestive periodic BINARY representations

$$\begin{aligned} 1092 &= 10001000100_2 \\ 3510 &= 110110110110_2. \end{aligned}$$

A PRIME factor p of a MERSENNE NUMBER $M_q = 2^q - 1$ is a Wieferich prime IFF $p^2 | 2^q - 1$. Therefore, MERSENNE PRIMES are *not* Wieferich primes.

If the first case of FERMAT'S LAST THEOREM is false for exponent p, then p must be a Wieferich prime (Wieferich 1909). If $p | 2^n \pm 1$ with p and n RELATIVELY PRIME, then p is a Wieferich prime IFF p^2 also divides $2^n \pm 1$. The CONJECTURE that there are no three POWERFUL NUMBERS implies that there are infinitely many Wieferich primes (Granville 1986, Vardi 1991). In addition, the ABC CONJECTURE implies that there are at least $C \ln x$ Wieferich primes $\leq x$ for some constant C (Silverman 1988, Vardi 1991).

see also ABC CONJECTURE, FERMAT'S LAST THEOREM, FERMAT QUOTIENT, MERSENNE NUMBER, MIRIMANOFF'S CONGRUENCE, POWERFUL NUMBER

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Wieferich, A. "Zum letzten Fermat'schen Theorem." *J. reine angew. Math.* **136**, 293–302, 1909.

Wielandt's Theorem

Let the $n \times n$ MATRIX A satisfy the conditions of the PERRON-FROBENIUS THEOREM and the $n \times n$ MATRIX $\mathsf{C} = c_{ij}$ satisfy

$$|c_{ij}| \leq a_{ij}$$

for $i, j = 1, 2, \ldots, n$. Then any EIGENVALUE λ_0 of C satisfies the inequality $|\lambda_0| \leq R$ with the equality sign holding only when there exists an $n \times n$ MATRIX $\mathsf{D} = \delta_{ij}$ (where δ_{ij} is the KRONECKER DELTA) and

$$\mathsf{C} = \frac{\lambda_0}{R} \mathsf{DAD}^{-1}.$$

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Wiener Filter

An optimal FILTER used for the removal of noise from a signal which is corrupted by the measuring process itself.

see also FILTER

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Wiener Function

see BROWN FUNCTION

Wiener-Khintchine Theorem

Recall the definition of the AUTOCORRELATION function $C(t)$ of a function $E(t)$,

$$C(t) \equiv \int_{-\infty}^{\infty} E^*(\tau) E(t+\tau)\, d\tau. \tag{1}$$

Also recall that the FOURIER TRANSFORM of $E(t)$ is defined by

$$E(\tau) = \int_{-\infty}^{\infty} E_\nu e^{-2\pi i \nu \tau}\, d\nu, \tag{2}$$

giving a COMPLEX CONJUGATE of

$$E^*(\tau) = \int_{-\infty}^{\infty} E_\nu^* e^{2\pi i \nu \tau}\, d\nu. \tag{3}$$

Plugging $E^*(\tau)$ and $E(t+\tau)$ into the AUTOCORRELATION function therefore gives

$$\begin{aligned}C(t) &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} E_\nu^* e^{2\pi i\nu\tau}\,d\nu\right]\\ &\quad\times\left[\int_{-\infty}^{\infty} E_{\nu'} e^{-2\pi i\nu'(t+\tau)}\,d\nu'\right]d\tau\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_\nu^* E_{\nu'} e^{-2\pi i\tau(\nu'-\nu)} e^{-2\pi i\nu' t}\,d\tau\,d\nu\,d\nu'\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_\nu^* E_{\nu'}\delta(\nu'-\nu) e^{-2\pi i\nu' t}\,d\nu\,d\nu'\\ &= \int_{-\infty}^{\infty} E_\nu^* E_\nu e^{-2\pi i\nu t}\,d\nu\\ &= \int_{-\infty}^{\infty} |E_\nu|^2 e^{-2\pi i\nu t}\,d\nu\\ &= \mathcal{F}[\,|E_\nu|^2], \qquad (4)\end{aligned}$$

so, amazingly, the AUTOCORRELATION is simply given by the FOURIER TRANSFORM of the ABSOLUTE SQUARE of $E(\nu)$,

$$C(t) = \mathcal{F}[|E(\nu)|^2]. \qquad (5)$$

The Wiener-Khintchine theorem is a special case of the CROSS-CORRELATION THEOREM with $f = g$.

see also AUTOCORRELATION, CROSS-CORRELATION THEOREM, FOURIER TRANSFORM

Wiener Measure

The distribution which arises whenever a central limit scaling procedure is carried out on path-space valued random variables.

Wiener Space

see MALLIAVIN CALCULUS

Wigner 3j-Symbol

The Wigner $3j$ symbols are written

$$\begin{pmatrix} j_1 & j_2 & j\\ m_1 & m_2 & m\end{pmatrix} \qquad (1)$$

and are sometimes expressed using the related CLEBSCH-GORDON COEFFICIENTS

$$C^j_{m_1m_2} = (j_1j_2m_1m_2|j_1j_2jm) \qquad (2)$$

(Condon and Shortley 1951, pp. 74–75; Wigner 1959, p. 206), or RACAH V-COEFFICIENTS

$$V(j_1j_2j;m_1m_2m). \qquad (3)$$

Connections among the three are

$$(j_1j_2m_1m_2|j_1j_2m) = (-1)^{-j_1+j_2-m}\sqrt{2j+1}\begin{pmatrix} j_1 & j_2 & j\\ m_1 & m_2 & -m\end{pmatrix} \qquad (4)$$

$$(j_1j_2m_1m_2|j_1j_2jm) = (-1)^{j+m}\sqrt{2j+1}V(j_1j_2j;m_1m_2\ -m) \qquad (5)$$

$$V(j_1j_2j;m_1m_2m) = (-1)^{-j_1+j_2+j}\begin{pmatrix} j_1 & j_2 & j_1\\ m_2 & m_1 & m_2\end{pmatrix}. \qquad (6)$$

The Wigner $3j$-symbols have the symmetries

$$\begin{aligned}\begin{pmatrix} j_1 & j_2 & j\\ m_1 & m_2 & m\end{pmatrix} &= \begin{pmatrix} j_2 & j & j_1\\ m_2 & m & m_1\end{pmatrix}\\ &= \begin{pmatrix} j & j_1 & j_2\\ m & m_1 & m_2\end{pmatrix} = (-1)^{j_1+j_2+j}\begin{pmatrix} j_2 & j_1 & j\\ m_2 & m_1 & m\end{pmatrix}\\ &= (-1)^{j_1+j_2+j}\begin{pmatrix} j_1 & j & j_2\\ m_1 & m & m_2\end{pmatrix}\\ &= (-1)^{j_1+j_2+j}\begin{pmatrix} j & j_2 & j_1\\ m & m_2 & m_1\end{pmatrix}\\ &= (-1)^{j_1+j_2+j}\begin{pmatrix} j_1 & j_2 & j\\ -m_1 & -m_2 & -m\end{pmatrix}. \qquad (7)\end{aligned}$$

The symbols obey the orthogonality relations

$$\sum_{j,m}(2j+1)\begin{pmatrix} j_1 & j_2 & j\\ m_1 & m_2 & m\end{pmatrix}\begin{pmatrix} j_1 & j_2 & j\\ m_1' & m_2' & m\end{pmatrix} = \delta_{m_1m_1'}\delta_{m_2m_2'} \qquad (8)$$

$$\sum_{m_1,m_2}\begin{pmatrix} j_1 & j_2 & j\\ m_1 & m_2 & m\end{pmatrix}\begin{pmatrix} j_1 & j_2 & j'\\ m_1 & m_2 & m'\end{pmatrix} = \delta_{jj'}\delta_{m_1m_1'}, \qquad (9)$$

where δ_{ij} is the KRONECKER DELTA.

General formulas are very complicated, but some specific cases are

$$\begin{aligned}\begin{pmatrix} j_1 & j_2 & j_1+j_2\\ m_1 & m_2 & -m_1-m_2\end{pmatrix} &= (-1)^{j_1-j_2+m_1+m_2}\\ &\times\left[\frac{(2j_1)!(2j_2)!}{(2j_1+2j_2+1)!(j_1+m_1)!}\right.\\ &\quad\times\left.\frac{(j_1+j_2+m_1+m_2)!(j_1+j_2-m_1-m_2)!}{(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}\right]^{1/2} \qquad (10)\end{aligned}$$

$$\begin{aligned}\begin{pmatrix} j_1 & j_2 & j\\ j_1 & -j_1- & m\end{pmatrix} &= (-1)^{-j_1+j_2+m}\\ &\times\left[\frac{(2j_1)!(-j_1+j_2+j)!}{(j_1+j_2+j+1)!(j_1-j_2+j)!}\right.\\ &\quad\left.\frac{(j_1+j_2+m)!(j-m)!}{(j_1+j_2-j)!(-j_1+j_2-m)!(j+m)!}\right]^{1/2} \qquad (11)\end{aligned}$$

$$\begin{pmatrix} j_1 & j_2 & j \\ 0 & 0 & 0 \end{pmatrix} = \begin{cases} (-1)^g \sqrt{\frac{(2g-2j_1)(2g-2j_2)!(2g-2j)!}{(2g+1)!}} \frac{g!}{(g-j_1)!(g-j_2)!(g-j)!} \\ \quad \text{if } J = 2g \\ 0 \\ \quad \text{if } J = 2g+1, \end{cases} \tag{12}$$

for $J \equiv j_1 + j_2 + j$.

For SPHERICAL HARMONICS $Y_{lm}(\theta, \phi)$,

$$Y_{l_1 m_1}(\theta,\phi) Y_{l_2 m_2}(\theta,\phi) = \sum_{l,m} \sqrt{\frac{(2l_1+1)(2l_2+1)(2l+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} \times Y^*_{lm}(\theta,\psi) \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

For values of l_3 obeying the TRIANGLE CONDITION $\Delta(l_1 l_2 l_3)$,

$$\int Y_{l_1 m_1}(\theta,\phi) Y_{l_2 m_2}(\theta,\phi) Y_{l_3 m_3}(\theta,\phi) \sin\theta \, d\theta \, d\phi = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \times \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{14}$$

and

$$\frac{1}{2} \int P_{l_1}(\cos\theta) P_{l_2}(\cos\theta) P_{l_3}(\cos\theta) \sin\theta \, d\theta = \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}^2. \tag{15}$$

see also CLEBSCH-GORDON COEFFICIENT, RACAH V-COEFFICIENT, RACAH W-COEFFICIENT, WIGNER $6j$-SYMBOL, WIGNER $9j$-SYMBOL

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Wigner $6j$-Symbol

A generalization of CLEBSCH-GORDON COEFFICIENTS and WIGNER $3j$-SYMBOL which arises in the coupling of three angular momenta. Let tensor operators $T^{(k)}$ and $U^{(k)}$ act, respectively, on subsystems 1 and 2 of a system, with subsystem 1 characterized by angular momentum $\mathbf{j}_1$ and subsystem 2 by the angular momentum $\mathbf{j}_2$. Then the matrix elements of the scalar product of these two tensor operators in the coupled basis $\mathbf{J} = \mathbf{j}_1 + \mathbf{j}_2$ are given by

$$(\tau_1' j_1' \tau_2' j_2' J' M' | T^{(k)} \cdot U^{(k)} | \tau_1 j_1 \tau_2 j_2 J M) = \delta_{JJ'} \delta_{MM'} (-1)^{j_1 + j_2' + J} \begin{Bmatrix} J & j_2' & j_1' \\ k & j_1 & j_2 \end{Bmatrix} \times (\tau_1' j_1' || T^{(k)} || \tau_1 j_1)(\tau_2' j_2' || U^{(k)} || \tau_2 j_2), \tag{1}$$

where $\begin{Bmatrix} J & j_2' & j_1' \\ k & j_1 & j_2 \end{Bmatrix}$ is the Wigner $6j$-symbol and τ_1 and τ_2 represent additional pertinent quantum numbers characterizing subsystems 1 and 2 (Gordy and Cook 1984).

Edmonds (1968) gives analytic forms of the $6j$-symbol for simple cases, and Shore and Menzel (1968) and Gordy and Cook (1984) give

$$\begin{Bmatrix} a & b & c \\ 0 & c & b \end{Bmatrix} = \frac{(-1)^s}{\sqrt{(2b+1)(2c+1)}} \tag{2}$$

$$\begin{Bmatrix} a & b & c \\ 1 & c & b \end{Bmatrix} = \frac{2(-1)^{s+1} X}{\sqrt{2b(2b+1)(2b+2)2c(2c+1)(2c+2)}} \tag{3}$$

$$\begin{Bmatrix} a & b & c \\ 2 & c & b \end{Bmatrix} = \frac{2(-1)^s[3X(X-1) - 4b(b+1)c(c+1)]}{\sqrt{(2b-1)2b(2b+1)(2b+2)(2b+3)}} \times \frac{1}{\sqrt{(2c-1)2c(2c+1)(2c+2)(2c+3)}}, \tag{4}$$

where

$$s \equiv a + b + c \tag{5}$$

$$X \equiv b(b+1) + c(c+1) - a(a+1). \tag{6}$$

see also CLEBSCH-GORDON COEFFICIENT, RACAH V-COEFFICIENT, RACAH W-COEFFICIENT, WIGNER $3j$-SYMBOL, WIGNER $9j$-SYMBOL

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Wigner 9j-Symbol

A generalization of CLEBSCH-GORDON COEFFICIENTS and WIGNER 3j- and 6j-SYMBOLS which arises in the coupling of four angular momenta and can be written in terms of the WIGNER 3j- and 6j-SYMBOLS. Let tensor operators $T^{(k_1)}$ and $U^{(k_2)}$ act, respectively, on subsystems 1 and 2. Then the reduced matrix element of the product $T^{(k_1)} \times U^{(k_2)}$ of these two irreducible operators in the coupled representation is given in terms of the reduced matrix elements of the individual operators in the uncoupled representation by

$$(\tau'\tau_1'j_1'\tau_2'j_2'J'||[T^{(k_1)} \times U^{(k_2)}]^{(k)}||\tau\tau_1j_1\tau_2j_2J) = \sqrt{(2J+1)(2J'+1)(2k+1)}\sum_{\tau''}\left\{\begin{matrix} j_1' & j_1 & k_1 \\ j_2' & j_2 & k_2 \\ J' & J & k \end{matrix}\right\} (\tau'\tau_1'j_1'||T^{(k_1)}||\tau''\tau_1j_1)(\tau''\tau_2'j_2'||U^{(k_2)}||\tau\tau_2j_2), \quad (1)$$

where $\left\{\begin{matrix} j_1' & j_1 & k_1 \\ j_2' & j_2 & k_2 \\ J' & J & k \end{matrix}\right\}$ is a Wigner 9j-symbol (Gordy and Cook 1984).

Shore and Menzel (1968) give the explicit formulas

$$\left\{\begin{matrix} a & b & C \\ d & e & F \\ G & H & J \end{matrix}\right\} = \sum_x (-1)^{2x}(2x+1) \times \left\{\begin{matrix} a & b & C \\ F & J & x \end{matrix}\right\}\left\{\begin{matrix} d & e & F \\ b & x & H \end{matrix}\right\}\left\{\begin{matrix} G & H & J \\ x & a & d \end{matrix}\right\} \quad (2)$$

$$\left\{\begin{matrix} a & b & J \\ c & d & J \\ K & K & 0 \end{matrix}\right\} = \frac{(-1)^{b+c+J+K}}{\sqrt{(2J+1)(2K+1)}}\left\{\begin{matrix} a & b & J \\ d & c & K \end{matrix}\right\} \quad (3)$$

$$\left\{\begin{matrix} S & S & 1 \\ L & L & 2 \\ J & J & 1 \end{matrix}\right\} = \frac{\left\{\begin{matrix} S & L & J \\ L & S & 1 \end{matrix}\right\}\left\{\begin{matrix} J & L & S \\ L & J & 1 \end{matrix}\right\}}{5\left\{\begin{matrix} 2 & L & L \\ L & 1 & 1 \end{matrix}\right\}} + \frac{(-1)^{S+L+J+1}}{15(2L+1)}\frac{\left\{\begin{matrix} S & J & L \\ J & S & 1 \end{matrix}\right\}}{\left\{\begin{matrix} 2 & L & L \\ L & 1 & 1 \end{matrix}\right\}}. \quad (4)$$

see also CLEBSCH-GORDON COEFFICIENT, RACAH V-COEFFICIENT, RACAH W-COEFFICIENT, WIGNER 3j-SYMBOL, WIGNER 6j-SYMBOL

References

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Messiah, A. "'9j' Symbols." Appendix C.III in *Quantum Mechanics, Vol. 2.* Amsterdam, Netherlands: North-Holland, pp. 567–569 and 1066–1068, 1962.
Shore, B. W. and Menzel, D. H. *Principles of Atomic Spectra.* New York: Wiley, pp. 279–284, 1968.

Wigner-Eckart Theorem

A theorem of fundamental importance in spectroscopy and angular momentum theory which provides both (1) an explicit form for the dependence of all matrix elements of irreducible tensors on the projection quantum numbers and (2) a formal expression of the conservation laws of angular momentum (Rose 1995).

The theorem states that the dependence of the matrix element $(j'm'|T_{LM}|jm)$ on the projection quantum numbers is entirely contained in the WIGNER 3j-SYMBOL (or, equivalently, the CLEBSCH-GORDON COEFFICIENT), given by

$$(j'm'|T_{LM}|jm) = C(jLj';mMm')(j'||T_L||j),$$

where $C(jLj';mMm')$ is a CLEBSCH-GORDON COEFFICIENT and T_{LM} is a set of tensor operators (Rose 1995, p. 85).

see also CLEBSCH-GORDON COEFFICIENT, WIGNER 3j-SYMBOL

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Wilbraham-Gibbs Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let a piecewise smooth function f with only finitely many discontinuities (which are all jumps) be defined on $[-\pi, \pi]$ with FOURIER SERIES

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt \qquad (1)$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt, \qquad (2)$$

$$S_n(f,x) = \tfrac{1}{2}a_0 + \left\{\sum_{k=1}^{n}[a_k\cos(kx) + b_k\sin(kx)]\right\}. \qquad (3)$$

Let a discontinuity be at $x = c$, with

$$\lim_{x\to c^-} f(x) > \lim_{x\to c^+} f(x), \qquad (4)$$

so

$$D \equiv \left[\lim_{x\to c^-} f(x)\right] - \left[\lim_{x\to c^+} f(x)\right] > 0. \qquad (5)$$

Define

$$\phi(c) = \frac{1}{2}\left[\lim_{x\to c^-} f(x) + \lim_{x\to c^+} f(x)\right], \qquad (6)$$

and let $x = x_n < c$ be the first local minimum and $x = \xi_n > c$ the first local maximum of $S_n(f,x)$ on either side of x_n. Then

$$\lim_{n\to\infty} S_n(f, x_n) = \phi(c) + \frac{D}{\pi}G' \qquad (7)$$

$$\lim_{n\to\infty} S_n(f, \xi_n) = \phi(c) - \frac{D}{\pi}G', \qquad (8)$$

where

$$G' \equiv \int_0^{\pi} \operatorname{sinc}\theta\,d\theta = 1.851937052\ldots. \qquad (9)$$

Here, $\operatorname{sinc} x \equiv \sin x/x$ is the SINC FUNCTION. The FOURIER SERIES of $y = x$ therefore does not converge to $-\pi$ and π at the ends, but to $-2G'$ and $2G'$. This phenomenon was observed by Wilbraham (1848) and Gibbs (1899). Although Wilbraham was the first to note the phenomenon, the constant G' is frequently (and unfairly) credited to Gibbs and known as the GIBBS CONSTANT. A related constant sometimes also called the GIBBS CONSTANT is

$$G \equiv \frac{2}{\pi}G' = \frac{2}{\pi}\int_0^{\pi} \operatorname{sinc} x\,dx = 1.17897974447216727\ldots \qquad (10)$$

(Le Lionnais 1983).

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Wilcoxon Rank Sum Test

A nonparametric alternative to the two-sample t-test.

see also PAIRED t-TEST, PARAMETRIC TEST

Wilcoxon Signed Rank Test

A nonparametric alternative to the PAIRED t-TEST which is similar to the FISHER SIGN TEST. This test assumes that there is information in the magnitudes of the differences between paired observations, as well as the signs. Take the paired observations, calculate the differences, and rank them from smallest to largest by ABSOLUTE VALUE. Add all the ranks associated with POSITIVE differences, giving the T_+ statistic. Finally, the P-VALUE associated with this statistic is found from an appropriate table. The Wilcoxon test is an R-ESTIMATE.

see also FISHER SIGN TEST, HYPOTHESIS TESTING, PAIRED t-TEST, PARAMETRIC TEST

Wild Knot

A KNOT which is not a TAME KNOT.

see also TAME KNOT

References

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Wild Point

For any point P on the boundary of an ordinary BALL, find a NEIGHBORHOOD of P in which the intersection with the BALL's boundary cuts the NEIGHBORHOOD into two parts, each HOMEOMORPHIC to a BALL. A wild point is a point on the boundary that has no such NEIGHBORHOOD.

see also BALL, HOMEOMORPHIC, NEIGHBORHOOD

Wilf-Zeilberger Pair

A pair of CLOSED FORM functions (F, G) is said to be a Wilf-Zeilberger pair if

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k). \qquad (1)$$

The Wilf-Zeilberger formalism provides succinct proofs of *known* identities and allows new identities to be discovered whenever it succeeds in finding a proof certificate for a known identity. However, if the starting point is an unknown hypergeometric sum, then the Wilf-Zeilberger method cannot discover a closed form solution, while ZEILBERGER'S ALGORITHM can.

Wilf-Zeilberger pairs are very useful in proving HYPERGEOMETRIC IDENTITIES of the form

$$\sum_k t(n,k) = \operatorname{rhs}(n) \qquad (2)$$

for which the SUMMAND $t(n,k)$ vanishes for all k outside some finite interval. Now divide by the right-hand side to obtain

$$\sum_k F(n,k) = 1, \qquad (3)$$

where

$$F(n,k) \equiv \frac{t(n,k)}{\mathrm{rhs}(n)}. \tag{4}$$

Now use a RATIONAL FUNCTION $R(n,k)$ provided by ZEILBERGER'S ALGORITHM, define

$$G(n,k) \equiv R(n,k)F(n,k). \tag{5}$$

The identity (1) then results. Summing the relation over all integers then telescopes the right side to 0, giving

$$\sum_k F(n+1,k) = \sum_k F(n,k). \tag{6}$$

Therefore, $\sum_k F(n,k)$ is independent of n, and so must be a constant. If F is properly normalized, then it will be true that $\sum_k F(0,k) = 1$.

For example, consider the BINOMIAL COEFFICIENT identity

$$\sum_k = \sum_{k=0}^{n} \binom{n}{k} = 2^n, \tag{7}$$

the function $R(n,k)$ returned by ZEILBERGER'S ALGORITHM is

$$R(n,k) = \frac{k}{2(k-n-1)}. \tag{8}$$

Therefore,

$$F(n,k) = \binom{n}{k} 2^{-n} \tag{9}$$

and

$$\begin{aligned} G(n,k) \equiv R(n,k)F(n,k) &= \frac{k}{2(k-n-1)}\binom{n}{k}2^{-n} \\ &= -\frac{kn!2^{-n}}{2(n+1-k)k!(n-k)!} = -\binom{n}{k-1}2^{-n-1}. \end{aligned} \tag{10}$$

Taking

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k) \tag{11}$$

then gives the alleged identity

$$\begin{aligned} &\binom{n+1}{k}2^{-n-1} - \binom{n}{k}2^{-n} \\ &\quad = -\binom{n}{k}2^{-n-1} + \binom{n}{k-1}2^{-n-1}? \end{aligned} \tag{12}$$

Expanding and evaluating shows that the identity does actually hold, and it can also be verified that

$$F(0,k) = \binom{0}{k} = \begin{cases} 1 & \text{for } k=0 \\ 0 & \text{otherwise,} \end{cases} \tag{13}$$

so $\sum_k F(0,k) = 1$ (Petkovšek *et al.* 1996, pp. 25–27).

For any Wilf-Zeilberger pair (F,G),

$$\sum_{n=0}^{\infty} G(n,0) = \sum_{n=1}^{\infty} [F(n,n-1) + G(n-1,n-1)] \tag{14}$$

whenever either side converges (Zeilberger 1993). In addition,

$$\sum_{n=0}^{\infty} G(n,0) = \sum_{n=0}^{\infty} \left[F(s(n+1),n) + \sum_{i=0}^{s-1} G(sn+i,n)\right], \tag{15}$$

$$\sum_{k=0}^{\infty} F(0,k) = \sum_{n=0}^{\infty} G(n,0), \tag{16}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} G(n,0) = \sum_{n=0}^{\infty} &\left[\sum_{j=0}^{t-1} F(s(n+1),tn+j)\right. \\ &\left. + \sum_{i=0}^{s-1} G(sn+i,tn)\right], \end{aligned} \tag{17}$$

where

$$F_{s,t}(n,k) = \sum_{j=0}^{t-1} F(sn,tk+j) \tag{18}$$

$$G_{s,t}(n,k) = \sum_{i=0}^{s-1} G(sn+i,tk) \tag{19}$$

(Amdeberhan and Zeilberger 1997). The latter identity has been used to compute APÉRY'S CONSTANT to a large number of decimal places (Plouffe).

see also APÉRY'S CONSTANT, CONVERGENCE IMPROVEMENT, ZEILBERGER'S ALGORITHM

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Wilkie's Theorem

Let $\phi(x_1, \ldots, x_n)$ be an $\mathcal{L}_{\text{exp}}$ formula, where $\mathcal{L}_{\text{exp}} \equiv \mathcal{L} \cup \{e^x\}$ and $\mathcal{L}$ is the language of ordered rings $\mathcal{L} = \{+, -, \cdot, <, 0, 1\}$. Then there are $n \geq m$ and $f_1, \ldots, f_s \in \mathbb{Z}[x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}]$ such that $\phi(x_1, \ldots, x_n)$ is equivalent to

$$\exists x_{m+1} \cdots \exists x_n f_1(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) = \ldots$$
$$= f_s(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) = 0$$

(Wilkie 1996). In other words, every formula is equivalent to an existential formula and every definable set is the projection of an exponential variety (Marker 1996).

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Williams $p+1$ Factorization Method

A variant of the POLLARD $p-1$ METHOD which uses LUCAS SEQUENCES to achieve rapid factorization if some factor p of N has a decomposition of $p+1$ in small PRIME factors.

see also LUCAS SEQUENCE, POLLARD $p-1$ METHOD, PRIME FACTORIZATION ALGORITHMS

References
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Wilson Plug

A 3-D surface with constant VECTOR FIELD on its boundary which traps at least one trajectory which enters it.

see also VECTOR FIELD

Wilson's Primality Test

see WILSON'S THEOREM

Wilson Prime

A PRIME satisfying

$$W(p) \equiv 0 \pmod{p},$$

where $W(p)$ is the WILSON QUOTIENT, or equivalently,

$$(p-1)! \equiv -1 \left(\text{mod } p^2\right).$$

5, 13, and 563 are the only Wilson primes less than 5×10^8 (Crandall *et al.* 1997).

References
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Wilson Quotient

$$W(p) \equiv \frac{(p-1)! - 1}{p}.$$

Wilson's Theorem

IFF p is a PRIME, then $(p-1)! + 1$ is a multiple of p, that is

$$(p-1)! \equiv -1 \pmod{p}.$$

This theorem was proposed by John Wilson in 1770 and proved by Lagrange in 1773. Unlike FERMAT'S LITTLE THEOREM, Wilson's theorem is both NECESSARY and SUFFICIENT for primality. For a COMPOSITE NUMBER, $(n-1)! \equiv 0 \pmod{n}$ except when $n = 4$.

see also FERMAT'S LITTLE THEOREM, WILSON'S THEOREM COROLLARY, WILSON'S THEOREM (GAUSS'S GENERALIZATION)

References
Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 61, 1987.
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Ore, Ø. *Number Theory and Its History.* New York: Dover, pp. 259–261, 1988.
Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, pp. 37–38, 1993.

Wilson's Theorem Corollary

Iff a PRIME p is of the form $4x+1$, then

$$[(2x)!]^2 \equiv -1 \pmod{p}.$$

Wilson's Theorem (Gauss's Generalization)

Let P be the product of INTEGERS less than or equal to n and RELATIVELY PRIME to n. Then

$$P \equiv \prod_{\substack{k=2 \\ k \nmid n}}^{n} = \begin{cases} -1 \pmod{n} & \text{for } n = 4, p^\alpha, 2p^\alpha \\ 1 \pmod{n} & \text{otherwise.} \end{cases}$$

When $m = 2$, this reduces to $P \equiv 1 \pmod{2}$ which is equivalent to $P \equiv -1 \pmod{2}$.

see also WILSON'S THEOREM, WILSON'S THEOREM COROLLARY

Winding Number (Contour)

Denoted $n(\gamma, z_0)$ and defined as the number of times a path γ curve passes around a point.

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}.$$

The contour winding number was part of the inspiration for the idea of the DEGREE of a MAP between two COMPACT, oriented MANIFOLDS of the same DIMENSION. In the language of the DEGREE of a MAP, if $\gamma : [0,1] \to \mathbb{C}$

is a closed curve (i.e., $\gamma(0) = \gamma(1)$), then it can be considered as a FUNCTION from $\mathbb{S}^1$ to $\mathbb{C}$. In that context, the winding number of γ around a point p in $\mathbb{C}$ is given by the degree of the MAP

$$\frac{\gamma - p}{|\gamma - p|}$$

from the CIRCLE to the CIRCLE.

Winding Number (Map)

The winding number of a map is defined by

$$W \equiv \lim_{n\to\infty} \frac{f^n(\theta) - \theta}{n},$$

which represents the average increase in the angle θ per unit time (average frequency). A system with a RATIONAL winding number $W = p/q$ is MODE-LOCKED, whereas a system with an IRRATIONAL winding number is QUASIPERIODIC. Note that since the RATIONALS are a set of zero MEASURE on any finite interval, almost all winding numbers will be irrational, so almost all maps will be QUASIPERIODIC.

Windmill

One name for the figure used by Euclid to prove the PYTHAGOREAN THEOREM.

see BRIDE'S CHAIR, PEACOCK'S TAIL

Window Function

see RECTANGLE FUNCTION

Winkler Conditions

Conditions arising in the study of the ROBBINS EQUATION and its connection with BOOLEAN ALGEBRA. Winkler studied Boolean conditions (such as idempotence or existence of a zero) which would make a ROBBINS ALGEBRA become a BOOLEAN ALGEBRA. Winkler showed that each of the conditions

$$\exists C, \exists D, C + D = C$$

$$\exists C, \exists D, n(C + D) = n(C),$$

known as the first and second Winkler conditions, SUFFICES. A computer proof demonstrated that every ROBBINS ALGEBRA satisfies the second Winkler condition, from which it follows immediately that all ROBBINS ALGEBRAS are BOOLEAN.

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McCune, W. "Robbins Algebras are Boolean." `http://www.mcs.anl.gov/home/mccune/ar/robbins/`.

Winkler, S. "Robbins Algebra: Conditions that Make a Near-Boolean Algebra Boolean." *J. Automated Reasoning* **6**, 465–489, 1990.

Winkler, S. "Absorption and Idempotency Criteria for a Problem in Near-Boolean Algebra." *J. Algebra* **153**, 414–423, 1992.

Winograd Transform

A discrete FAST FOURIER TRANSFORM ALGORITHM which can be implemented for $N = 2, 3, 4, 5, 7, 8, 11, 13$, and 16 points.

see also FAST FOURIER TRANSFORM

Wirtinger's Inequality

If y has period 2π, y' is L^2, and

$$\int_0^{2\pi} y\,dx = 0,$$

then

$$\int_0^{2\pi} y^2\,dx < \int_0^{2\pi} y'^2\,dx$$

unless

$$y = A\cos x + B\sin x.$$

References

Hardy, G. H.; Littlewood, J. E.; and Pólya, G. *Inequalities, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 184–187, 1988.

Wirtinger-Sobolev Isoperimetric Constants

Constants γ such that

$$\left[\int_\Omega |f|^q\,dx\right]^{1/q} \leq \gamma \left[\int_\Omega \sum_{i=1}^N \left|\frac{\partial f}{\partial x_i}\right|^p\,dx\right]^{1/p},$$

where f is a real-valued smooth function on a region Ω satisfying some BOUNDARY CONDITIONS.

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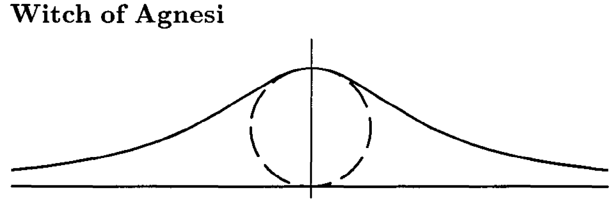
Witch of Agnesi



A curve studied and named "versiera" (Italian for "she-devil" or "witch") by Maria Agnesi in 1748 in her book *Istituzioni Analitiche* (MacTutor Archive). It is also known as CUBIQUE D'AGNESI or AGNÉSIENNE. Some suggest that Agnesi confused an old Italian word meaning "free to move" with another meaning "witch." The curve had been studied earlier by Fermat and Guido Grandi in 1703.

It is the curve obtained by drawing a line from the origin through the CIRCLE of radius $2a$ (OB), then picking the point with the y coordinate of the intersection with the circle and the x coordinate of the intersection of the extension of line OB with the line $y = 2a$. The curve has INFLECTION POINTS at $y = 3a/2$. The line $y = 0$ is an ASYMPTOTE to the curve.

In parametric form,

$$x = 2a \cot\theta \quad (1)$$

$$y = a[1 - \cos(2\theta)], \quad (2)$$

or

$$x = 2at \quad (3)$$

$$y = \frac{2a}{1+t^2}. \quad (4)$$

In rectangular coordinates,

$$y = \frac{8a^3}{x^2 + 4a^2}. \quad (5)$$

see also LAMÉ CURVE

References
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 90–93, 1972.
Lee, X. "Witch of Agnesi." `http://www.best.com/~xah/SpecialPlaneCurves_dir/WitchOfAgnesi_dir/witchOfAgnesi.html`.
MacTutor History of Mathematics Archive. "Witch of Agnesi." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Witch.html`.
Yates, R. C. "Witch of Agnesi." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 237–238, 1952.

Witness

A witness is a number which, as a result of its number theoretic properties, guarantees either the compositeness or primality of a number n. Witnesses are most commonly used in connection with FERMAT'S LITTLE THEOREM CONVERSE. A PRATT CERTIFICATE uses witnesses to prove primality, and MILLER'S PRIMALITY TEST uses witnesses to prove compositeness.

see also ADLEMAN-POMERANCE-RUMELY PRIMALITY TEST, FERMAT'S LITTLE THEOREM CONVERSE, MILLER'S PRIMALITY TEST, PRATT CERTIFICATE, PRIMALITY CERTIFICATE

Witten's Equations

Also called the SEIBERG-WITTEN INVARIANTS. For a connection A and a POSITIVE SPINOR $\phi \in \Gamma(V_+)$,

$$D_A\phi = 0$$

$$F_+^A = i\sigma(\phi,\phi).$$

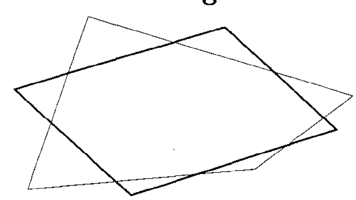
The solutions are called monopoles and are the minima of the functional

$$\int_X (|F_+^A - i\sigma(\phi,\phi)|^2 + |D_A\phi|^2).$$

see also LICHNEROWICZ FORMULA, LICHNEROWICZ-WEITZENBOCK FORMULA, SEIBERG-WITTEN EQUATIONS

References
Cipra, B. "A Tale of Two Theories." *What's Happening in the Mathematical Sciences, 1995-1996, Vol. 3.* Providence, RI: Amer. Math. Soc., pp. 14–25, 1996.
Donaldson, S. K. "The Seiberg-Witten Equations and 4-Manifold Topology." *Bull. Amer. Math. Soc.* **33**, 45–70, 1996.
Kotschick, D. "Gauge Theory is Dead!—Long Live Gauge Theory!" *Not. Amer. Math. Soc.* **42**, 335–338, 1995.
Seiberg, N. and Witten, E. "Monopoles, Duality, and Chiral Symmetry Breaking in $N = 2$ Supersymmetric QCD." *Nucl. Phys. B* **431**, 581–640, 1994.
Witten, E. "Monopoles and 4-Manifolds." *Math. Res. Let.* **1**, 769–796, 1994.

Wittenbauer's Parallelogram



Divide the sides of a QUADRILATERAL into three equal parts. The figure formed by connecting and extending adjacent points on either side of a VERTEX is a PARALLELOGRAM known as Wittenbauer's parallelogram.

see also QUADRILATERAL, WITTENBAUER'S THEOREM

Wittenbauer's Theorem

The CENTROID of a QUADRILATERAL LAMINA is the center of its WITTENBAUER'S PARALLELOGRAM.

see also CENTROID (GEOMETRIC), LAMINA, QUADRILATERAL, WITTENBAUER'S PARALLELOGRAM

Wolstenholme's Theorem

If p is a PRIME > 3, then the NUMERATOR of

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \tfrac{1}{p-1}$$

is divisible by p^2 and the NUMERATOR of

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{(p-1)^2}$$

is divisible by p. These imply that if $p \geq 5$ is PRIME, then

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.$$

References
Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 85, 1994.
Ribenboim, P. *The Book of Prime Number Records, 2nd ed.* New York: Springer-Verlag, p. 21, 1989.

Woodall Number

Numbers of the form

$$W_n = 2^n n - 1.$$

The first few are 1, 7, 23, 63, 159, 383, ... (Sloane's A003261). The only Woodall numbers W_n for $n < 100,000$ which are PRIME are for $n = 5312$, 7755, 9531, 12379, 15822, 18885, 22971, 23005, 98726, ... (Sloane's A014617; Ballinger).

see also CULLEN NUMBER, CUNNINGHAM NUMBER, FERMAT NUMBER, MERSENNE NUMBER, SIERPIŃSKI NUMBER OF THE FIRST KIND

References

Ballinger, R. "Cullen Primes: Definition and Status." http://ballingerr.xray.ufl.edu/proths/cullen.html.

Guy, R. K. "Cullen Numbers." §B20 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 77, 1994.

Leyland, P. ftp://sable.ox.ac.uk/pub/math/factors/woodall.

Ribenboim, P. *The New Book of Prime Number Records.* New York: Springer-Verlag, pp. 360–361, 1996.

Sloane, N. J. A. Sequences A014617 and A003261/M4379 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Woodbury Formula

$$(\mathsf{A}+\mathsf{UV}^{\mathrm{T}})^{-1} = \mathsf{A}^{-1} - [\mathsf{A}^{-1}\mathsf{U}(1+\mathsf{V}^{\mathrm{T}}\mathsf{A}^{-1}\mathsf{U})^{-1}\mathsf{V}^{\mathrm{T}}\mathsf{A}^{-1}].$$

Word

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

A finite sequence of n letters from some ALPHABET is said to be an n-ary word. A "square" word consists of two identical subwords (for example, *acbacb*). A squarefree word contains *no* square words as subwords (for example, *abcacbabcb*). The only squarefree binary words are *a*, *b*, *ab*, *ba*, *aba*, and *bab*. However, there are arbitrarily long ternary squarefree words. The number of ternary squarefree words of length n is bounded by

$$6 \cdot 1.032^n \leq s(n) \leq 6 \cdot 1.379^n \tag{1}$$

(Brandenburg 1983). In addition,

$$S \equiv \lim_{n\to\infty} [s(n)]^{1/n} = 1.302\ldots \tag{2}$$

(Brinkhuis 1983, Noonan and Zeilberger). Binary cubefree words satisfy

$$2 \cdot 1.080^n \leq c(n) \leq 2 \cdot 1.522^n. \tag{3}$$

A word is said to be overlapfree if it has no subwords of the form *xyxyx*. A squarefree word is overlapfree, and an overlapfree word is cubefree. The number $t(n)$ of binary overlapfree words of length n satisfies

$$p \cdot n^{1.155} \leq t(n) \leq q \cdot n^{1.587} \tag{4}$$

for some constants p and q (Restivo and Selemi 1985, Kobayashi 1988). In addition, while

$$\lim_{n\to\infty} \frac{\ln t(n)}{\ln n} \tag{5}$$

does not exist,

$$1.155 < T_L < 1.276 < 1.332 < T_U < 1.587, \tag{6}$$

where

$$T_L \equiv \liminf_{n\to\infty} \frac{\ln t(n)}{\ln n} \tag{7}$$

$$T_U \equiv \limsup_{n\to\infty} \frac{\ln t(n)}{\ln n} \tag{8}$$

(Cassaigne 1993).

see also ALPHABET

References

Brandenburg, F.-J. "Uniformly Growing kth Power-Free Homomorphisms." *Theor. Comput. Sci.* **23**, 69–82, 1983.

Brinkhuis, J. "Non-Repetitive Sequences on Three Symbols." *Quart. J. Math. Oxford Ser. 2* **34**, 145–149, 1983.

Cassaigne, J. "Counting Overlap-Free Binary Words." *STACS '93: Tenth Annual Symposium on Theoretical Aspects of Computer Science, Würzburg, Germany, February 25–27, 1993 Proceedings* (Ed. G. Goos, J. Hartmanis, A. Finkel, P. Enjalbert, K. W. Wagner). New York: Springer-Verlag, pp. 216–225, 1993.

Finch, S. "Favorite Mathematical Constants." http://www.mathsoft.com/asolve/constant/words/words.html.

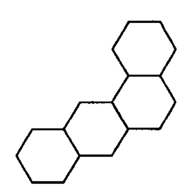
Kobayashi, Y. "Enumeration of Irreducible Binary Words." *Discrete Appl. Math.* **20**, 221–232, 1988.

Noonan, J. and Zeilberger, D. "The Goulden-Jackson Cluster Method: Extensions, Applications, and Implementations." Submitted.

World Line

The path of an object through PHASE SPACE.

Worm



A 4-POLYHEX.

References

Gardner, M. *Mathematical Magic Show: More Puzzles, Games, Diversions, Illusions and Other Mathematical Sleight-of-Mind from Scientific American.* New York: Vintage, p. 147, 1978.

Worpitzky's Identity

$$x^n = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{n+k}{n},$$

where $\left\langle {n \atop k} \right\rangle$ is an EULERIAN NUMBER and $\binom{n}{k}$ is a BINOMIAL COEFFICIENT.

Writhe

Also called the TWIST NUMBER. The sum of crossings p of a LINK L,

$$w(L) = \sum_{p \in C(L)} \epsilon(p),$$

where $\epsilon(p)$ defined to be ± 1 if the overpass slants from top left to bottom right or bottom left to top right and $C(L)$ is the set of crossings of an oriented LINK. If a KNOT K is AMPHICHIRAL, then $w(K) = 0$ (Thistlethwaite). Letting Lk be the LINKING NUMBER of the two components of a ribbon, Tw be the TWIST, and Wr be the writhe, then

$$\mathrm{Lk}(K) = \mathrm{Tw}(K) + \mathrm{Wr}(K).$$

(Adams 1994, p. 187).

see also SCREW, TWIST

References

Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, 1994.

Wronskian

$$W(\phi_1, \ldots, \phi_n) \equiv \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & \ddots & \vdots \\ {\phi_1}^{(n-1)} & {\phi_2}^{(n-1)} & \cdots & {\phi_n}^{(n-1)} \end{vmatrix}.$$

If the Wronskian is NONZERO in some region, the functions ϕ_i are LINEARLY INDEPENDENT. If $W = 0$ over some range, the functions are linearly dependent somewhere in the range.

see also ABEL'S IDENTITY, GRAM DETERMINANT, LINEARLY DEPENDENT FUNCTIONS

References

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 524–525, 1953.

Wulff Shape

An equilibrium MINIMAL SURFACE for a crystal which has the least anisotropic surface energy for a given volume. It is the anisotropic analog of a SPHERE.

see also SPHERE

Wynn's Epsilon Method

A method for numerical evaluation of SUMS and PRODUCTS which samples a number of additional terms in the series and then tries to fit them to a POLYNOMIAL multiplied by a decaying exponential.

see also EULER-MACLAURIN INTEGRATION FORMULAS

Wythoff Array

A INTERSPERSION array given by

1	2	3	5	8	13	21	34	55	⋯
4	7	11	18	29	47	76	123	199	⋯
6	10	16	26	42	68	110	178	288	⋯
9	15	24	39	63	102	165	267	432	⋯
12	20	32	52	84	136	220	356	576	⋯
14	23	37	60	97	157	254	411	665	⋯
17	28	45	73	118	191	309	500	809	⋯
19	31	50	81	131	212	343	555	898	⋯
22	36	58	94	152	246	398	644	1042	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱	

the first row of which is the FIBONACCI NUMBERS.

see also FIBONACCI NUMBER, INTERSPERSION, STOLARSKY ARRAY

References

Kimberling, C. "Fractal Sequences and Interspersions." *Ars Combin.* **45**, 157–168, 1997.

Wythoff Construction

A method of constructing UNIFORM POLYHEDRA.

see also UNIFORM POLYHEDRON

References

Har'El, Z. "Uniform Solution for Uniform Polyhedra." *Geometriae Dedicata* **47**, 57–110, 1993.

Wythoff's Game

A game played with two heaps of counters in which a player may take any number from either heap or the same number from both. The player taking the last counter wins. The rth SAFE combination is $(x, x+r)$, where $x = \lfloor \phi r \rfloor$, with ϕ the GOLDEN RATIO and $\lfloor x \rfloor$ the FLOOR FUNCTION. It is also true that $x + r = \lfloor \phi^2 r \rfloor$. The first few SAFE combinations are (1, 2), (3, 5), (4, 7), (6, 10), ...

see also NIM, SAFE

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 39–40, 1987.

Coxeter, H. S. M. "The Golden Section, Phyllotaxis, and Wythoff's Game." *Scripta Math.* **19**, 135–143, 1953.

O'Beirne, T. H. *Puzzles and Paradoxes.* Oxford, England: Oxford University Press, pp. 109 and 134–138, 1965.

Wythoff Symbol

A symbol used to describe UNIFORM POLYHEDRA. For example, the Wythoff symbol for the TETRAHEDRON is $3\,|\,2\,3$. There are three types of Wythoff symbols $p\,|\,q\,r$, $p\,q\,|\,r$ and $p\,q\,r\,|$, and one exceptional symbol $|\,\frac{3}{2}\,\frac{5}{3}\,3\,\frac{5}{2}$ used for the GREAT DIRHOMBICOSIDODECAHEDRON. Some special cases in terms of SCHLÄFLI SYMBOLS are

$$p\,|\,q\,2 = p\,|\,2\,q = \{q,p\}$$
$$2\,|\,p\,q = \begin{Bmatrix} p \\ q \end{Bmatrix}$$
$$p\,q\,|\,2 = r\begin{Bmatrix} p \\ q \end{Bmatrix}$$
$$2\,q\,|\,p = \mathrm{t}\{p,q\}$$
$$2\,p\,q\,| = \mathrm{t}\begin{Bmatrix} p \\ q \end{Bmatrix}$$
$$|\,2\,p\,q = \mathrm{s}\begin{Bmatrix} p \\ q \end{Bmatrix}.$$

For the symbol $p\,q\,r\,|$, permuting the letters gives the same POLYHEDRON.

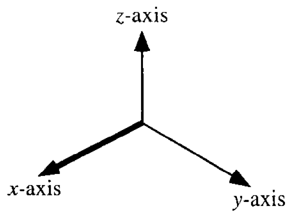
see also UNIFORM POLYHEDRON

References

Har'El, Z. "Uniform Solution for Uniform Polyhedra." *Geometriae Dedicata* **47**, 57–110, 1993.

X

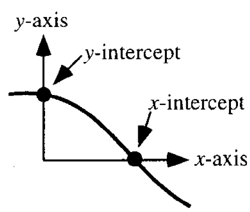
x-Axis



The horizontal axis of a 2-D plot in CARTESIAN COORDINATES, also called the ABSCISSA.

see also ABSCISSA, ORDINATE, y-AXIS, z-AXIS

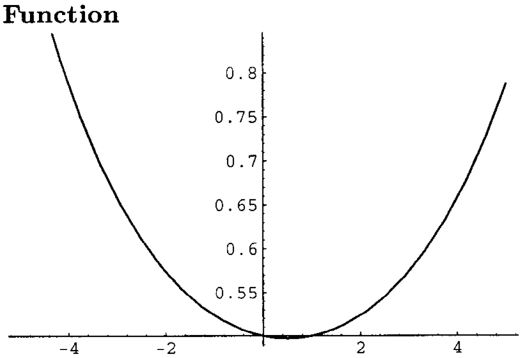
x-Intercept

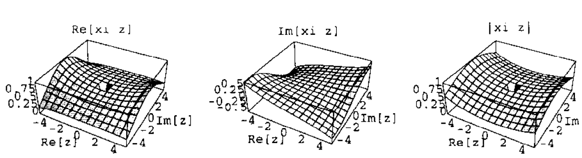


The point at which a curve or function crosses the x-AXIS (i.e., when $y = 0$ in 2-D).

see also LINE, y-INTERCEPT

Xi Function





$$\xi(z) \equiv \tfrac{1}{2} z(z-1) \frac{\Gamma(\tfrac{1}{2}z)}{\pi^{z/2}} \zeta(z) = \frac{(z-1)\Gamma(\tfrac{1}{2}z+1)\zeta(z)}{\sqrt{\pi^z}}, \tag{1}$$

where $\zeta(z)$ is the RIEMANN ZETA FUNCTION and $\Gamma(z)$ is the GAMMA FUNCTION (Gradshteyn and Ryzhik 1980, p. 1076). The ξ function satisfies the identity

$$\xi(1-z) = \xi(z). \tag{2}$$

The zeros of $\xi(z)$ and of its DERIVATIVES are all located on the CRITICAL STRIP $z = \sigma + it$, where $0 < \sigma < 1$. Therefore, the nontrivial zeros of the RIEMANN ZETA FUNCTION exactly correspond to those of $\xi(z)$. The function $\xi(z)$ is related to what Gradshteyn and Ryzhik (1980, p. 1074) call $\Xi(t)$ by

$$\Xi(t) \equiv \xi(z), \tag{3}$$

where $z \equiv \frac{1}{2} + it$. This function can also be defined as

$$\Xi(it) \equiv \tfrac{1}{2}(t^2 - \tfrac{1}{4})\pi^{-t/2-1/4}\Gamma(\tfrac{1}{2}t + \tfrac{1}{4})\zeta(t + \tfrac{1}{2}), \tag{4}$$

giving

$$\Xi(t) = -\tfrac{1}{2}(t^2 + \tfrac{1}{4})\pi^{it/2-1/4}\Gamma(\tfrac{1}{4} - \tfrac{1}{2}it)\zeta(\tfrac{1}{2} - it). \tag{5}$$

The DE BRUIJN-NEWMAN CONSTANT is defined in terms of the $\Xi(t)$ function.

see also DE BRUIJN-NEWMAN CONSTANT

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, corr. enl. 4th ed.* San Diego, CA: Academic Press, 1980.

XOR

An operation in LOGIC known as EXCLUSIVE OR. It yields true if exactly one (but not both) of two conditions is true. The BINARY XOR operator has the following TRUTH TABLE.

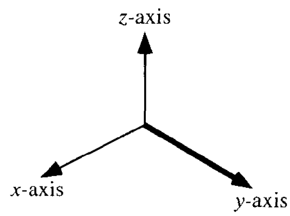
A	B	A XOR B
F	F	F
F	T	T
T	F	T
T	T	F

The BINOMIAL COEFFICIENT $\binom{m}{n}$ mod 2 can be computed using the XOR operation n XOR m, making PASCAL'S TRIANGLE mod 2 very easy to construct.

see also AND, BINARY OPERATOR, BOOLEAN ALGEBRA, LOGIC, NOT, OR, PASCAL'S TRIANGLE, TRUTH TABLE

Y

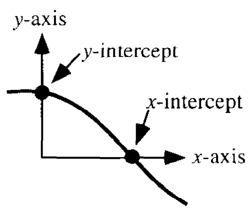
y-Axis



The vertical axis of a 2-D plot in CARTESIAN COORDINATES, also called the ORDINATE.

see also ABSCISSA, ORDINATE, x-AXIS, z-AXIS

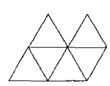
y-Intercept



The point at which a curve or function crosses the y-AXIS (i.e., when $x = 0$ in 2-D).

see also LINE, x-INTERCEPT

Yacht



A 6-POLYIAMOND.

References

Golomb, S. W. *Polyominoes: Puzzles, Patterns, Problems, and Packings, 2nd ed.* Princeton, NJ: Princeton University Press, p. 92, 1994.

Yanghui Triangle

see PASCAL'S TRIANGLE

Yff Center of Congruence

Let three ISOSCELIZERS, one for each side, be constructed on a TRIANGLE such that the four interior triangles they determine are congruent. Now parallel-displace these ISOSCELIZERS until they concur in a single point. This point is called the Yff center of congruence and has TRIANGLE CENTER FUNCTION

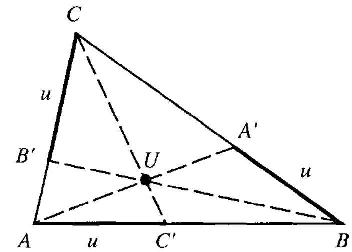
$$\alpha = \sec(\tfrac{1}{2}A).$$

see also CONGRUENT ISOSCELIZERS POINT, ISOSCELIZER

References

Kimberling, C. "Yff Center of Congruence." `http://www.evansville.edu/~ck6/tcenters/recent/yffcc.html`.

Yff Points



Let points A', B', and C' be marked off some fixed distance x along each of the sides BC, CA, and AB. Then the lines AA', BB', and CC' concur in a point U known as the first Yff point if

$$x^3 = (a-x)(b-x)(c-x). \tag{1}$$

This equation has a single real root u, which can by obtained by solving the CUBIC EQUATION

$$f(x) = 2x^3 - px^2 + qx - r = 0, \tag{2}$$

where

$$p = a + b + c \tag{3}$$

$$q = ab + ac + bc \tag{4}$$

$$r = abc. \tag{5}$$

The ISOTOMIC CONJUGATE POINT U' is called the second Yff point. The TRIANGLE CENTER FUNCTIONS of the first and second points are given by

$$\alpha = \frac{1}{a}\left(\frac{c-u}{b-u}\right)^{1/3} \tag{6}$$

and

$$\alpha' = \frac{1}{a}\left(\frac{b-u}{c-u}\right)^{1/3}, \tag{7}$$

respectively. Analogous to the inequality $\omega \leq \pi/6$ for the BROCARD ANGLE ω, $u \leq p/6$ holds for the Yff points, with equality in the case of an EQUILATERAL TRIANGLE. Analogous to

$$\omega < \alpha_i < \pi - 3\omega \tag{8}$$

for $i = 1, 2, 3$, the Yff points satisfy

$$u < a_i < p - 3u. \tag{9}$$

Yff (1963) gives a number of other interesting properties. The line UU' is PERPENDICULAR to the line containing the INCENTER I and CIRCUMCENTER O, and its length is given by

$$\overline{UU'} = \frac{4u\overline{IO}\Delta}{u^3 + abc}, \tag{10}$$

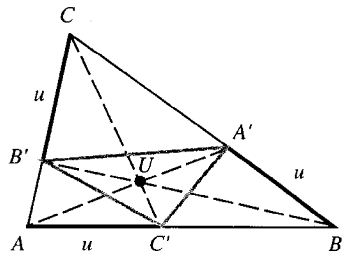
where Δ is the AREA of the TRIANGLE.

see also BROCARD POINTS, YFF TRIANGLES

References

Yff, P. "An Analog of the Brocard Points." *Amer. Math. Monthly* **70**, 495–501, 1963.

Yff Triangles



The TRIANGLE $\Delta A'B'C'$ formed by connecting the points used to construct the YFF POINTS is called the first Yff triangle. The AREA of the triangle is

$$\Delta = \frac{u^3}{2R},$$

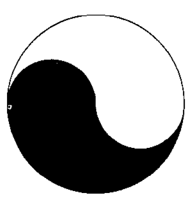
where R is the CIRCUMRADIUS of the original TRIANGLE ΔABC. The second Yff triangle is formed by connecting the ISOTOMIC CONJUGATE POINTS of A', B', and C'.

see also YFF POINTS

References

Yff, P. "An Analog of the Brocard Points." *Amer. Math. Monthly* **70**, 495–501, 1963.

Yin-Yang



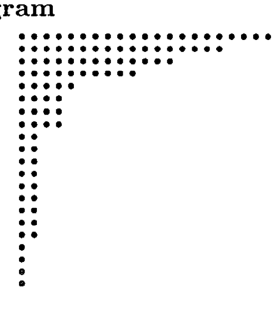
A figure used in many Asian cultures to symbolize the unity of the two "opposite" male and female elements, the "yin" and "yang." The solid and hollow parts composing the symbol are similar and combine to make a CIRCLE. Each part consists of two equal oppositely oriented SEMICIRCLES of radius 1/2 joined at their edges, plus a SEMICIRCLE of radius 1 joining the other edges.

see also BASEBALL COVER, CIRCLE, PIECEWISE CIRCULAR CURVE, SEMICIRCLE

References

Dixon, R. *Mathographics.* New York: Dover, p. 11, 1991.
Gardner, M. "Mathematical Games: A New Collection of 'Brain-Teasers.'" *Sci. Amer.* **203**, 172–180, Oct. 1960.
Gardner, M. "Mathematical Games: More About the Shapes that Can Be Made with Complex Dominoes." *Sci. Amer.* **203**, 186–198, Nov. 1960.

Young Diagram



A Young diagram, also called a FERRERS DIAGRAM, represents PARTITIONS as patterns of dots, with the nth row having the same number of dots as the nth term in the PARTITION. A Young diagram of the PARTITION

$$n = a + b + \ldots + c,$$

for a list a, b, ..., c of k POSITIVE INTEGERS with $a \geq b \geq \ldots \geq c$ is therefore the arrangement of n dots or square boxes in k rows, such that the dots or boxes are left-justified, the first row is of length a, the second row is of length b, and so on, with the kth row of length c. The above diagram corresponds to one of the possible partitions of 100.

see also DURFEE SQUARE, HOOK LENGTH FORMULA, PARTITION, PARTITION FUNCTION P, YOUNG TABLEAU

References

Messiah, A. Appendix D in *Quantum Mechanics, 2 vols.* Amsterdam, Netherlands: North-Holland, p. 1113, 1961–62.

Young Girl-Old Woman Illusion

A perceptual ILLUSION in which the brain switches between seeing a young girl and an old woman.

see also RABBIT-DUCK ILLUSION

References

Pappas, T. *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 173, 1989.

Young Inequality

For $0 < p < 1$,

$$ab \leq \frac{a^p}{p} + \left(1 - \frac{1}{p}\right) b^{1/(1-1/p)}.$$

Young's Integral

Let $f(x)$ be a REAL continuous monotonic strictly increasing function on the interval $[0, a]$ with $f(0) = 0$ and $b \leq f(a)$, then

$$ab \leq \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy,$$

where $f^{-1}(y)$ is the INVERSE FUNCTION. Equality holds IFF $b = f(a)$.

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Young Tableau

The YOUNG TABLEAU of a YOUNG DIAGRAM is obtained by placing the numbers 1, ..., n in the n boxes of the diagram. A "standard" Young tableau is a Young tableau in which the numbers form a nondecreasing sequence along each line and along each column. The standard Young tableaux of size three are given by $\{\{1,2,3\}\}$, $\{\{1,3\},\{2\}\}$, $\{\{1,2\},\{3\}\}$, and $\{\{1\},\{2\},\{3\}\}$. The number of standard Young tableaux of size 1, 2, 3, ... are 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, ... (Sloane's A000085). These numbers can be generated by the RECURRENCE RELATION

$$a(n) = a(n-1) + (n-1)a(n-2)$$

with $a(1) = 1$ and $a(2) = 2$.

There is a correspondence between a PERMUTATION and a pair of Young tableaux, known as the SCHENSTED CORRESPONDENCE. The number of all standard Young tableaux with a given shape (corresponding to a given YOUNG DIAGRAM) is calculated with the HOOK LENGTH FORMULA. The BUMPING ALGORITHM is used to construct a standard Young tableau from a permutation of $\{1, \ldots, n\}$.

see also BUMPING ALGORITHM, HOOK LENGTH FORMULA, INVOLUTION (SET), SCHENSTED CORRESPONDENCE, YOUNG DIAGRAM

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Z

$\mathbb{Z}$

The RING of INTEGERS ..., -2, -1, 0, 1, 2, ..., also denoted $\mathbb{I}$.

see also $\mathbb{C}$, $\mathbb{C}^*$, COUNTING NUMBER, $\mathbb{I}$, $\mathbb{N}$, NATURAL NUMBER, $\mathbb{Q}$, $\mathbb{R}$, WHOLE NUMBER, $\mathbb{Z}^-$, $\mathbb{Z}^-$

$\mathbb{Z}^-$

The NEGATIVE INTEGERS ..., -3, -2, -1.

see also COUNTING NUMBER, NATURAL NUMBER, NEGATIVE, WHOLE NUMBER, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^*$

$\mathbb{Z}^+$

The POSITIVE INTEGERS 1, 2, 3, ..., equivalent to $\mathbb{N}$.

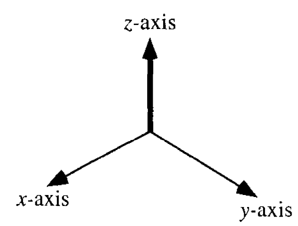
see also COUNTING NUMBER, $\mathbb{N}$, NATURAL NUMBER, POSITIVE, WHOLE NUMBER, $\mathbb{Z}$, $\mathbb{Z}^-$, $\mathbb{Z}^*$

$\mathbb{Z}^*$

The NONNEGATIVE INTEGERS 0, 1, 2, ...

see also COUNTING NUMBER, NATURAL NUMBER, NONNEGATIVE, WHOLE NUMBER $\mathbb{Z}$, $\mathbb{Z}^-$, $\mathbb{Z}^+$

z-Axis



The axis in 3-D CARTESIAN COORDINATES which is usually oriented vertically. CYLINDRICAL COORDINATES are defined such that the z-axis is the axis about which the azimuthal coordinate θ is measured.

see also AXIS, x-AXIS, y-AXIS

z-Distribution

see FISHER'S z-DISTRIBUTION, STUDENT'S z-DISTRIBUTION

Z-Number

A Z-number is a REAL NUMBER ξ such that

$$0 \leq \operatorname{frac}\left[\left(\frac{3}{2}\right)^k \xi\right] < \tfrac{1}{2}$$

for all $k = 1, 2, \ldots$, where $\operatorname{frac}(x)$ is the fractional part of x. Mahler (1968) showed that there is at most one Z-number in each interval $[n, n+1)$ for integral n. Mahler (1968) therefore concluded that it is unlikely that any Z-numbers exist. The Z-numbers arise in the analysis of the COLLATZ PROBLEM.

see also COLLATZ PROBLEM

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z-Score

The z-score associated with the ith observation of a random variable x is given by

$$z_i \equiv \frac{x_i - \bar{x}}{\sigma},$$

where $\bar{x}$ is the MEAN and σ the STANDARD DEVIATION of all observations $x_1, \ldots, x_n$.

z-Transform

The discrete z-transform is defined as

$$\mathcal{Z}[a] = \sum_{n=0}^{N-1} a_n z^{kn}. \tag{1}$$

The DISCRETE FOURIER TRANSFORM is a special case of the z-transform with

$$z \equiv e^{-2\pi i/N}. \tag{2}$$

A z-transform with

$$z \equiv e^{-2\pi i\alpha/N} \tag{3}$$

for $\alpha \neq \pm 1$ is called a FRACTIONAL FOURIER TRANSFORM.

see also DISCRETE FOURIER TRANSFORM, FRACTIONAL FOURIER TRANSFORM

References

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z-Transform (Population)

see POPULATION COMPARISON

Z-Transform

The Z-transform of $F(t)$ is defined by

$$Z[F(t)] = \mathcal{L}[F^*(t)], \tag{1}$$

where

$$F^*(t) = F(t)\delta_T(t) = \sum_{n=0}^{\infty} F(nT)\delta(t - nT), \tag{2}$$

$\delta(t)$ is the DELTA FUNCTION, T is the sampling period, and $\mathcal{L}$ is the LAPLACE TRANSFORM. An alternative definition is

$$Z[F(t)] = \sum_{\text{residues}} \left(\frac{1}{1 - e^{Tz}z^{-1}}\right) f(z), \tag{3}$$

where

$$f(z) = \sum_{n=0}^{\infty} F(nT)z^{-n}. \tag{4}$$

The inverse Z-transform is

$$Z^{-1}[f(z)] = F^*(t) = \frac{1}{2\pi i}\oint f(z)z^{n-1}\,dz. \tag{5}$$

It satisfies

$$Z[aF(t) + bG(t)] = aZ[F(t)] + bZ[F(t)] \tag{6}$$
$$Z[F(t+T)] = zZ[F(t)] - zF(0) \tag{7}$$
$$Z[F(t+2T)] = z^2 Z[F(t)] - z^2 F(0) - zF(t) \tag{8}$$
$$Z[F(t+mT)] = z^m Z[F(t)] - \sum_{r=0}^{m-1} z^{m-r}F(rt) \tag{9}$$
$$Z[F(t-mT)] = z^{-m}Z[F(t)] \tag{10}$$
$$Z[e^{at}F(t)] = Z[e^{-aT}z] \tag{11}$$
$$Z[e^{-at}F(t)] = Z[e^{aT}z] \tag{12}$$
$$tF(t) = -Tz\frac{d}{dz}Z[F(t)] \tag{13}$$
$$t^{-1}F(t) = -\frac{1}{T}\int_0^z \frac{f(z)}{z}\,dz. \tag{14}$$

Transforms of special functions (Beyer 1987, pp. 426–427) include

$$Z[\delta(t)] = 1 \tag{15}$$
$$Z[\delta(t-mT)] = z^{-m} \tag{16}$$
$$Z[H(t)] = \frac{z}{z-1} \tag{17}$$
$$Z[H(t-mT)] = \frac{z}{z^m(z-1)} \tag{18}$$
$$Z[t] = \frac{Tz}{(z-1)^2} \tag{19}$$
$$Z[t^2] = \frac{T^2 z(z+1)}{(z-1)^3} \tag{20}$$
$$Z[t^3] = \frac{T^3 z(z^2+4z+1)}{(z-1)^4} \tag{21}$$
$$Z[a^{\omega t}] = \frac{z}{z-a^{\omega T}} \tag{22}$$
$$Z[\cos(\omega t)] = \frac{z\sin(\omega T)}{z^2 - 2z\cos(\omega T) + 1} \tag{23}$$
$$Z[\sin(\omega t)] = \frac{z[z - \cos(\omega T)]}{z^2 - 2z\cos(\omega T) + 1}, \tag{24}$$

where $H(t)$ is the HEAVISIDE STEP FUNCTION. In general,

$$Z[t^n] = (-1)^n \lim_{x\to 0}\frac{\partial^n}{\partial x^n}\left(\frac{z}{z-e^{-xT}}\right) \tag{25}$$
$$= \frac{T^n z\sum_{k=1}^n \left\langle {n \atop k} \right\rangle z^{k-1}}{(z-1)^{n+1}}, \tag{26}$$

where the $\left\langle {n \atop k} \right\rangle$ are EULERIAN NUMBERS. Amazingly, the Z-transforms of t^n are therefore generators for EULER'S TRIANGLE.

see also EULER'S TRIANGLE, EULERIAN NUMBER

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Zag Number

An EVEN ALTERNATING PERMUTATION number, more commonly called a TANGENT NUMBER.

see also ALTERNATING PERMUTATION, TANGENT NUMBER, ZIG NUMBER

Zarankiewicz's Conjecture

The CROSSING NUMBER for a COMPLETE BIGRAPH is

$$\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION. This has been shown to be true for all $m, n \leq 7$. Zarankiewicz has shown that, in general, the FORMULA provides an upper bound to the actual number.

see also COMPLETE BIGRAPH, CROSSING NUMBER (GRAPH)

Zariski Topology

A TOPOLOGY of an infinite set whose OPEN SETS have finite complements.

Zaslavskii Map

The 2-D map

$$x_{n+1} = [x_n + \nu(1+\mu y_n) + \epsilon\nu\mu\cos(2\pi x_n)] \pmod 1$$
$$y_{n+1} = e^{-\Gamma}[y_n + \epsilon\cos(2\pi x_n)],$$

where

$$\mu \equiv \frac{1-e^{-\Gamma}}{\Gamma}$$

(Zaslavskii 1978). It has CORRELATION EXPONENT $\nu \approx 1.5$ (Grassberger and Procaccia 1983) and CAPACITY DIMENSION 1.39 (Russell *et al.* 1980).

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Zaslavskii, G. M. "The Simplest Case of a Strange Attractor." *Phys. Let.* **69A**, 145–147, 1978.

Zassenhaus-Berlekamp Algorithm

A method for factoring POLYNOMIALS.

Zeckendorf Representation

A number written as a sum of nonconsecutive FIBONACCI NUMBERS,

$$n = \sum_{k=0}^{L} \epsilon_k F_k,$$

where ϵ_k are 0 or 1 and

$$\epsilon_k \epsilon_{k+1} = 0.$$

Every POSITIVE INTEGER can be written uniquely in such a form.

see also ZECKENDORF'S THEOREM

References

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Zeckendorf's Theorem

The SEQUENCE $\{F_n - 1\}$ is COMPLETE even if restricted to subsequences which contain no two consecutive terms, where F_n is a FIBONACCI NUMBER.

see also FIBONACCI DUAL THEOREM, ZECKENDORF REPRESENTATION

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Zeeman's Paradox

There is only one point in front of a PERSPECTIVE drawing where its three mutually PERPENDICULAR VANISHING POINTS appear in mutually PERPENDICULAR directions, but such a drawing nonetheless appears realistic from a variety of distances and angles.

see also LEONARDO'S PARADOX, PERSPECTIVE, VANISHING POINT

References

Dixon, R. *Mathographics.* New York: Dover, p. 82, 1991.

Zeilberger's Algorithm

An ALGORITHM which finds a POLYNOMIAL recurrence for a terminating HYPERGEOMETRIC IDENTITIES of the form

$$\sum_k \binom{n}{k} \frac{\prod_{i=1}^{A}(a_i n + a_i' k + a_i'')!}{\prod_{i=1}^{B}(b_i n + b_i' k + b_i'')!} z_k = C \frac{\prod_{i=1}^{\bar{A}}(\bar{a}_i n + \bar{a}_i')!}{\prod_{i=1}^{\bar{B}}(\bar{b}_i n + \bar{b}_i')} \bar{x}^n,$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT, a_i, a_i', $\bar{a}_i$, b_i, b_i', $\bar{b}_i$ are constant integers and a_i'', $\bar{a}_i'$, b_i'', $\bar{b}_i'$, C, x, and z are complex numbers (Zeilberger 1990). The method was called CREATIVE TELESCOPING by van der Poorten (1979), and led to the development of the amazing machinery of WILF-ZEILBERGER PAIRS.

see also BINOMIAL SERIES, GOSPER'S ALGORITHM, HYPERGEOMETRIC IDENTITY, SISTER CELINE'S METHOD, WILF-ZEILBERGER PAIR

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Zeilberger, D. "The Method of Creative Telescoping." *J. Symb. Comput.* **11**, 195–204, 1991.

Zeisel Number

A number $N = p_1 p_2 \cdots p_k$ (where the p_is are distinct PRIMES) such that

$$p_n = Ap_{n-1} + B,$$

with A and B constants and $p_0 \equiv 1$. For example, $1885 = 1 \cdot 5 \cdot 13 \cdot 29$ and $114985 = 1 \cdot 5 \cdot 13 \cdot 29 \cdot 61$ are Zeisel numbers with $(A, B) = (2, 3)$.

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Brown, K. S. "Zeisel Numbers." http://www.seanet.com/~ksbrown/kmath015.htm.

Zeno's Paradoxes

A set of four PARADOXES dealing with counterintuitive aspects of continuous space and time.

1. Dichotomy paradox: Before an object can travel a given distance d, it must travel a distance $d/2$. In order to travel $d/2$, it must travel $d/4$, etc. Since this sequence goes on forever, it therefore appears that the distance d cannot be traveled. The resolution of the paradox awaited CALCULUS and the proof that infinite GEOMETRIC SERIES such as $\sum_{i=1}^{\infty}(1/2)^i = 1$ can converge, so that the infinite number of "half-steps" needed is balanced by the increasingly short amount of time needed to traverse the distances.
2. Achilles and the tortoise paradox: A fleet-of-foot Achilles is unable to catch a plodding tortoise which has been given a head start, since during the time it takes Achilles to catch up to a given position, the tortoise has moved forward some distance. But this is obviously fallacious since Achilles will clearly pass the tortoise! The resolution is similar to that of the dichotomy paradox.
3. Arrow paradox: An arrow in flight has an instantaneous position at a given instant of time. At that instant, however, it is indistinguishable from a motionless arrow in the same position, so how is the motion of the arrow perceived?
4. Stade paradox: A paradox arising from the assumption that space and time can be divided only by a definite amount.

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Zermelo's Axiom of Choice

see AXIOM OF CHOICE

Zermelo-Fraenkel Axioms

The Zermelo-Fraenkel axioms are the basis for ZERMELO-FRAENKEL SET THEORY. In the following, $\exists$ stands for EXISTS, $\in$ for "is an element of," $\forall$ for FOR ALL, $\Rightarrow$ for IMPLIES, $\neg$ for NOT (NEGATION), $\wedge$ for AND, $\vee$ for OR, $\rightleftharpoons$ for "is EQUIVALENT to," and $\mathcal{S}$ denotes the union y of all the sets that are the elements of x.

1. Existence of the empty set: $\exists x \forall u \neg(u \in x)$.
2. Extensionality axiom: $\forall x \forall y (\forall u (u \in x \rightleftharpoons u \in y) \to x = y)$.
3. Unordered pair axiom: $\forall x \forall y \exists z \forall u (u \in z \rightleftharpoons u = x \vee u = y)$.
4. Union (or "sum-set") axiom: $\forall x \exists y \forall u (u \in y \rightleftharpoons \exists v (u \in v \wedge v \in x))$.
5. Subset axiom: $\forall x \exists y \forall u (u \in y \rightleftharpoons \forall v (v \in u \to v \in x))$.
6. Replacement axiom: For any set-theoretic formula $A(u, v)$,

$$\begin{aligned}&\forall u \forall v \forall w (A(u,v) \wedge A(u,w) \to v = w)\\ &\qquad \to \forall x \exists y \forall v (v \in y \rightleftharpoons \exists u (u \in x \wedge A(u,v))).\end{aligned}$$

7. Regularity axiom: For any set-theoretic formula $A(u)$, $\exists x A(x) \to \exists x (A(x) \wedge \neg \exists y (A(y) \wedge y \in x))$.
8. AXIOM OF CHOICE:

$$\begin{aligned}&\forall x [\forall u (u \in x \to \exists v (v \in u))\\ &\qquad \wedge \forall u \forall v ((u \in x \wedge v \in x \wedge \neg u = v)\\ &\qquad \to \neg \exists w (w \in u \wedge w \in v)) \to \exists y \{y \subset \mathcal{S}(x)\\ &\qquad \wedge \forall u (u \in x \to \exists z (z \in u \wedge z \in y\\ &\qquad \wedge \forall w (w \in u \wedge w \in y \to w = z)))\}]\end{aligned}$$

9. Infinity axiom: $\exists x (\exists u (u \in x) \wedge \forall u (u \in x \to \exists v (v \in x \wedge u \subset v \wedge \neg v = u)))$.

If Axiom 6 is replaced by

6'. Axiom of subsets: for any set-theoretic formula $A(u)$, $\forall x \exists y \forall u (u \in y \rightleftharpoons u \in x \wedge A(u))$,

which can be deduced from Axiom 6, then the set theory is called ZERMELO SET THEORY instead of ZERMELO-FRAENKEL SET THEORY.

Abian (1969) proved CONSISTENCY and independence of four of the Zermelo-Fraenkel axioms.

see also ZERMELO-FRAENKEL SET THEORY

References

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Zermelo-Fraenkel Set Theory

A version of SET THEORY which is a formal system expressed in first-order predicate LOGIC. Zermelo-Fraenkel set theory is based on the ZERMELO-FRAENKEL AXIOMS.

see also LOGIC, SET THEORY, ZERMELO-FRAENKEL AXIOMS, ZERMELO SET THEORY

Zermelo Set Theory

The version of set theory obtained if Axiom 6 of ZERMELO-FRAENKEL SET THEORY is replaced by

6'. Axiom of subsets: for any set-theoretic formula $A(u)$, $\forall x \exists y \forall u (u \in y \rightleftharpoons u \in x \wedge A(u))$,

which can be deduced from Axiom 6.

see also ZERMELO-FRAENKEL SET THEORY

References

Iyanaga, S. and Kawada, Y. (Eds.). "Zermelo-Fraenkel Set Theory." §35B in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 135, 1980.

Zernike Polynomial

ORTHOGONAL POLYNOMIALS which arise in the expansion of a wavefront function for optical systems with circular pupils. The ODD and EVEN Zernike polynomials are given by

$$\begin{matrix} {}^oU_n^m(\rho,\phi) \\ {}^eU_n^m(\rho,\phi) \end{matrix} = R_n^m(\rho) \begin{matrix}\sin\\ \cos\end{matrix}(m\phi) \quad (1)$$

with radial function

$$R_n^m(\rho) = \sum_{l=0}^{(n-m)/2} \frac{(-1)^l (n-l)!}{l![\frac{1}{2}(n+m)-l]![\frac{1}{2}(n-m)-l]!}\rho^{n-2l} \quad (2)$$

for n and m integers with $n \geq m \geq 0$ and $n-m$ EVEN. Otherwise,

$$R_n^m(\rho) = 0. \quad (3)$$

Here, ϕ is the azimuthal angle with $0 \leq \phi < 2\pi$ and ρ is the radial distance with $0 \leq \rho \leq 1$ (Prata and Rusch 1989). The radial functions satisfy the orthogonality relation

$$\int_0^1 R_n^m(\rho) R_{n'}^m(\rho)\rho\, d\rho = \frac{1}{2(n+1)}\delta_{nn'}, \quad (4)$$

where δ_{ij} is the KRONECKER DELTA, and are related to the BESSEL FUNCTION OF THE FIRST KIND by

$$\int_0^1 R_n^m(\rho) J_m(v\rho)\rho\, d\rho = (-1)^{(n-m)/2}\frac{J_{n+1}(v)}{v} \quad (5)$$

(Born and Wolf 1989, p. 466). The radial Zernike polynomials have the GENERATING FUNCTION

$$\frac{[1+z-\sqrt{1-2z(1-2\rho^2)+z^2}\,]^m}{(2z\rho)^m\sqrt{1-2z(1-2\rho^2)+z^2}} = \sum_{s=0}^\infty z^s R_{m+2s}^{\pm m}(\rho), \quad (6)$$

and are normalized so that

$$R_n^{\pm m}(1) = 1 \quad (7)$$

(Born and Wolf 1989, p. 465). The first few NONZERO radial polynomials are

$$\begin{aligned} R_0^0(\rho) &= 1 \\ R_1^1(\rho) &= \rho \\ R_2^0(\rho) &= 2\rho^2 - 1 \\ R_2^2(\rho) &= \rho^2 \\ R_3^1(\rho) &= 3\rho^3 - 2\rho \\ R_3^3(\rho) &= \rho^3 \\ R_4^0(\rho) &= 6\rho^4 - 6\rho^2 + 1 \\ R_4^2(\rho) &= 4\rho^4 - 3\rho^2 \\ R_4^4(\rho) &= \rho^4 \end{aligned}$$

(Born and Wolf 1989, p. 465).

The Zernike polynomial is a special case of the JACOBI POLYNOMIAL with

$$P_{n'}^{(\alpha,\beta)}(x) = (-1)^{n'}\frac{R_n^m(\rho)}{\rho^\alpha} \quad (8)$$

and

$$x = 1 - 2\rho^2 \quad (9)$$
$$\beta = 0 \quad (10)$$
$$\alpha = m \quad (11)$$
$$n' = \tfrac{1}{2}(n-m). \quad (12)$$

The Zernike polynomials also satisfy the RECURRENCE RELATIONS

$$\rho R_n^m(\rho) = \frac{1}{2(n+1)}[(n+m+2)R_{n+1}^{m+1}(\rho) + (n-m)R_{n-1}^{m+1}(\rho)] \quad (13)$$

$$R_{n+2}^m(\rho) = \frac{n+2}{(n+2)^2-m^2}\left\{\left[4(n+1)\rho^2 - \frac{(n+m)^2}{n} - \frac{(n-m+2)^2}{n+2}\right]R_n^m(\rho) - \frac{n^2-m^2}{n}R_{n-2}^m(\rho)\right. \quad (14)$$

$$R_n^m(\rho) + R_n^{m+2}(\rho) = \frac{1}{n+1}\frac{d[R_{n+1}^{m+1}(\rho) - R_{n-1}^{m+1}(\rho)]}{d\rho} \quad (15)$$

(Prata and Rusch 1989). The coefficients A_n^m and B_n^m in the expansion of an arbitrary radial function $F(\rho,\phi)$ in terms of Zernike polynomials

$$F(\rho,\phi) = \sum_{m=0}^\infty \sum_{n=m}^\infty [A_n^m\, {}^oU_n^m(\rho,\phi) + B_n^m\, {}^eU_n^m(\rho,\phi)] \quad (16)$$

are given by

$$\frac{A_n^m}{B_n^m} = \frac{(n+1)}{\epsilon_{mn}{}^2\pi}\int_0^1\int_0^{2\pi} F(\rho,\phi)\,\frac{{}^oU_n^m(\rho,\phi)}{{}^eU_n^m(\rho,\phi)}\,\rho\,d\phi\,d\rho, \tag{17}$$

where

$$\epsilon_{mn} \equiv \begin{cases} \epsilon \equiv \frac{1}{\sqrt{2}} & \text{for } m=0,\ n\neq 0 \\ 1 & \text{otherwise} \end{cases} \tag{18}$$

Let a "primary" aberration be given by

$$\Phi = a'_{lmn} Y_1^{2l+m^*}(\theta,\phi)\rho^n \cos^m\theta \tag{19}$$

with $2l+m+n=4$ and where Y^* is the COMPLEX CONJUGATE of Y, and define

$$A'_{lmn} = a'_{lmn} Y_1^{2l+m^*}(\theta,\phi), \tag{20}$$

giving

$$\Phi = \frac{1}{\epsilon_{nm}{}^2} A_{lmn} R_n^m(\rho)\cos(m\theta). \tag{21}$$

Then the types of primary aberrations are given in the following table (Born and Wolf 1989, p. 470).

Aberration	l	m	n	A	A'
spherical aberration	0	4	0	$A'_{040}\rho^4$	$\epsilon A_{040} R_4^0(\rho)$
coma	0	3	1	$A'_{031}\rho^3\cos\theta$	$A_{031}R_3^1(\rho)\cos\theta$
astigmatism	0	2	2	$A'_{022}\rho^2\cos^2\theta$	$A_{022}R_2^2(\rho)\cos(2\theta)$
field curvature	1	2	0	$A'_{120}\rho^2$	$\epsilon A_{120}R_2^0(\rho)$
distortion	1	1	1	$A'_{111}\rho\cos\theta$	$A_{111}R_1^1(\rho)\cos\theta$

see also JACOBI POLYNOMIAL

References

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Zernike, F. "Beugungstheorie des Schneidenverfahrens und seiner verbesserten Form, der Phasenkontrastmethode." *Physica* **1**, 689–704, 1934.

Zhang, S. and Shannon, R. R. "Catalog of Spot Diagrams." Ch. 4 in *Applied Optics and Optical Engineering, Vol. 11.* New York: Academic Press, p. 201, 1992.

Zero

The INTEGER denoted 0 which, when used as a counting number, means that no objects are present. It is the only INTEGER (and, in fact, the only REAL NUMBER) which is neither NEGATIVE nor POSITIVE. A number which is not zero is said to be NONZERO.

Because the number of PERMUTATIONS of 0 elements is 1, 0! (zero FACTORIAL) is often defined as 1. This definition is useful in expressing many mathematical identities in simple form. A number *other than 0* taken to the POWER 0 is defined to be 1. 0^0 is undefined, but defining $0^0 = 1$ allows concise statement of the beautiful analytical formula for the integral of the generalized SINC FUNCTION

$$\int_0^\infty \frac{\sin^a x}{x^b}\,dx = \frac{\pi^{1-c}(-1)^{\lfloor(a-b)/2\rfloor}}{2^{a-c}(b-1)!} \times \sum_{k=0}^{\lfloor a/2\rfloor-c} (-1)^k \binom{a}{k}(a-2k)^{b-1}[\ln(a-2k)]^c$$

given by Kogan, where $a \geq b > c$, $c \equiv a-b \pmod 2$, and $\lfloor x\rfloor$ is the FLOOR FUNCTION.

The following table gives the first few numbers n such that n^k contains no zeros, for small k. The largest known n for which 2^n contain no zeros is 86 (Madachy 1979), with no other $n \leq 4.6\times 10^7$ (M. Cook), improving the 3.0739×10^7 limit obtained by Beeler *et al.* (1972). The values $a(n)$ such that the positions of the right-most zero in $2^{a(n)}$ increases are 10, 20, 30, 40, 46, 68, 93, 95, 129, 176, 229, 700, 1757, 1958, 7931, 57356, 269518, ... (Sloane's A031140). The positions in which the right-most zeros occur are 2, 5, 8, 11, 12, 13, 14, 23, 36, 38, 54, 57, 59, 93, 115, 119, 120, 121, 136, 138, 164, ... (Sloane's A031141). The right-most zero of $2^{781,717,865}$ occurs at the 217th decimal place, the farthest over for powers up to 2.5×10^9.

k	Sloane	n such that n^k contains no 0s
2	007377	1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 14, 15, 16, ...
3	030700	1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, ...
4	030701	1, 2, 3, 4, 7, 8, 9, 12, 14, 16, 17, 18, ...
5	008839	1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 17, 18, 30, ...
6	030702	1, 2, 3, 4, 5, 6, 7, 8, 12, 17, 24, 29, 44, ...
7	030703	1, 2, 3, 6, 7, 10, 11, 19, 35, ...
8	030704	1, 2, 3, 5, 6, 8, 9, 11, 12, 13, 17, 24, 27, ...
9	030705	1, 2, 3, 4, 6, 7, 12, 13, 14, 17, 34, ...
11	030706	1, 2, 3, 4, 6, 7, 8, 9, 12, 13, 14, 15, 16, ...

While it has not been proven that the numbers listed above are the only ones without zeros for a given base, the probability that any additional ones exist is vanishingly small. Under this assumption, the sequence of largest n such that k^n contains no zeros for $k = 2$, 3, ... is then given by 86, 68, 43, 58, 44, 35, 27, 34, 0, 41, ... (Sloane's A020665).

see also 10, NAUGHT, NEGATIVE, NONNEGATIVE, NONZERO, ONE, POSITIVE, TWO

References

Beeler, M.; Gosper, R. W.; and Schroeppel, R. *HAKMEM.* Cambridge, MA: MIT Artificial Intelligence Laboratory, Memo AIM-239, Item 57, Feb. 1972.

Kogan, S. "A Note on Definite Integrals Involving Trigonometric Functions." `http://www.mathsoft.com/asolve/constant/pi/sin/sin.html`.

Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, pp. 127–128, 1979.

Pappas, T. "Zero–Where & When." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 162, 1989.

Sloane, N. J. A. Sequence A007377/M0485 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Zero Divisor

A NONZERO element x of a RING for which $x \cdot y = 0$, where y is some other NONZERO element and the vector multiplication $x \cdot y$ is assumed to be BILINEAR. A RING with no zero divisors is known as an INTEGRAL DOMAIN. Let A denote an $\mathbb{R}$-algebra, so that A is a VECTOR SPACE over R and

$$A \times A \to A$$

$$(x, y) \mapsto x \cdot y.$$

Now define

$$Z \equiv \{x \in A : x \cdot y = 0 \text{ for some NONZERO } y \in A\},$$

where $0 \in Z$. A is said to be m-ASSOCIATIVE if there exists an m-dimensional SUBSPACE S of A such that $(y \cdot x) \cdot z = y \cdot (x \cdot z)$ for all $y, z \in A$ and $x \in S$. A is said to be TAME if Z is a finite union of SUBSPACES of A.

References

Finch, S. "Zero Structures in Real Algebras." `http://www.mathsoft.com/asolve/zerodiv/zerodiv.html`.

Zero (Root)

see ROOT

Zero-Sum Game

A GAME in which players make payments only to each other. One player's loss is the other player's gain, so the total amount of "money" available remains constant.

see also FINITE GAME, GAME

References

Dresher, M. *The Mathematics of Games of Strategy: Theory and Applications.* New York: Dover, p. 2, 1981.

Zeta Fuchsian

A class of functions discovered by Poincaré which are related to the AUTOMORPHIC FUNCTIONS.

see also AUTOMORPHIC FUNCTION

Zeta Function

A function satisfying certain properties which is computed as an INFINITE SUM of NEGATIVE POWERS. The most commonly encountered zeta function is the RIEMANN ZETA FUNCTION,

$$\zeta(n) \equiv \sum_{k=1}^{\infty} \frac{1}{k^n}.$$

see also DEDEKIND FUNCTION, DIRICHLET BETA FUNCTION, DIRICHLET ETA FUNCTION, DIRICHLET L-SERIES, DIRICHLET LAMBDA FUNCTION, EPSTEIN ZETA FUNCTION, JACOBI ZETA FUNCTION, NINT ZETA FUNCTION, PRIME ZETA FUNCTION, RIEMANN ZETA FUNCTION

References

Ireland, K. and Rosen, M. "The Zeta Function." Ch. 11 in *A Classical Introduction to Modern Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 151–171, 1990.

Zeuthen's Rule

On an ALGEBRAIC CURVE, the sum of the number of coincidences at a noncuspidal point C is the sum of the orders of the infinitesimal distances from a nearby point P to the corresponding points when the distance PC is taken as the principal infinitesimal.

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 131, 1959.

Zeuthen's Theorem

If there is a (ν, ν') correspondence between two curves of GENUS p and p' and the number of BRANCH POINTS properly counted are β and β', then

$$\beta + 2\nu'(p-1) = \beta' + 2\nu(p'-1).$$

see also CHASLES-CAYLEY-BRILL FORMULA

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 246, 1959.

Zig Number

An ODD ALTERNATING PERMUTATION number, more commonly called an EULER NUMBER or SECANT NUMBER.

see also ALTERNATING PERMUTATION, EULER NUMBER, ZAG NUMBER

Zig-Zag Triangle

see also SEIDEL-ENTRINGER-ARNOLD TRIANGLE

Zigzag Permutation

see ALTERNATING PERMUTATION

Zillion

A generic word for a very LARGE NUMBER. The term has no well-defined mathematical meaning. Conway and Guy (1996) define the nth zillion as 10^{3n+3} in the American system (million $= 10^6$, billion $= 10^9$, trillion $= 10^{12}$, ...) and 10^{6n} in the British system (million $= 10^6$, billion $= 10^{12}$, trillion $= 10^{18}$, ...). Conway and Guy (1996) also define the words n-PLEX and n-MINEX for 10^n and 10^{-n}, respectively.

see also LARGE NUMBER

References

Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 13–16, 1996.

Zipf's Law

In the English language, the probability of encountering the rth most common word is given roughly by $P(r) = 0.1/r$ for r up to 1000 or so. The law breaks down for less frequent words, since the HARMONIC SERIES diverges. Pierce's (1980, p. 87) statement that $\sum P(r) > 1$ for $r = 8727$ is incorrect. Goetz states the law as follows: The frequency of a word is inversely proportional to its RANK r such that

$$P(r) \approx \frac{1}{r \ln(1.78R)},$$

where R is the number of different words.

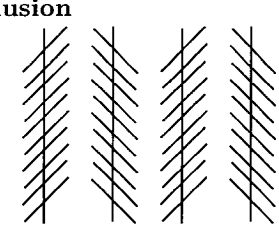
see also HARMONIC SERIES, RANK (STATISTICS)

References

Goetz, P. "Phil's Good Enough Complexity Dictionary." http://www.cs.buffalo.edu/~goetz/dict.html.

Pierce, J. R. *Introduction to Information Theory: Symbols, Signals, and Noise, 2nd rev. ed.* New York: Dover, pp. 86–87 and 238–239, 1980.

Zollner's Illusion



In this ILLUSION, the VERTICAL lines in the above figure are PARALLEL, but appear to be tilted at an angle.

see also ILLUSION

References

Jablan, S. "Some Visual Illusions Occurring in Interrupted Systems." http://members.tripod.com/~modularity/interr.htm.

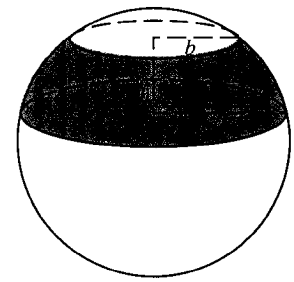
Pappas, T. *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 172, 1989.

Zonal Harmonic

A SPHERICAL HARMONIC which is a product of factors linear in x^2, y^2, and z^2, with the product multiplied by z when n is ODD.

see also TESSERAL HARMONIC

Zone



The SURFACE AREA of a SPHERICAL SEGMENT. Call the RADIUS of the SPHERE R, the upper and lower RADII b and a, respectively, and the height of the SPHERICAL SEGMENT h. The zone is a SURFACE OF REVOLUTION about the z-AXIS, so the SURFACE AREA is given by

$$S = 2\pi \int x\sqrt{1+x'^2}\,dz. \tag{1}$$

In the xz-plane, the equation of the zone is simply that of a CIRCLE,

$$x = \sqrt{R^2 - z^2}, \tag{2}$$

so

$$x' = -z(R^2 - z^2)^{-1/2} \tag{3}$$

$$x'^2 = \frac{z^2}{R^2 - z^2}, \tag{4}$$

and

$$\begin{aligned} S &= 2\pi \int_{\sqrt{R^2-a^2}}^{\sqrt{R^2-b^2}} \sqrt{R^2 - z^2}\sqrt{1 + \frac{z^2}{R^2 - z^2}}\,dz \\ &= 2\pi R \int_{\sqrt{R^2-a^2}}^{\sqrt{R^2-b^2}} dz = 2\pi R(\sqrt{R^2 - b^2} - \sqrt{R^2 - a^2}\,) \\ &= 2\pi Rh. \end{aligned} \tag{5}$$

This result is somewhat surprising since it depends only on the *height* of the zone, not its vertical position with respect to the SPHERE.

see also SPHERE, SPHERICAL CAP, SPHERICAL SEGMENT, ZONOHEDRON

References

Beyer, W. H. (Ed.). *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 130, 1987.

Zonohedron

A convex POLYHEDRON whose faces are PARALLEL-sided $2m$-gons. There exist $n(n-1)$ PARALLELOGRAMS in a nonsingular zonohedron, where n is the number of different directions in which EDGES occur (Ball and Coxeter

1987, pp. 141–144). Zonohedra include the CUBE, ENNEACONTAHEDRON, GREAT RHOMBIC TRIACONTAHEDRON, MEDIAL RHOMBIC TRIACONTAHEDRON, RHOMBIC DODECAHEDRON, RHOMBIC ICOSAHEDRON, RHOMBIC TRIACONTAHEDRON, RHOMBOHEDRON, and TRUNCATED CUBOCTAHEDRON, as well as the entire class of PARALLELEPIPEDS.

Regular zonohedra have bands of PARALLELOGRAMS which form equators and are called "ZONES." Every convex polyhedron bounded solely by PARALLELOGRAMS is a zonohedron (Coxeter 1973, p. 27). Plate II (following p. 32 of Coxeter 1973) illustrates some equilateral zonohedra. Equilateral zonohedra can be regarded as 3-dimensional projections of n-D HYPERCUBES (Ball and Coxeter 1987).

see also HYPERCUBE

References
Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 141–144, 1987.
Coxeter, H. S. M. "Zonohedra." §2.8 in *Regular Polytopes, 3rd ed.* New York: Dover, pp. 27–30, 1973.
Coxeter, H. S. M. Ch. 4 in *Twelve Geometric Essays.* Carbondale, IL: Southern Illinois University Press, 1968.
Eppstein, D. "Ukrainian Easter Egg." http://www.ics.uci.edu/~eppstein/junkyard/ukraine.
Fedorov, E. S. *Zeitschr. Krystallographie und Mineralogie* **21**, 689, 1893.
Fedorov, E.W. *Nachala Ucheniya o Figurakh.* Leningrad, 1953.
Hart, G. W. "Zonohedra." http://www.li.net/~george/virtual-polyhedra/zonohedra-info.html.

Zonotype

The MINKOWSKI SUM of line segments.

Zorn's Lemma

If S is any nonempty PARTIALLY ORDERED SET in which every CHAIN has an upper bound, then S has a maximal element. This statement is equivalent to the AXIOM OF CHOICE.

see also AXIOM OF CHOICE

Zsigmondy Theorem

If $1 \leq b < a$ and $(a, b) = 1$ (i.e., a and b are RELATIVELY PRIME), then $a^n - b^n$ has a PRIMITIVE PRIME FACTOR with the following two possible exceptions:

1. $2^6 - 1^6$.
2. $n = 2$ and $a + b$ is a POWER of 2.

Similarly, if $a > b \geq 1$, then $a^n + b^n$ has a PRIMITIVE PRIME FACTOR with the exception $2^3 + 1^3 = 9$.

References
Ribenboim, P. *The Little Book of Big Primes.* New York: Springer-Verlag, p. 27, 1991.